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INEQUALITIES FOR D-SYNCHRONOUS FUNCTIONS AND RELATED FUNCTIONALS

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ABSTRACT. We intoduce in this paper the concept of quadruple D-synchronous functions that generalizes the concept of a pair of synchronous functions, establish an inequality similar to Čebyšev inequality and provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are provided. Discrete inequalities are also stated.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for ν -a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\nu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\nu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\nu$ instead of $\int_\Omega w(x)\,d\nu(x)$. Assume also that $\int_\Omega w d\nu = 1$.

We say that the pair of measurable functions (f,g) are *synchronous* on Ω if

$$(f(x)-f(y))(g(x)-g(y))\geq 0 \tag{1.1}$$

for ν-a.e. $x,y\in\Omega$. If the inequality reverses in (1.1), the functions are called *asynchronous* on Ω.

If (f,g) are synchronous on Ω and $f,\ g,\ fg\in L_w(\Omega,\nu)$ then the following inequality, that is known in the literature as *Čebyšev's Inequality,* holds

$$\int_\Omega wfg d\nu \geq \int_\Omega wf d\nu \int_\Omega wg d\nu, \tag{1.2}$$

where $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$ and $\int_\Omega w d\nu = 1$.

If $f,\ g:\Omega\to\mathbb{R}$ are ν-measurable functions and $f,\ g,\ fg\in L_w(\Omega,\nu)$, then we may consider the *Čebyšev functional*

$$T_w(f,g) := \int_\Omega wfg d\nu - \int_\Omega wf d\nu \int_\Omega wg d\nu.$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.3}$$

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provided

$$-\infty < \gamma \le f(x) \le \Gamma < \infty, \quad -\infty < \delta \le g(x) \le \Delta < \infty \tag{1.4}$$

for ν-a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If $f \in L_w(\Omega, \nu)$, then we may define

$$D_w(f) := \int_\Omega w(x) \left| f(x) - \int_\Omega w(y) f(y) d\nu(y) \right| d\nu(x). \tag{1.5}$$

The following refinement of Grüss inequality in the general setting of measure spaces is due to Cerone & Dragomir [1]:

Theorem 1. *Let w, f, $g : \Omega \to \mathbb{R}$ be ν-measurable functions with $w \ge 0$ ν-a.e. on Ω and $\int_\Omega w d\nu = 1$. If f, g, $fg \in L_w(\Omega, \nu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \le g(x) \le \Delta < \infty \quad \text{for } \nu - a.e. \ x \in \Omega, \tag{1.6}$$

then we have the inequality

$$|T_w(f, g)| \le \frac{1}{2}(\Delta - \delta) D_w(f). \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Motivated by the above results, we introduce in this paper the concept of quadruple D-synchronous functions that generalizes the concept of a pair of synchronous functions, establish an inequality similar to Čebyšev inequality and provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are provided. Discrete inequalities are also stated.

2. D-Synchronous Functions

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space and f, g, h, $\ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω.

Definition 1. *The quadruple (f, g, h, ℓ) is called D-Synchronous (D-Asynchronous) on Ω if*

$$\det \begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det \begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \ge (\le) \, 0 \tag{2.1}$$

for ν-a.e. (almost every) x, $y \in \Omega$.

This concept is a generalization of synchronous functions, since for $g = 1, \ell = 1$ the quadruple (f, g, h, ℓ) is D-Synchronous if and only if (f, h) is synchronous on Ω..

We observe that

$$\begin{aligned} &\det \begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det \begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \\ &= (f(x) g(y) - g(x) f(y)) (h(x) \ell(y) - \ell(x) h(y)) \end{aligned} \tag{2.2}$$

for every $x,\ y\in\Omega$.

If $g,\ell\neq 0$ ν-a.e on Ω, then

$$\begin{aligned}(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y))\\ =g(x)\ell(x)g(y)\ell(y)\left(\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right)\left(\frac{h(x)}{\ell(x)}-\frac{h(y)}{\ell(y)}\right)\end{aligned}\tag{2.3}$$

for ν-a.e. $x,y\in\Omega$, showing that for the functions g, ℓ with $g\ell>0$ ν-a.e on Ω the quadruple (f,g,h,ℓ) is D-Synchronous if and only if $\left(\frac{f}{g}\frac{h}{\ell},\right)$ is synchronous on Ω.

Theorem 2. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω and such that the quadruple (f,g,h,ℓ) is D-Synchronous (D-Aynchronous), $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$, then*

$$\det\begin{pmatrix}\int_\Omega wfhd\nu & \int_\Omega wghd\nu\\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu\end{pmatrix}\geq(\leq)\,0.\tag{2.4}$$

Proof. Since the quadruple (f,g,h,ℓ) is D-Synchronous, then

$$\begin{aligned}0&\leq(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y))\\&=f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y)\\&-f(x)\ell(x)g(y)h(y)-g(x)h(x)f(y)\ell(y)\end{aligned}\tag{2.5}$$

for ν-a.e. $x,\ y\in\Omega$.

This is equivalent to

$$\begin{aligned}f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y)\\ \geq f(x)\ell(x)g(y)h(y)+g(x)h(x)f(y)\ell(y)\end{aligned}\tag{2.6}$$

for ν-a.e. $x,\ y\in\Omega$.

Multiply (2.6) by $w(x)w(y)\geq 0$ to get

$$\begin{aligned}w(x)f(x)h(x)w(y)g(y)\ell(y)+w(x)g(x)\ell(x)w(y)f(y)h(y)\\ \geq w(x)f(x)\ell(x)w(y)g(y)h(y)+w(x)g(x)h(x)w(y)f(y)\ell(y)\end{aligned}\tag{2.7}$$

for ν-a.e. $x,\ y\in\Omega$.

If we integrate the inequality (2.7) over $x\in\Omega$, then we get

$$\begin{aligned}w(y)g(y)\ell(y)\int_\Omega wfhd\nu+w(y)f(y)h(y)\int_\Omega wg\ell d\nu\\ \geq w(y)g(y)h(y)\int_\Omega wf\ell d\nu+w(y)f(y)\ell(y)\int_\Omega wghd\nu\end{aligned}\tag{2.8}$$

for ν-a.e. $y\in\Omega$.

Finally, if we integrate the inequality (2.8) over $y\in\Omega$, then we get

$$\begin{aligned}\int_\Omega wfhd\nu\int_\Omega wg\ell d\nu+\int_\Omega wg\ell d\nu\int_\Omega wfhd\nu\\ \geq\int_\Omega wf\ell d\nu\int_\Omega wghd\nu+\int_\Omega wghd\nu\int_\Omega wf\ell d\nu,\end{aligned}$$

which is equivalent to the desired result (2.4). $\square$

Corollary 1. *Let* $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ *be four* ν*-measurable functions on* Ω *and such that* $g\ell>0$ ν*-a.e on* Ω, $\left(\frac{f}{g},\frac{h}{\ell}\right)$ *is synchronous (asynchronous) on* Ω, $w\geq 0$ *a.e. on* Ω *with* $\int_\Omega wd\nu=1$ *and* $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$, *then the inequality (2.4) is valid.*

Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω , $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$, then we can consider the functionals

$$\begin{aligned}(2.9)\qquad \mathcal{D}(f,g,h,\ell;w,\Omega)&:=\det\begin{pmatrix}\int_\Omega wfhd\nu & \int_\Omega wghd\nu\\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu\end{pmatrix}\\ &=\int_\Omega wfhd\nu\int_\Omega wg\ell d\nu-\int_\Omega wf\ell d\nu\int_\Omega wghd\nu\end{aligned}$$

and, for $(f,g)=(h,\ell)$,

$$\begin{aligned}(2.10)\qquad \mathcal{D}(f,g;w,\Omega)&:=\mathcal{D}(f,g,f,g;w,\Omega)\\ &=\det\begin{pmatrix}\int_\Omega wf^2d\nu & \int_\Omega wfgd\nu\\ \int_\Omega wfgd\nu & \int_\Omega wg^2d\nu\end{pmatrix}\\ &=\int_\Omega wf^2d\nu\int_\Omega wg^2d\nu-\left(\int_\Omega wfgd\nu\right)^2,\end{aligned}$$

provided $f^2,\ g^2\in L_w(\Omega,\nu)$.

We can improve the inequality (2.4) as follows:

Theorem 3. *Let* $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ *be four* ν*-measurable functions on* Ω *and such that the quadruple* (f,g,h,ℓ) *is D-Synchronous,* $w\geq 0$ *a.e. on* Ω *with* $\int_\Omega wd\nu=1$ *and* $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$, *then*

$$\begin{aligned}(2.11)\qquad \mathcal{D}(f,g,h,\ell;w,\Omega)&\geq\max\{|\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|,\\ &\qquad|\mathcal{D}(f,g,|h|,|\ell|;w,\Omega)|,|\mathcal{D}(|f|,|g|,|h|,|\ell|;w,\Omega)|\}\\ &\geq 0.\end{aligned}$$

Proof. We use the continuity property of the modulus, namely

$$|a-b|\geq||a|-|b||,\ a,b\in\mathbb{R}.$$

Since (f,g,h,ℓ) is D-Synchronous, then

$$\begin{aligned}(2.12)\qquad &(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y))\\ &=|f(x)g(y)-g(x)f(y)||h(x)\ell(y)-\ell(x)h(y)|\\ &\geq\begin{cases}|(|f(x)||g(y)|-|g(x)||f(y)|)(h(x)\ell(y)-\ell(x)h(y))|\\ \\ |(f(x)g(y)-g(x)f(y))(|h(x)||\ell(y)|-|\ell(x)||h(y)|)|\\ \\ |(|f(x)||g(y)|-|g(x)||f(y)|)(|h(x)||\ell(y)|-|\ell(x)||h(y)|)|\end{cases}\end{aligned}$$

for ν-a.e. $x,\ y\in\Omega$.

As in the proof of Theorem 2, we have the identity

$$\begin{aligned}(2.13)\qquad \mathcal{D}(f,g,h,\ell;w,\Omega)&=\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)(f(x)g(y)-g(x)f(y))\\ &\qquad\times(h(x)\ell(y)-\ell(x)h(y))d\nu(x)d\nu(y).\end{aligned}$$

By using the identity (2.13) and the first branch in (2.12) we have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &\geq \frac{1}{2}\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,|(|f(x)|\,|g(y)| - |g(x)|\,|f(y)|)\\ &\quad\times (h(x)\,\ell(y) - \ell(x)\,h(y))|\,d\nu(x)\,d\nu(y)\\ &\geq \frac{1}{2}\left|\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(|f(x)|\,|g(y)| - |g(x)|\,|f(y)|)\right.\\ &\quad\left.\times (h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right|\\ &= |\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|,\end{aligned}$$

which proves the first part of (2.11).

The second and third part of (2.11) can be proved in a similar way and the details are omitted. □

3. Further Results for the functional $\mathcal{D}$

We have the following Schwarz's type inequality for the functional $\mathcal{D}$:

Theorem 4. *Let f, g, h, $\ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω , $w\geq 0$ a.e. on Ω with $\int_{\Omega} wd\nu = 1$ and f^2, g^2, h^2, $\ell^2 \in L_w(\Omega,\nu)$. Then we have*

$$\mathcal{D}^2(f,g,h,\ell;w,\Omega) \leq \mathcal{D}(f,g;w,\Omega)\,\mathcal{D}(h,\ell;w,\Omega). \tag{3.1}$$

Proof. As in the proof of Theorem 3, we have the identities

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &= \frac{1}{2}\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y))\\ &\quad\times (h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y),\end{aligned}$$

$$\mathcal{D}(f,g;w,\Omega) = \frac{1}{2}\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y))^2\,d\nu(x)\,d\nu(y)$$

and

$$\mathcal{D}(h,\ell;w,\Omega) = \frac{1}{2}\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(h(x)\,\ell(y) - \ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y).$$

By the Cauchy-Bunyakovsky-Schwarz double integral inequality we have

$$\begin{aligned}&\left(\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y))\,(h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right)^2\\ &\leq \int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(h(x)\,\ell(y) - \ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y)\\ &\times \int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(h(x)\,\ell(y) - \ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y),\end{aligned}$$

which produces the desired result (3.1). □

Corollary 2. *Let f, g, h, $\ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω with g^2, $\ell^2\in L_w(\Omega,\nu)$, $w\geq 0$ a.e. on Ω with $\int_{\Omega} wd\nu = 1$ and a, A, b, $B\in\mathbb{R}$ such that*

$$ag \leq f \leq Ag \text{ and } b\ell \leq h \leq B\ell \tag{3.2}$$

ν-a.e. on Ω. Then we have

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{4}(A-a)(B-b)\int_{\Omega} wg^2 d\nu \int_{\Omega} w\ell^2 d\nu. \tag{3.3}$$

Proof. In [2] (see also [4, p. 8]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \le \frac{1}{4}(A-a)^2 \left(\int_\Omega wg^2 d\nu\right)^2$$

provided that $ag \le f \le Ag$ ν-a.e. on Ω and $g^2 \in L_w(\Omega, \nu)$.

Since, we also have

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \le \frac{1}{4}(B-b)^2 \left(\int_\Omega w\ell^2 d\nu\right)^2$$

provided that $b\ell \le h \le B\ell$ ν-a.e. on Ω and $\ell^2 \in L_w(\Omega, \nu)$, then by (3.1) we have

$$\mathcal{D}^2(f, g, h, \ell; w, \Omega) \le \frac{1}{16}(A-a)^2 (B-b)^2 \left(\int_\Omega wg^2 d\nu\right)^2 \left(\int_\Omega w\ell^2 d\nu\right)^2$$

that is equivalent to the desired result (3.3). □

For positive margins we also have:

Corollary 3. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω with $g^2, \ell^2 \in L_w(\Omega, \nu)$, $w \ge 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and $a, A, b, B > 0$ such that*

$$ag \le f \le Ag \text{ and } b\ell \le h \le B\ell \tag{3.4}$$

ν-a.e. on Ω. Then we have

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \le \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}} \int_\Omega wfg d\nu \int_\Omega wh\ell d\nu. \tag{3.5}$$

Proof. In [3] (see also [4, p. 16]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \le \frac{(A-a)^2}{4aA} \left(\int_\Omega wfg d\nu\right)^2$$

provided $ag \le f \le Ag$ ν-a.e. on Ω.

Since

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \le \frac{(B-b)^2}{4bB} \left(\int_\Omega wh\ell d\nu\right)^2$$

provided $b\ell \le h \le B\ell$ ν-a.e. on Ω, then by (3.1) we get the desired result (3.5). □

If bounds for the sum and difference are available, then we have:

Corollary 4. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω with g^2, $\ell^2 \in L_w(\Omega, \nu)$, $w \ge 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants P_1, Q_1, P_2, Q_2 such that*

$$|g - f| \le P_1, \ \ |g + f| \le Q_1, \ \ |h - \ell| \le P_2, \ \ |h + \ell| \le Q_2 \tag{3.6}$$

a.e. on Ω, then

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \le \frac{1}{4}P_1 Q_1 P_2 Q_2. \tag{3.7}$$

Proof. In the recent paper [5] we obtained amongst other the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_{\Omega} wf^2 d\nu \int_{\Omega} wg^2 d\nu - \left(\int_{\Omega} wfg d\nu \right)^2 \leq \frac{1}{4} P_1^2 Q_1^2,$$

provided $|g - f| \leq P_1$, $|g + f| \leq Q_1$ a.e. on Ω.

Since

$$\int_{\Omega} wh^2 d\nu \int_{\Omega} w\ell^2 d\nu - \left(\int_{\Omega} wh\ell d\nu \right)^2 \leq \frac{1}{4} P_2^2 Q_2^2,$$

provided $|h - \ell| \leq P_2$, $|h + \ell| \leq Q_2$ a.e. on Ω, then by (3.1) we get the desired result (3.7). □

If bounds for each function are available, then we have:

Corollary 5. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω and $w \geq 0$ a.e. on Ω with $\int_{\Omega} wd\nu = 1$. Assume that there exists the constants a_i, A_i, b_i and B_i with $i \in \{1, 2\}$ such that*

$$0 < a_1 \leq f \leq A_1 < \infty, \ 0 < a_2 \leq g \leq A_2 < \infty, \tag{3.8}$$

and

$$0 < b_1 \leq h \leq B_1 < \infty, \ 0 < b_2 \leq \ell \leq B_2 < \infty, \tag{3.9}$$

a.e. on Ω, then

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \leq \frac{1}{3} (A_1 A_2 - a_1 a_2)(B_1 B_2 - b_1 b_2). \tag{3.10}$$

Proof. We use the following Ozeki's type inequality obtained in [7]

$$\int_{\Omega} wf^2 d\nu \int_{\Omega} wg^2 d\nu - \left(\int_{\Omega} wfg d\nu \right)^2 \leq \frac{1}{3} (A_1 A_2 - a_1 a_2)^2,$$

provided $0 < a_1 \leq f \leq A_1 < \infty$, $0 < a_2 \leq g \leq A_2 < \infty$ a.e. on Ω.

Since

$$\int_{\Omega} wh^2 d\nu \int_{\Omega} w\ell^2 d\nu - \left(\int_{\Omega} wh\ell d\nu \right)^2 \leq \frac{1}{3} (B_1 B_2 - b_1 b_2)^2$$

provided $0 < b_1 \leq h \leq B_1 < \infty$, $0 < b_2 \leq \ell \leq B_2 < \infty$ a.e. on Ω, then by (3.1) we get the desired result (3.10). □

4. Results for Univariate Functions

Let $\Omega = [a, b]$ be an interval of real numbers and assume that $f, g, h, \ell : [a, b] \to \mathbb{R}$ are measurable D-Synchronous (D-Aynchronous), $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t) dt = 1$ and fh, $g\ell$, gh, $f\ell \in L_w([a, b])$, then

$$\begin{aligned} &\int_a^b w(t) f(t) h(t) dt \int_a^b w(t) g(t) \ell(t) dt \\ &\geq (\leq) \int_a^b w(t) g(t) h(t) dt \int_a^b w(t) f(t) \ell(t) dt. \end{aligned} \tag{4.1}$$

Now, assume that $[a, b] \subset (0, \infty)$ and take $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$ with $p, q, r, s \in \mathbb{R}$. Then

$$\frac{f(t)}{g(t)} = t^{p-q} \text{ and } \frac{h(t)}{\ell(t)} = t^{r-s}.$$

If $(p-q)(r-s)>0$, then the functions $\left(\frac{f}{g},\frac{h}{\ell}\right)$ have the same monotonicity on $[a,b]$ while if $(p-q)(r-s)<0$ then $\left(\frac{f}{g},\frac{h}{\ell}\right)$ have opposite monotonicity on $[a,b]$. Therefore by (4.1) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt=1$ that

$$\int_a^b w(t)\,t^{p+r}dt\int_a^b w(t)\,t^{q+s}dt \geq (\leq) \int_a^b w(t)\,t^{q+r}dt\int_a^b w(t)\,t^{p+s}dt, \tag{4.2}$$

provided $(p-q)(r-s)>(<)\,0$.

Assume that $[a,b]\subset(0,\infty)$ and take $f(t)=\exp(\alpha t)$, $g(t)=\exp(\beta t)$, $h(t)=\exp(\gamma t)$ and $\ell(t)=\exp(\delta t)$ with $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbb{R}$. Then

$$\frac{f(t)}{g(t)}=\exp[(\alpha-\beta)t] \text{ and } \frac{h(t)}{\ell(t)}=\exp[(\gamma-\delta)t].$$

If $(\alpha-\beta)(\gamma-\delta)>0$, then the functions $\left(\frac{f}{g},\frac{h}{\ell}\right)$ have the same monotonicity on $[a,b]$ while if $(\alpha-\beta)(\gamma-\delta)<0$ then $\left(\frac{f}{g},\frac{h}{\ell}\right)$ have opposite monotonicity on $[a,b]$. Therefore by (4.1) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt=1$ that

$$\begin{aligned}&\int_a^b w(t)\exp[(\alpha+\gamma)t]\,dt\int_a^b w(t)\exp[(\beta+\delta)t]\,dt\\&\geq(\leq)\int_a^b w(t)\exp[(\beta+\gamma)t]\,dt\int_a^b w(t)\exp[(\alpha+\delta)t]\,dt\end{aligned} \tag{4.3}$$

provided $(\alpha-\beta)(\gamma-\delta)>(<)\,0$.

Consider the functional

$$\begin{aligned}\mathcal{D}_{p,q,r,s}(w):=&\int_a^b w(t)\,t^{p+r}dt\int_a^b w(t)\,t^{q+s}dt\\&-\int_a^b w(t)\,t^{q+r}dt\int_a^b w(t)\,t^{p+s}dt,\end{aligned} \tag{4.4}$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt=1$ and $p,\ q,\ r,\ s\in\mathbb{R}$.

We observe that for $t\in[a,b]\subset(0,\infty)$ we have

$$\begin{aligned}k_{p,q}(a,b):=&\begin{cases}a^{p-q} \text{ if } p\geq q\\ b^{p-q} \text{ if } p<q\end{cases}\leq\frac{f(t)}{g(t)}=t^{p-q}\\&\leq K_{p,q}(a,b):=\begin{cases}b^{p-q} \text{ if } p\geq q\\ a^{p-q} \text{ if } p<q\end{cases}\end{aligned} \tag{4.5}$$

and, similarly

$$k_{r,s}(a,b)\leq\frac{h(t)}{\ell(t)}=t^{r-s}\leq K_{r,s}(a,b).$$

Using the inequality (3.3) we have

$$\begin{aligned}(4.6)\qquad |\mathcal{D}_{p,q,r,s}(w)| &\le \frac{1}{4}\left[K_{p,q}(a,b)-k_{p,q}(a,b)\right]\left[K_{r,s}(a,b)-k_{r,s}(a,b)\right]\\ &\quad\times\int_a^b w(t)\,t^{2q}dt\int_a^b w(t)\,t^{2s}dt,\end{aligned}$$

while from (3.5) we have

$$\begin{aligned}(4.7)\qquad |\mathcal{D}_{p,q,r,s}(w)| &\le \frac{1}{4}\frac{\left[K_{p,q}(a,b)-k_{p,q}(a,b)\right]\left[K_{r,s}(a,b)-k_{r,s}(a,b)\right]}{\sqrt{k_{p,q}(a,b)\,k_{r,s}(a,b)\,K_{p,q}(a,b)\,K_{r,s}(a,b)}}\\ &\quad\times\int_a^b w(t)\,t^{p+q}dt\int_a^b w(t)\,t^{r+s}dt.\end{aligned}$$

We also have for $t\in[a,b]\subset(0,\infty)$ that

$$\begin{aligned}u_p(a,b) &:= \begin{cases} a^p \text{ if } p\ge 0\\ \\ b^p \text{ if } p<0\end{cases} \le f(t)=t^p\\ &\le U_p(a,b) := \begin{cases} b^p \text{ if } p\ge 0\\ \\ a^p \text{ if } p<0\end{cases}\end{aligned}$$

and the corresponding bounds for $g(t)=t^q$, $h(t)=t^r$ and $\ell(t)=t^s$ with p, q, r, $s\in\mathbb{R}$.

Making use of the inequality (3.10) we get

$$\begin{aligned}(4.8)\qquad |\mathcal{D}_{p,q,r,s}(w)| &\le \frac{1}{3}\left(U_p(a,b)\,U_q(a,b)-u_p(a,b)\,u_q(a,b)\right)\\ &\quad\times\left(U_r(a,b)\,U_s(a,b)-u_r(a,b)\,u_s(a,b)\right).\end{aligned}$$

Similar results may be stated for the functional

$$\begin{aligned}\mathcal{D}_{\alpha,\beta,\gamma,\delta}(w) &:= \int_a^b w(t)\exp\left[(\alpha+\gamma)t\right]dt\int_a^b w(t)\exp\left[(\beta+\delta)t\right]dt\\ &\quad-\int_a^b w(t)\exp\left[(\beta+\gamma)t\right]dt\int_a^b w(t)\exp\left[(\alpha+\delta)t\right]dt\end{aligned}$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt=1$, for α, β, γ, $\delta\in\mathbb{R}$ and $[a,b]\subset(0,\infty)$. The details are omitted.

We say that the function $\varphi:[a,b]\to\mathbb{R}$ is Lipschitzian with the constant $L>0$ if

$$|\varphi(t)-\varphi(s)|\le L\,|t-s|$$

for any t, $s\in[a,b]$.

Define the functional

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) &:= \int_a^b w(t)\,f(t)\,h(t)\,dt\int_a^b w(t)\,g(t)\,\ell(t)\,dt\\ &\quad-\int_a^b w(t)\,g(t)\,h(t)\,dt\int_a^b w(t)\,f(t)\,\ell(t)\,dt.\end{aligned}$$

We have:

Theorem 5. *Let $f,\ g,\ h,\ \ell:[a,b]\to\mathbb{R}$ be measurable functions and $w\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$.*

(i) If $g(t),\ell(t)\neq 0$ for any $t\in[a,b]$ and $\frac{f}{g}$ is Lipschitzian with the constant $L>0$ and $\frac{h}{\ell}$ is Lipschitzian with the constant $K>0$ and $g\ell,\ g\ell e^2\in L_w([a,b])$ with $e(t)=t,\ t\in[a,b]$, then

$$\begin{aligned}(4.9)\quad &\left|\mathcal{D}(f,g,h,\ell;w,[a,b])\right|\\ &\leq LK\left[\int_a^b w(s)|g(s)||\ell(s)|\,ds\int_a^b w(t)|\ell(t)||g(t)|\,t^2dt\right.\\ &\qquad\left.-\left(\int_a^b w(t)|g(t)||\ell(t)|\,tdt\right)^2\right].\end{aligned}$$

(ii) If, in addition, we have that $wg\ell\in L_\infty[a,b]$ and $\|wg\ell\|_\infty=\operatorname{essup}_{t\in[a,b]}|w(t)g(t)\ell(t)|<\infty$, then we have

$$(4.10)\qquad \left|\mathcal{D}(f,g,h,\ell;w,[a,b])\right|\leq\frac{1}{12}(b-a)^4LK\|wg\ell\|_\infty^2.$$

Proof. We have

$$\begin{aligned}&\mathcal{D}(f,g,h,\ell;w,[a,b])\\ &=\frac{1}{2}\int_a^b\int_a^b w(t)w(s)(f(t)g(s)-g(t)f(s))\\ &\times(h(t)\ell(s)-\ell(t)h(s))\,dtds\\ &=\frac{1}{2}\int_a^b\int_a^b w(t)w(s)g(t)g(s)\ell(t)\ell(s)\\ &\times\left(\frac{f(t)}{g(t)}-\frac{f(s)}{g(s)}\right)\left(\frac{h(t)}{\ell(t)}-\frac{h(s)}{\ell(s)}\right)dtds.\end{aligned}$$

By taking the modulus in this equality, we get

$$\begin{aligned}(4.11)\quad &\left|\mathcal{D}(f,g,h,\ell;w,[a,b])\right|\\ &\leq\frac{1}{2}\int_a^b\int_a^b w(t)w(s)|g(t)||g(s)||\ell(t)||\ell(s)|\\ &\times\left|\frac{f(t)}{g(t)}-\frac{f(s)}{g(s)}\right|\left|\frac{h(t)}{\ell(t)}-\frac{h(s)}{\ell(s)}\right|dtds\\ &\leq\frac{1}{2}LK\int_a^b\int_a^b w(t)w(s)|g(t)||g(s)||\ell(t)||\ell(s)|(t-s)^2\,dtds.\end{aligned}$$

Now, observe that

$$\begin{aligned}(4.12)\quad &\int_a^b\int_a^b w(t)w(s)|g(t)||g(s)||\ell(t)||\ell(s)|(t-s)^2\,dtds\\ &=\int_a^b\int_a^b w(t)w(s)|g(t)||g(s)||\ell(t)||\ell(s)|\left(t^2-2ts+s^2\right)dtds\end{aligned}$$

$$= 2\left(\int_a^b \int_a^b w(t) w(s) |g(t)| |g(s)| |\ell(t)| |\ell(s)| t^2 dtds - \int_a^b \int_a^b w(t) w(s) |g(t)| |g(s)| |\ell(t)| |\ell(s)| tsdtds\right)$$

$$= 2\left[\int_a^b w(s) |g(s)| |\ell(s)| ds \int_a^b w(t) |g(t)| |\ell(t)| t^2 dt - \left(\int_a^b w(t) |g(t)| |\ell(t)| tdt\right)^2\right].$$

On making use of (4.11) and (4.12) we get the desired result (4.9).

If $wg\ell \in L_\infty [a, b]$, then

$$\int_a^b \int_a^b w(t) w(s) |g(t)| |g(s)| |\ell(t)| |\ell(s)| (t-s)^2 dtds \leq \|wg\ell\|_\infty^2 \int_a^b \int_a^b (t-s)^2 dtds = \frac{1}{6}(b-a)^2 \|wg\ell\|_\infty^2. \tag{4.13}$$

Therefore, by the inequalities (4.11) and (4.13) we obtain the desired result (4.10). □

5. Discrete Inequalities

Consider the n-tuples of real numbers $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $z = (z_1, ..., z_n)$ and $u = (u_1, ..., u_n)$. We say that the quadruple (x, y, z, u) is *D-Synchronous* if

$$0 \leq \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \det\begin{pmatrix} z_i & z_j \\ u_i & u_j \end{pmatrix} = (x_i y_j - x_j y_i)(z_i u_j - z_j u_i) \tag{5.1}$$

for any $i, j \in \{1, ..., n\}$.

If $p = (p_1, ..., p_n)$ is a probability distribution, namely, $p_i \geq 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$ and the quadruple (x, y, z, u) is D-Synchronous then by (2.4) we have:

$$\mathcal{D}_n(x, y, z, u; p) := \det\begin{pmatrix} \sum_{i=1}^n p_i x_i z_i & \sum_{i=1}^n p_i y_i z_i \\ \sum_{i=1}^n p_i x_i u_i & \sum_{i=1}^n p_i y_i u_i \end{pmatrix} = \sum_{i=1}^n p_i x_i z_i \sum_{i=1}^n p_i y_i u_i - \sum_{i=1}^n p_i x_i u_i \sum_{i=1}^n p_i y_i z_i \geq 0. \tag{5.2}$$

For an n-tuples of real numbers $x = (x_1, ..., x_n)$, we denote by $|x| := (|x_1|, ..., |x_n|)$. On making use of the inequality (2.11), then for any D-Synchronous quadruple (x, y, z, u) and for any probability distribution $p = (p_1, ..., p_n)$ we have

$$\mathcal{D}_n(x, y, z, u; p) \geq \max\{|\mathcal{D}_n(|x|, y, z, u; p)|, |\mathcal{D}_n(x, |y|, z, u; p)|, |\mathcal{D}_n(|x|, |y|, z, u; p)|\} \geq 0. \tag{5.3}$$

Observe that if we consider

$$\mathcal{D}_n(x,y;p) := \mathcal{D}_n(x,y,x,y;p) = \sum_{i=1}^{n} p_i x_i^2 \sum_{i=1}^{n} p_i y_i^2 - \left(\sum_{i=1}^{n} p_i x_i y_i\right)^2,$$

then by (3.1) we have

$$|\mathcal{D}_n(x,y,z,u;p)|^2 \le \mathcal{D}_n(x,y;p)\,\mathcal{D}_n(z,u;p) \tag{5.4}$$

for any quadruple (x,y,z,u) and any probability distribution $p=(p_1,...,p_n)$.

If a, A, b, $B \in \mathbb{R}$ and (x,y,z,u) are such that

$$ay_i \le x_i \le Ay_i \text{ and } bu_i \le z_i \le Bu_i \tag{5.5}$$

for any $i \in \{1,...,n\}$, then by (3.3) we have

$$|\mathcal{D}_n(x,y,z,u;p)| \le \frac{1}{4}(A-a)(B-b)\sum_{i=1}^{n} p_i y_i^2 \sum_{i=1}^{n} p_i u_i^2. \tag{5.6}$$

If a, A, b, $B > 0$ and the condition (5.5) is valid, then by (3.5) we have

$$|\mathcal{D}_n(x,y,z,u;p)| \le \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\sum_{i=1}^{n} p_i x_i y_i \sum_{i=1}^{n} p_i z_i u_i. \tag{5.7}$$

Now, if we use the *Klamkin-McLenaghan's inequality*

$$\sum_{i=1}^{n} p_i x_i^2 \sum_{i=1}^{n} p_i y_i^2 - \left(\sum_{i=1}^{n} p_i x_i y_i\right)^2 \le \left(\sqrt{A}-\sqrt{a}\right)^2 \sum_{i=1}^{n} p_i x_i y_i \sum_{i=1}^{n} p_i x_i^2$$

that holds for x, y satisfying the condition (5.5) with A, $a > 0$, then by (5.4) we get

$$\begin{aligned} &|\mathcal{D}_n(x,y,z,u;p)| \\ &\le \left(\sqrt{A}-\sqrt{a}\right)\left(\sqrt{B}-\sqrt{b}\right) \\ &\times \left(\sum_{i=1}^{n} p_i x_i y_i\right)^{1/2} \left(\sum_{i=1}^{n} p_i x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i z_i u_i\right)^{1/2} \left(\sum_{i=1}^{n} p_i z_i^2\right)^{1/2} \end{aligned} \tag{5.8}$$

provided (x,y,z,u) satisfy (5.5) with a, A, b, $B > 0$.

Now, assume that

$$0 < a_1 \le x_i \le A_1 < \infty,\ 0 < a_2 \le y_i \le A_2 < \infty, \tag{5.9}$$

and

$$0 < b_1 \le x_i \le B_1 < \infty,\ 0 < b_2 \le u_i \le B_2 < \infty, \tag{5.10}$$

for any $i \in \{1,...,n\}$, then by (3.10) we get

$$|\mathcal{D}_n(x,y,z,u;p)| \le \frac{1}{3}(A_1A_2 - a_1a_2)(B_1B_2 - b_1b_2), \tag{5.11}$$

for any probability distribution $p=(p_1,...,p_n)$.

References

[1] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38** (2007), No. 1, 37-49. Preprint RGMIA Res. Rep. Coll., 5(2) (2002), Art. 14. [http://rgmia.org/papers/v5n2/RGIApp.pdf].

[2] S. S. Dragomir, A counterpart of Schwarz's inequality in inner product spaces, *East Asian Math. J.*, **20** (1) (2004), 1-10. Preprint *RGMIA Res. Rep Coll.* **6** (2003), Supplement Art. 18 [http://rgmia.org/papers/v6e/CSIIPS.pdf], [http://arxiv.org/abs/math/0305373].

[3] S. S. Dragomir, A generalization of the Cassels and Greub-Reinboldt inequalities in inner product spaces. *Nonlinear Anal. Forum* **8** (2003), no. 2, 169–178. [http://prof.ks.ac.kr/bslee/naf/table/vol-0802/v0802-07a.pdf]

[4] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*. Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.

[5] S. S. Dragomir, Reverses of Schwarz inequality in inner product spaces with applications. *Math. Nachr.* **288** (2015), no. 7, 730–742.

[6] S. S. Dragomir, Inequalities for synchronous functions and applications, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. .

[7] S. Izumino, H. Mori and Y. Seo, On Ozeki's Inequality, *J. lnequal. & Appl.*, 1998, Vol. **2**, pp. 235-253.

[8] M. S. Klamkin and K. G. McLenaghan, An ellipse inequality, *Math. Mag.*, **50** (1977), 261-263.

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