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MULTIPLICATIVE REFINEMENTS AND REVERSES OF YOUNG'S OPERATOR INEQUALITY WITH APPLICATIONS

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Abstract. In this paper we obtain some new multiplicative refinements and reverses of Young's operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

1. Introduction

Throughout this paper A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean.* When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-*weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [11]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1, \infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b} \right)^r \right) a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \leq S\left(\frac{a}{b} \right) a^{1-\nu} b^\nu, \tag{1.3}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [12] while the first one is due to Furuichi [6].

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The operator version is as follows [6], [12] :

Theorem 1. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*
(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$,
(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$,
we have

$$S\left((h')^r\right) A\sharp_\nu B \leq A\nabla_\nu B \leq S(h) A\sharp_\nu B, \tag{1.4}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ *and* $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \tag{1.6}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [13] while the second by Liao et al. [9].

The operator version is as follows [13], [9]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*
(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$,
(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$,
we have

$$K^r(h') A\sharp_\nu B \leq A\nabla_\nu B \leq K^R(h) A\sharp_\nu B, \tag{1.7}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$.

In [1] we proved the following multiplicative reverse of Young's inequality

$$1 \leq \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^\nu} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{1.8}$$

for any a, $b > 0$ and $\nu \in [0,1]$, where K is Kantorovich's constant defined by (1.5).

In [2] we also obtained the following multiplicative refinement and reverse of Young's inequality

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ &\leq \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^\nu} \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right] \end{aligned} \tag{1.9}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Other recent results for operators may be found in [1]-[5].

Motivated by the above results we establish in this paper some new multiplicative refinements and reverses of Young's operator inequality. Applications related to the Hölder-McCarthy inequality for positive operators and for trace class operators on Hilbert spaces are given as well.

2. Multiplicative Reverses

We consider the function $g_\nu : (0,\infty) \to (0,\infty)$ defined for $\nu \in (0,1)$ by

$$g_\nu(x) = \frac{1-\nu+\nu x}{x^\nu} = (1-\nu)x^{-\nu} + \nu x^{1-\nu}. \tag{2.1}$$

For $[m,M] \subset (0,\infty)$ define the quantities

$$\Gamma_\nu(m,M) := \begin{cases} g_\nu(m) \text{ if } M<1, \\ \\ \max\{g_\nu(m), g_\nu(M)\} \text{ if } m \le 1 \le M, \\ \\ g_\nu(M) \text{ if } 1<m \end{cases} \tag{2.2}$$

$$= \begin{cases} (1-\nu)m^{-\nu} + \nu m^{1-\nu} \text{ if } M<1, \\ \\ \max\left\{(1-\nu)m^{-\nu} + \nu m^{1-\nu}, (1-\nu)M^{-\nu} + \nu M^{1-\nu}\right\} \\ \text{if } m \le 1 \le M, \\ \\ (1-\nu)M^{-\nu} + \nu M^{1-\nu} \text{ if } 1<m \end{cases}$$

and

$$\gamma_\nu(m,M) := \begin{cases} g_\nu(M) \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ g_\nu(m) \text{ if } 1<m. \end{cases} \tag{2.3}$$

$$= \begin{cases} (1-\nu)M^{-\nu} + \nu M^{1-\nu} \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ (1-\nu)m^{-\nu} + \nu m^{1-\nu} \text{ if } 1<m. \end{cases}$$

The following lemma holds.

Lemma 1. *For any $x \in [m,M] \subset (0,\infty)$ we have*

$$\max_{x\in[m,M]} g_\nu(x) = \Gamma_\nu(m,M) \tag{2.4}$$

and

$$\min_{x\in[m,M]} g_\nu(x) = \gamma_\nu(m,M). \tag{2.5}$$

Proof. The function g_ν is differentiable and

$$g_\nu'(x) = (1-\nu)\nu x^{-\nu-1}(x-1),$$

which shows that the function g_ν is decreasing on $(0,1)$ and increasing on $[1,\infty)$. We have $g_\nu(1)=1$, $\lim_{x\to 0+} g_\nu(x)=+\infty$, $\lim_{x\to\infty} g_\nu(x)=+\infty$ and $g_\nu\left(\frac{1}{x}\right)=g_{1-\nu}(x)$ for any $x>0$ and $\nu\in(0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[m,M]$ and the number 1 we get the desired bounds (2.4) and (2.5). □

The following result provides a multiplicative refinement and reverse for the operator Young's inequality:

Theorem 3. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that*

$$mA\le B\le MA. \tag{2.6}$$

Let $\nu\in[0,1]$, then we have the inequalities

$$\gamma_\nu(m,M)\,A\sharp_\nu B\le A\nabla_\nu B\le \Gamma_\nu(m,M)\,A\sharp_\nu B, \tag{2.7}$$

where $\Gamma_\nu(m,M)$ and $\gamma_\nu(m,M)$ are defined by (2.2) and (2.3), respectively.

Proof. From Lemma 1 we have the double inequality

$$\gamma_\nu(m,M)\,x^\nu\le 1-\nu+\nu x\le \Gamma_\nu(m,M)\,x^\nu \tag{2.8}$$

for any $x\in[m,M]$.

If X is an operator such that $mI\le X\le MI$, then by (2.8) and the continuous functional calculus, we have

$$\gamma_\nu(m,M)\,X^\nu\le(1-\nu)I+\nu X\le\Gamma_\nu(m,M)\,X^\nu. \tag{2.9}$$

If the condition (2.6) holds, then by multiplying in both sides with $A^{-1/2}$ we get $mI\le A^{-1/2}BA^{-1/2}\le MI$ and by taking $X=A^{-1/2}BA^{-1/2}$ in (2.9) we get

$$\begin{aligned}\gamma_\nu(m,M)\left(A^{-1/2}BA^{-1/2}\right)^\nu&\le(1-\nu)I+\nu A^{-1/2}BA^{-1/2}\\&\le\Gamma_\nu(m,M)\left(A^{-1/2}BA^{-1/2}\right)^\nu.\end{aligned} \tag{2.10}$$

Now, if we multiply (2.10) in both sides with $A^{1/2}$ we get the desired result (2.7). □

Corollary 1. *For two positive operators A, B and positive real numbers m, m', M, M' put $h=\frac{M}{m}$ and $h'=\frac{M'}{m'}$. Let $\nu\in(0,1)$.*
If

(i) $0<mI\le A\le m'I<M'I\le B\le MI$,

then

$$g_\nu(h')\,A\sharp_\nu B\le A\nabla_\nu B\le g_\nu(h)\,A\sharp_\nu B. \tag{2.11}$$

If

(ii) $0<mI\le B\le m'I<M'I\le A\le MI$,

then

$$g_{1-\nu}(h')\,A\sharp_\nu B\le A\nabla_\nu B\le g_{1-\nu}(h)\,A\sharp_\nu B. \tag{2.12}$$

Proof. If *(i)* is valid, then we have

$$A<\frac{M'}{m'}A=h'A\le B\le hA=\frac{M}{m}A,$$

and by (2.4) for $1<h'\le h$ we have

$$g_\nu(h')\,A\sharp_\nu B\le A\nabla_\nu B\le g_\nu(h)\,A\sharp_\nu B,$$

and the inequality (2.11) is proved.

If *(ii)* is valid, then we have

$$\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A$$

and by (2.11) for $\frac{1}{h} \leq \frac{1}{h'} < 1$ we also have

$$g_\nu\left(\frac{1}{h'}\right)A\sharp_\nu B \leq A\nabla_\nu B \leq g_\nu\left(\frac{1}{h}\right)A\sharp_\nu B,$$

and since

$$g_\nu\left(\frac{1}{h}\right) = g_{1-\nu}(h),\ g_\nu\left(\frac{1}{h'}\right) = g_{1-\nu}(h'),$$

the inequality (2.12) is proved. □

Remark 1. *By making use of (2.7) and (1.6) we have the following upper and lower bounds in terms of Specht's ratio S*

$$\tag{2.13} \begin{cases} S(M^r)A\sharp_\nu B \text{ if } M < 1, \\ A\sharp_\nu B \text{ if } m \leq 1 \leq M, \\ S(m^r)A\sharp_\nu B \text{ if } 1 < m. \end{cases}$$

$$\leq A\nabla_\nu B$$

$$\leq \begin{cases} S(m)A\sharp_\nu B \text{ if } M < 1, \\ \max\{S(m), S(M)\}A\sharp_\nu B \text{ if } m \leq 1 \leq M, \\ S(M)A\sharp_\nu B \text{ if } 1 < m \end{cases},$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

With the assumptions of Corollary 1, we have, either in the case (i) or in the case (ii) that

$$\tag{2.14} S\left((h')^r\right)A\sharp_\nu B \leq A\nabla_\nu B \leq S(h)A\sharp_\nu B$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$. *We recapture in this way the result from Theorem 1.*

Remark 2. *By making use of (2.7) and (1.3) we have the following upper and lower bounds in terms of Kantorovich's constant K*

$$\begin{cases} K^{r}(M)\,A\sharp_{\nu}B \text{ if } M<1, \\ A\sharp_{\nu}B \text{ if } m\leq 1\leq M, \\ K^{r}(m)\,A\sharp_{\nu}B \text{ if } 1<m. \end{cases} \tag{2.15}$$

$$\leq A\nabla_{\nu}B$$

$$\leq \begin{cases} K^{R}(m)\,A\sharp_{\nu}B \text{ if } M<1, \\ \max\left\{K^{R}(m),K^{R}(M)\right\}A\sharp_{\nu}B \text{ if } m\leq 1\leq M, \\ K^{R}(M)\,A\sharp_{\nu}B \text{ if } 1<m, \end{cases},$$

where $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

With the assumptions of Corollary 1, we have, either in the case (i) or in the case (ii) that

$$K^{r}(h')\,A\sharp_{\nu}B\leq A\nabla_{\nu}B\leq K^{R}(h)\,A\sharp_{\nu}B, \tag{2.16}$$

where $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$. *We recapture in this way the result from Theorem 2.*

Remark 3. *By making use of (2.7) and (1.8) we have the following exponential upper bound*

$$A\nabla_{\nu}B\leq \begin{cases} \exp\left[4\nu(1-\nu)(K(m)-1)\right]A\sharp_{\nu}B \text{ if } M<1, \\ \max\{\exp\left[4\nu(1-\nu)(K(m)-1)\right], \\ \quad \exp\left[4\nu(1-\nu)(K(M)-1)\right]\}A\sharp_{\nu}B \text{ if } m\leq 1\leq M, \\ \exp\left[4\nu(1-\nu)(K(M)-1)\right]A\sharp_{\nu}B \text{ if } 1<m. \end{cases} \tag{2.17}$$

With the assumptions of Corollary 1, we have either in the case (i) or in the case (ii) that

$$A\nabla_{\nu}B\leq \exp\left[4\nu(1-\nu)(K(h)-1)\right]A\sharp_{\nu}B,$$

where $\nu\in[0,1]$.

Remark 4. *By making use of (2.7) and (1.8) we have the following exponential lower and upper bounds*

$$\begin{cases} \exp\left[\frac{1}{2}\nu(1-\nu)(1-M)^2\right] A\sharp_\nu B \text{ if } M<1, \\ \\ A\sharp_\nu B \text{ if } m\le 1\le M, \\ \\ \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{m}\right)^2\right] A\sharp_\nu B \text{ if } 1<m. \end{cases} \tag{2.18}$$

$$\le A\nabla_\nu B$$

$$\le \begin{cases} \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{M}-1\right)^2\right] A\sharp_\nu B \text{ if } M<1, \\ \\ \max\left\{\exp\left[\frac{1}{2}\nu(1-\nu)(m-1)^2\right],\right. \\ \left.\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{1}{M}-1\right)^2\right]\right\} A\sharp_\nu B \text{ if } m\le 1\le M, \\ \\ \exp\left[\frac{1}{2}\nu(1-\nu)(m-1)^2\right] A\sharp_\nu B \text{ if } 1<m. \end{cases}$$

With the assumptions of Corollary 1, we have either in the case (i) or in the case (ii) that

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2\right] A\sharp_\nu B \tag{2.19}$$
$$\le A\nabla_\nu B \le \exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^2\right] A\sharp_\nu B,$$

where $\nu\in[0,1]$.

Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that the positive invertible operators A, B satisfy the condition

$$mA^p \le B^q \le MA^p. \tag{2.20}$$

Then by replacing A with A^p, B with B^q and $\nu=\frac{1}{q}$ in (2.7) we have

$$\gamma_{\frac{1}{q}}(m,M)\, A^p\sharp_{\frac{1}{q}}B^q \le \frac{1}{p}A^p+\frac{1}{q}B^q \le \Gamma_{\frac{1}{q}}(m,M)\, A^p\sharp_{\frac{1}{q}}B^q, \tag{2.21}$$

where $\Gamma_{\frac{1}{q}}(m,M)$ and $\gamma_{\frac{1}{q}}(m,M)$ are defined by (2.2) and (2.3), respectively.

Assume that A and B satisfy the conditions

$$m_1 I\le A\le M_1 I,\ m_2 I\le B\le M_2 I \tag{2.22}$$

for some $0<m_1<M_1$ and $0<m_2<M_2$. We have from (2.22) that

$$m_1^p I\le A^p\le M_1^p I.$$

Then by (2.22) we also have

$$m_1^p M_2^{-q} I \le m_1^p B^{-q} \le B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \le M_1^p B^{-q} \le M_1^p m_2^{-q} I,$$

which implies that

$$m_1 M_2^{-\frac{q}{p}} I \le \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \le M_1 m_2^{-\frac{q}{p}} I.$$

Now, on using the inequality (2.21) for $m = m_1 M_2^{-\frac{q}{p}}$ and $M = M_1 m_2^{-\frac{q}{p}}$, we get

$$\begin{aligned} &\gamma_{\frac{1}{q}} \left(m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}} \right) A^p \sharp_{\frac{1}{q}} B^q \\ &\le \frac{1}{p} A^p + \frac{1}{q} B^q \\ &\le \Gamma_{\frac{1}{q}} \left(m_1 M_2^{-\frac{q}{p}}, M_1 m_2^{-\frac{q}{p}} \right) A^p \sharp_{\frac{1}{q}} B^q, \end{aligned} \tag{2.23}$$

where $\Gamma_{\frac{1}{q}}(\cdot,\cdot)$ and $\gamma_{\frac{1}{q}}(\cdot,\cdot)$ are defined by (2.2) and (2.3), respectively.

Further bounds may be stated as in Remarks 1-4, however the details are not provided here.

3. Inequalities Related to McCarthy's

By the use of the spectral resolution of $P \ge 0$ and the Hölder inequality, C. A. McCarthy [10] proved that

$$\langle Px, x \rangle^p \le \langle P^p x, x \rangle, \; p \in (1, \infty) \tag{3.1}$$

and

$$\langle P^p x, x \rangle \le \langle Px, x \rangle^p, \; p \in (0, 1) \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$.

From the previous section, for positive numbers a, b with $\frac{b}{a} \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$ we can state the following scalar inequalities

$$\gamma_\nu (m, M) a^{1-\nu} b^\nu \le (1-\nu) a + \nu b \le \Gamma_\nu (m, M) a^{1-\nu} b^\nu, \tag{3.3}$$

where $\gamma_\nu (m, M)$ and $\Gamma_\nu (m, M)$ are defined by (2.2) and (2.3), respectively.

This inequality can be written explicitly as

$$\begin{aligned} &\begin{cases} g_\nu (M) a^{1-\nu} b^\nu \text{ if } M < 1, \\ \\ a^{1-\nu} b^\nu \text{ if } m \le 1 \le M, \\ \\ g_\nu (m) a^{1-\nu} b^\nu \text{ if } 1 < m. \end{cases} \\ &\le (1-\nu) a + \nu b \\ &\le \begin{cases} g_\nu (m) a^{1-\nu} b^\nu \text{ if } M < 1, \\ \\ \max \{ g_\nu (m), g_\nu (M) \} a^{1-\nu} b^\nu \text{ if } m \le 1 \le M, \\ \\ g_\nu (M) a^{1-\nu} b^\nu \text{ if } 1 < m, \end{cases} , \end{aligned} \tag{3.4}$$

where

$$g_\nu (x) = \frac{1 - \nu + \nu x}{x^\nu} = (1-\nu) x^{-\nu} + \nu x^{1-\nu}, \; x > 0. \tag{3.5}$$

We have the following reverse of McCarthy's inequality:

Theorem 4. *Let P and operator such that*

$$zI \leq P \leq ZI \tag{3.6}$$

for some constants $Z > z > 0$.

Then for any $x \in H$ with $\|x\| = 1$ we have

$$(1 \leq) \frac{\langle Px, x\rangle^{\lambda}}{\langle P^{\lambda}x, x\rangle} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\}, \tag{3.7}$$

where $\lambda \in [0,1]$ and the function $g_{\lambda} : (0,\infty) \to (0,\infty)$ is defined by (3.5).

Proof. If $u, v \in [z, Z]$ then $\frac{u}{v} \in \left[\frac{z}{Z}, \frac{Z}{z}\right]$ and by (3.4) we have

$$\left(v^{1-\lambda}u^{\lambda} \leq\right) (1-\lambda)\, v + \lambda u \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}u^{\lambda}$$

for any $\lambda \in [0,1]$.

Fix $v \in [z, Z]$, then by using the functional calculus for the operator P with $zI \leq P \leq ZI$ we have

$$(1-\lambda)\, vI + \lambda P \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda}P^{\lambda} \tag{3.8}$$

for any $\lambda \in [0,1]$.

The inequality (3.8) implies that

$$(1-\lambda)\, vI + \lambda \langle Px, x\rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda} \left\langle P^{\lambda}x, x\right\rangle \tag{3.9}$$

for any $x \in H$ with $\|x\| = 1$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

If we take in (3.9) $v = \langle Px, x\rangle \in [z, Z]$, for $x \in H$ with $\|x\| = 1$, then we have

$$\langle Px, x\rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} \langle Px, x\rangle^{1-\lambda} \left\langle P^{\lambda}x, x\right\rangle,$$

which, by division with $\langle Px, x\rangle^{1-\lambda} > 0$ produces the desired result (3.7). □

Corollary 2. *With the assumptions of Theorem 4 and if $T = \max\{\lambda, 1-\lambda\}$ for $\lambda \in (0,1)$, then we have*

$$(1 \leq) \frac{\langle Px, x\rangle^{\lambda}}{\langle P^{\lambda}x, x\rangle} \leq \begin{cases} S\left(\frac{Z}{z}\right), \\ K^{T}\left(\frac{Z}{z}\right), \\ \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{Z}{z}\right)-1\right)\right], \\ \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{Z}{z}-1\right)^{2}\right] \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

We have:

Theorem 5. *Let A and B be two positive invertible operators, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and m, $M > 0$ such that*

$$m^{p}B^{q} \leq A^{p} \leq M^{p}B^{q}. \tag{3.10}$$

Then we have

$$(1\leq)\,\frac{\left\langle A^{p}x,x\right\rangle^{1/p}\left\langle B^{q}x,x\right\rangle^{1/q}}{\left\langle B^{q}\sharp_{1/p}A^{p}x,x\right\rangle}\leq\max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right),g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}, \tag{3.11}$$

where the function $g_\lambda:(0,\infty)\to(0,\infty)$ *is defined by (3.5).*

Proof. From the inequality (3.7) for $x=\frac{y}{\|y\|}$, $y\neq 0$ we have

$$(1\leq)\,\frac{\langle y,y\rangle^{1-\lambda}\langle Py,y\rangle^{\lambda}}{\langle P^{\lambda}y,y\rangle}\leq\max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right),g_{\lambda}\left(\frac{Z}{z}\right)\right\}, \tag{3.12}$$

provided that P satisfy the condition (3.6).

Now, from (3.10) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^{p}I\leq B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\leq M^{p}I$. By writing the inequality (3.12) for $P=B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}$, $z=m^{p}$, $Z=M^{p}$, $\lambda=\frac{1}{p}$ and $y=B^{\frac{q}{2}}x$, with $x\in H$, $x\neq 0$, we have

$$\begin{aligned}&(1\leq)\,\frac{\left\langle B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{q}}\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle^{\frac{1}{p}}}{\left\langle\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x,B^{\frac{q}{2}}x\right\rangle}\\&\leq\max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right),g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}\end{aligned}$$

that is equivalent to

$$(1\leq)\,\frac{\left\langle B^{q}x,x\right\rangle^{\frac{1}{q}}\left\langle A^{p}x,x\right\rangle^{\frac{1}{p}}}{\left\langle B^{\frac{q}{2}}\left(B^{-\frac{q}{2}}A^{p}B^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{\frac{q}{2}}x,x\right\rangle}\leq\max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^{p}\right),g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^{p}\right)\right\}$$

with $x\in H$, $x\neq 0$. □

Corollary 3. *With the assumptions of Theorem 5 we have for* $x\in H$, $x\neq 0$, *that*

$$(1\leq)\,\frac{\left\langle A^{p}x,x\right\rangle^{1/p}\left\langle B^{q}x,x\right\rangle^{1/q}}{\left\langle B^{q}\sharp_{1/p}A^{p}x,x\right\rangle}\leq\begin{cases}S\left(\left(\frac{M}{m}\right)^{p}\right),\\[2mm] K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^{p}\right),\\[2mm] \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^{p}\right)-1\right)\right],\\[2mm] \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^{p}-1\right)^{2}\right],\end{cases}$$

where $T_{p,q}=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

4. Trace Inequalities

In the general case of Hilbert spaces $(H;\langle\cdot,\cdot\rangle)$, if $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that a bounded linear operator $A\in\mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{4.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \|A^*\|_1$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a *Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{4.2}$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{4.3}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{4.4}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

We have the following trace inequality:

Theorem 6. *Let C be an operator with the property that*

$$zI \leq C \leq ZI \tag{4.5}$$

for some constants z, Z with $Z > z > 0$ and $P \in \mathcal{B}_1(H)$, $P \geq 0$ with $\operatorname{tr}(P) > 0$. Then for any $\lambda \in [0,1]$ we have

$$(1 \leq) \frac{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^{\lambda}(PC)}{\operatorname{tr}(PC^{\lambda})} \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} \tag{4.6}$$

and the function g_λ is defined by (3.5).

Proof. From the proof of Theorem 4 we have

$$(1-\lambda) vI + \lambda \langle Cx, x \rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda} \langle C^{\lambda}x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

This inequality implies that

$$(1-\lambda) v \langle x, x \rangle + \lambda \langle Cx, x \rangle \leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda} \langle C^{\lambda}x, x \rangle \tag{4.7}$$

for any $x \in H$, for any $\lambda \in [0,1]$ and for any $v \in [z, Z]$.

Now, if we take in (4.7) $x = P^{1/2}e$, where $e \in H$, then

$$\begin{aligned} &(1-\lambda) v \langle Pe, e \rangle + \lambda \left\langle P^{1/2}CP^{1/2}e, e \right\rangle \\ &\leq \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right), g_{\lambda}\left(\frac{Z}{z}\right)\right\} v^{1-\lambda} \left\langle P^{1/2}C^{\lambda}P^{1/2}e, e \right\rangle \end{aligned} \tag{4.8}$$

for any $e \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (4.8) $e=e_i$, $i\in I$ and by summing over $i\in I$, then we get

$$(1-\lambda)v\sum_{i\in I}\langle Pe_i,e_i\rangle+\lambda\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle \le \max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right),g_\lambda\left(\frac{Z}{z}\right)\right\}v^{1-\lambda}\sum_{i\in I}\left\langle P^{1/2}C^\lambda P^{1/2}e_i,e_i\right\rangle, \tag{4.9}$$

and by the properties of trace we have

$$(1-\lambda)v\operatorname{tr}(P)+\lambda\operatorname{tr}(PC)\le\max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right),g_\lambda\left(\frac{Z}{z}\right)\right\}v^{1-\lambda}\operatorname{tr}\left(PC^\lambda\right),$$

for any $\lambda\in[0,1]$ and for any $v\in[z,Z]$.

This inequality can be written as

$$(1-\lambda)v+\lambda\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\le\max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right),g_\lambda\left(\frac{Z}{z}\right)\right\}v^{1-\lambda}\frac{\operatorname{tr}\left(PC^\lambda\right)}{\operatorname{tr}(P)}, \tag{4.10}$$

for any $\lambda\in[0,1]$ and for any $v\in[z,Z]$.

Now, if we take in (4.10) $v=\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\in[z,Z]$, then we get

$$\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\le\max\left\{g_{1-\lambda}\left(\frac{Z}{z}\right),g_\lambda\left(\frac{Z}{z}\right)\right\}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\lambda}\frac{\operatorname{tr}\left(PC^\lambda\right)}{\operatorname{tr}(P)},$$

that is equivalent to the desired result (4.6). □

In particular, we have:

Corollary 4. *With the assumptions of Theorem 6 and if* $T=\max\{\lambda,1-\lambda\}$ *for* $\lambda\in(0,1)$, *then we have*

$$(1\le)\frac{\operatorname{tr}^{1-\lambda}(P)\operatorname{tr}^\lambda(PC)}{\operatorname{tr}\left(PC^\lambda\right)}\le\begin{cases}S\left(\frac{Z}{z}\right),\\ K^T\left(\frac{Z}{z}\right),\\ \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{Z}{z}\right)-1\right)\right],\\ \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{Z}{z}-1\right)^2\right].\end{cases}$$

The following reverse of Hölder's trace inequality may be stated:

Theorem 7. *Let* A *and* B *be two positive invertible operators,* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* m, $M>0$ *such that*

$$m^pB^q\le A^p\le M^pB^q. \tag{4.11}$$

If $B^q\in\mathcal{B}_1(H)$, *then*

$$(1\le)\frac{\operatorname{tr}^{1/p}(A^p)\operatorname{tr}^{1/q}(B^q)}{\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right)}\le\max\left\{g_{\frac{1}{q}}\left(\left(\frac{M}{m}\right)^p\right),g_{\frac{1}{p}}\left(\left(\frac{M}{m}\right)^p\right)\right\}. \tag{4.12}$$

Proof. Now, from (4.11) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^pI\le B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\le M^pI$. By writing the inequality (4.6) for $C=B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}$, $z=m^p$, $Z=M^p$, $\lambda=\frac{1}{p}$ and $P=B^q$ we get the desired result (4.12). □

Finally, we have:

Corollary 5. *With the assumptions of Theorem 7 and if* $T_{p,q} = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, *then we have*

$$(1 \leq) \frac{\operatorname{tr}^{1/p}(A^p)\operatorname{tr}^{1/q}(B^q)}{\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right)} \leq \begin{cases} S\left(\left(\frac{M}{m}\right)^p\right), \\ K^{T_{p,q}}\left(\left(\frac{M}{m}\right)^p\right), \\ \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right) - 1\right)\right], \\ \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]. \end{cases} \tag{4.13}$$

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