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GENERALIZED OSTROWSKI TYPE INTEGRAL INEQUALITIES INVOLVING GENERALIZED MOMENTS VIA LOCAL FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain generalized Ostrowski type integral inequalities involving moments of a continuous random variables via local fractional integrals.

1. Introduction

Ostrowski [5] proved the following integral inequality which is well known in the literature as the Ostrowski's inequality.

Theorem 1. *Let mapping* $f:[a,b]\to R$ *be a continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f':[a,b]\to R$ *be bounded on* (a,b), *i.e.,* $|f'(x)|_{\infty}=\sup\limits_{t\in(a,b)}|f'(t)dt|\leq M\,(<\infty)$. *Then, for all* $x\in[a,b]$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{M}{b-a}\left(\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right). \tag{1.1}$$

Definition 1. *Let* X *be a random variable whose probability density function is* $f;[a,b]\to\mathbb{R}$ *and* $M_r(c)$ *represents the* r^{th} *moment about* $c\in\mathbb{R}$ *of* X *defined as* $M_r(c)=\int\limits_a^b(x-c)^r f(x)dx$, *for any positive integer* r. *It may be noted that for* $c=0$, $M_r(0)$ *produces moments about origin and for* $c=M_1(0)=\mu$, $M_r(\mu)$ *generates the central moments of* X.

In [2], Kumar proved the following version of the Ostrowski's inequalities using the Grüss inequality.

Theorem 2. *Let* X *be a random variable whose probability density function* $f:[a,b]\to R$ *is an absolutely continuous mapping with* $c\in R$ *and* $|f'(x)|\leq M$ *for all*

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$x \in [a,b]$, $a < b$. *Then, for any positive integer* r,

$$\begin{aligned} & M_r(c) \\ \leq \; & M\left(\frac{b-a}{2}+\frac{1}{M(b-a)}\right)\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) \\ & -M\left(\frac{(b-c)^{r+2}+(a-c)^{r+2}}{(r+1)(r+2)}\right)+\frac{2M}{b-a}\left(\frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+1)(r+2)(r+3)}\right). \end{aligned}$$

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [12, 13] and so on.

Recently, the theory of Yang's fractional sets [12] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, ..., \pm n^{\alpha}, ...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z,\ q \neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z,\ q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$.

If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha} + b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a+b)^{\alpha} = (b+a)^{\alpha}$;
(3) $a^{\alpha} + (b^{\alpha} + c^{\alpha}) = (a+b)^{\alpha} + c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha} = b^{\alpha}a^{\alpha} = (ab)^{\alpha} = (ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha}) = (a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha} + c^{\alpha}) = a^{\alpha}b^{\alpha} + a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha} + 0^{\alpha} = 0^{\alpha} + a^{\alpha} = a^{\alpha}$ and $a^{\alpha}1^{\alpha} = 1^{\alpha}a^{\alpha} = a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2. [12] *A non-differentiable function* $f : R \to R^{\alpha}$, $x \to f(x)$ *is called to be local fractional continuous at* x_0, *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$, *such that*

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}$$

holds for $|x - x_0| < \delta$, *where* $\varepsilon, \delta \in R$. *If* $f(x)$ *is local continuous on the interval* (a,b), *we denote* $f(x) \in C_{\alpha}(a,b)$.

Definition 3. [12] *The local fractional derivative of* $f(x)$ *of order* α *at* $x = x_0$ *is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}},$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha+1)(f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 4. [12] *Let $f(x) \in C_\alpha[a,b]$. Then the local fractional integral is defined by,*

$$ {}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, $$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ if $a = b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a < b$. If for any $x \in [a,b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha[a,b]$.

Lemma 1. [12]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x) = g(b) - g(a). $$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x). $$

Lemma 2. [12]

$$ \frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}; $$

$$ \frac{1}{\Gamma(\alpha+1)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right),\ k \in R. $$

In [6], Sarikaya and Budak proved the following generalized montgomery identity and generalized Ostrowski inequality for local fractionl integral:

Theorem 3 (Generalized Montgomery Identity)**.** *Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \to R^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^0$ with $a < b$ Then. for all $x \in [a,b]$, we have the identity*

$$ f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(t) = \frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha \tag{2.1} $$

where

$$ p(x,t) = \begin{cases} (t-a)^\alpha, & t \in [a,x] \\ (t-b)^\alpha, & t \in (x,b]. \end{cases} $$

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval, $f : I^\circ \subset \mathbb{R} \to \mathbb{R}^\alpha$ (I° is the interior of I) such that $f \in \mathbb{R}_\alpha(I^\circ)$ and $f^{(\alpha)} \in \mathbb{C}_\alpha[a,b]$ for $a, b \in I^\circ$ with $a < b$ and $\left\|f^{(\alpha)}\right\|_\infty < M(<\infty)$. Then. for all $x \in [a,b]$*

$$ \begin{aligned} \left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(t)\right| &\le 2^\alpha M(b-a)^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \\ &\times \left[\frac{1}{4^\alpha} + \frac{1}{(b-a)^{2\alpha}}\left(x - \frac{a+b}{2}\right)^{2\alpha}\right]. \end{aligned} \tag{2.2} $$

Sarikaya et al. gave the following generalized Grüss inequality for local fractional integrals in [11].

Theorem 5 (Generalized Grüss inequality)**.** *Let $f,g\in I_x^{\alpha}[a,b]$. Then, $\varphi\leq f(x)\leq \Phi$ and $\gamma\leq g(x)\leq\Gamma$, for all $x\in[a,b]$, φ,Φ,γ and $\Gamma\in\mathbb{R}^{\alpha}$, we have*

$$|T_\alpha(f,g)|\leq\frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^2(1+\alpha)}(\Phi-\varphi)(\Gamma-\gamma) \tag{2.3}$$

where

$$T_\alpha(f,g)=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\ {}_aI_b^{\alpha}f(x)g(x)-\left[\ {}_aI_b^{\alpha}f(x)\right]\left[\ {}_aI_b^{\alpha}g(x)\right].$$

The interested reader is refer to [1],[3],[4],[6]-[16] for local freactional theory.

The aim of the paper is to establish some generalized Ostrowski inequality involving generalized moments via local fractional theory.

3. Main Results

Let X be a random variable whose generalized probability density function is $f:[a,b]\to R^{\alpha}$ with ${}_aI_b^{\alpha}f(t)=1$ and M_r^{α} denotes the generalized r^{th} moment of X, $r\geq0$, defined as

$$M_r^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{r\alpha}f(x)\,(dx)^{\alpha}.$$

The generalized mean and generalized variance of X are

$$\begin{aligned}\mu_\alpha &= M_1^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{\alpha}f(x)\,(dx)^{\alpha}\\ \sigma_\alpha^2 &= M_2^{\alpha}-(M_1^{\alpha})^2=\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-\mu_\alpha)^{2\alpha}f(x)\,(dx)^{\alpha}.\end{aligned}$$

Also, denote by $M_r^{\alpha}(c)$ the generalized r^{th} moment abouth $c\in R$ of X defined as

$$M_r^{\alpha}(c)=\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-c)^{r\alpha}f(x)\,(dx)^{\alpha}.$$

Theorem 6. *Let X be a random variable whose generalized probability density function $f:[a,b]\to R^{\alpha}$ is local fractional continuous with $c\in R$ and $\left\|f^{(\alpha)}(x)\right\|_\infty<$*

M for all $x \in [a,b]$, $a<b$. Then, for any positive integer r, we have the inequality

$$
\begin{aligned}
& M_r^{\alpha}(c) \\
\leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)} \left[(b-c)^{(r+1)\alpha} - (a-c)^{(r+1)\alpha}\right] \\
& + 2^{\alpha} M (b-a)^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \\
& \times \left[\frac{1}{2^{\alpha}} \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)} \left[(b-c)^{(r+1)\alpha} - (a-c)^{(r+1)\alpha}\right] \right. \\
& - \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{1}{2^{\alpha}(b-a)^{\alpha}} \left[(b-c)^{(r+2)\alpha} + (a-c)^{(r+2)\alpha}\right] \\
& \left. + \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+3)\alpha)} \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \left[(b-c)^{(r+3)\alpha} - (a-c)^{(r+3)\alpha}\right] \right].
\end{aligned} \tag{3.1}
$$

Proof. Since f is generalized probability density function, we have ${}_aI_b^{\alpha} f(t) = 1$. From (2.2), we have

$$
\begin{aligned}
& f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \leq 2^{\alpha} M (b-a)^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \\
& \times \left[\frac{1}{4^{\alpha}} + \frac{1}{(b-a)^{2\alpha}} \left(x - \frac{a+b}{2} \right)^{2\alpha} \right]
\end{aligned} \tag{3.2}
$$

Multiplying both sides of (3.2) by $(x-c)^{r\alpha}$ and integrating the resulting inequality with respect to with respect to x from a to b, we have

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)} \int_a^b (x-c)^{r\alpha} f(x) (dx)^{\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_a^b (x-c)^{r\alpha} (dx)^{\alpha} \\
& \leq 2^{\alpha} M (b-a)^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^{\alpha}\Gamma(1+\alpha)} \int_a^b (x-c)^{r\alpha} (dx)^{\alpha} \right. \\
& \left. + \frac{1}{(b-a)^{2\alpha}\Gamma(1+\alpha)} \int_a^b (x-c)^{r\alpha} \left(x - \frac{a+b}{2} \right)^{2\alpha} (dx)^{\alpha} \right].
\end{aligned} \tag{3.3}
$$

Using the local fractional integration by parts twice, we have

$$
\begin{aligned}
J &= \frac{1}{\Gamma(1+\alpha)}\int_a^b (x-c)^{r\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}(dx)^\alpha \\
&= \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}(x-c)^{(r+1)\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}\Bigg|_a^b \\
&\quad -\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-c)^{(r+1)\alpha}\left(x-\frac{a+b}{2}\right)^{\alpha}(dx)^\alpha \\
&= \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left(\frac{b-a}{2}\right)^{2\alpha}\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right] \\
&\quad -\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(x-c)^{(r+2)\alpha}\left(x-\frac{a+b}{2}\right)^{\alpha}\Bigg|_a^b \\
&\quad +\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_a^b (x-c)^{(r+2)\alpha}(dx)^\alpha \\
&= \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left(\frac{b-a}{2}\right)^{2\alpha}\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right] \\
&\quad -\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^{\alpha}\left[(b-c)^{(r+2)\alpha}+(a-c)^{(r+2)\alpha}\right] \\
&\quad +\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+3)\alpha)}\Gamma(1+2\alpha)\left[(b-c)^{(r+3)\alpha}-(a-c)^{(r+3)\alpha}\right].
\end{aligned}
$$

Thus, (3.3) siplifies to

$$
\begin{aligned}
& M_r^\alpha(c)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right] \\
\leq\ & 2^\alpha M(b-a)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \\
& \times\left[\frac{1}{4^\alpha}\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right]+\frac{1}{(b-a)^{2\alpha}}J\right]
\end{aligned}
$$

which completes the proof. □

Corollary 1. *Under assunptions of Theorem 6 with $c = 0$, we have*

$$\begin{aligned} & M_r^\alpha(0) \\ \leq\ & \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[b^{(r+1)\alpha}-a^{(r+1)\alpha}\right] \\ & \times\left[\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}+M(b-a)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right] \\ & -M\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\left[b^{(r+2)\alpha}+a^{(r+2)\alpha}\right] \qquad (3.4) \\ & +\frac{2^\alpha M}{(b-a)^\alpha}\Gamma(1+\alpha)\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+3)\alpha)}\left[b^{(r+3)\alpha}-a^{(r+3)\alpha}\right]. \end{aligned}$$

Taking $r = 1$ in (3.4), we obtain

$$\begin{aligned} \mu_\alpha \leq\ & \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}+M(b-a)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]\left[b^{2\alpha}-a^{2\alpha}\right] \\ & -M\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[b^{3\alpha}+a^{3\alpha}\right]+\frac{2^\alpha M}{(b-a)^\alpha}\Gamma(1+\alpha)\frac{\Gamma(1+\alpha)}{\Gamma(1+4\alpha)}\left[b^{4\alpha}-a^{4\alpha}\right]. \end{aligned}$$

Corollary 2. *Under assunptions of Theorem 6 with $r = 2$ and $c = \mu_\alpha$, we have*

$$\begin{aligned} \sigma_\alpha^2 \leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[(b-\mu_\alpha)^{3\alpha}-(a-\mu_\alpha)^{3\alpha}\right] \\ & +2^\alpha M(b-a)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{1}{2^\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[(b-\mu_\alpha)^{3\alpha}-(a-\mu_\alpha)^{3\alpha}\right]\right. \\ & -\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{1}{2^\alpha(b-a)^\alpha}\left[(b-\mu_\alpha)^{4\alpha}+(a-\mu_\alpha)^{4\alpha}\right] \\ & \left.+\frac{\Gamma(1+2\alpha)}{\Gamma(1+5\alpha)}\frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}}\left[(b-\mu_\alpha)^{5\alpha}-(a-\mu_\alpha)^{5\alpha}\right]\right]. \end{aligned}$$

Theorem 7. *Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \to R^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a, b]$ for $a, b \in I^0$ with $a < b$. If ,*

$$\varphi \leq f^{(\alpha)}(x) \leq \Phi \text{ for all } x \in [a, b],\ \varphi \text{ and } \Phi \in R^\alpha$$

then we have the following Ostrowski-Grüss type inequality

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)-2^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(x-\frac{a+b}{2}\right)^\alpha\frac{f(b)-f(a)}{(b-a)^\alpha}\right| \qquad (3.5)$$

$$\leq \frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)}(\Phi-\varphi).$$

Proof. From (2.1), we have

$$f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)=\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha \qquad (3.6)$$

where

$$p(x,t)=\begin{cases}(t-a)^{\alpha}, & t\in[a,x]\\ (t-b)^{\alpha}, & t\in(x,b].\end{cases}$$

It is clear that for all $x\in[a,b]$ and $t\in[a,b]$ we have

$$(x-b)^{\alpha}\leq p(x,t)\leq(x-a)^{\alpha}.$$

Applying Theorem 5 to the mappings $p(x,.)$ and $f^{(\alpha)}(.)$, we obtain

$$\begin{aligned}(3.7)\qquad &\left|\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\ {}_aI_b^{\alpha}p(x,t)f^{(\alpha)}(t)-[\ {}_aI_b^{\alpha}p(x,t)]\left[\ {}_aI_b^{\alpha}f^{(\alpha)}(t)\right]\right|\\ \leq\ & \frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^{2}(1+\alpha)}((x-a)^{\alpha}-(x-b)^{\alpha})(\Phi-\varphi)\\ =\ & \frac{(b-a)^{3\alpha}}{4^{\alpha}\Gamma^{2}(1+\alpha)}(\Phi-\varphi).\end{aligned}$$

A simple calculation we get

$$\begin{aligned}(3.8)\quad {}_aI_b^{\alpha}p(x,t) &= \frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)(dt)^{\alpha}\\ &= \frac{1}{\Gamma(1+\alpha)}\int_a^x (t-a)^{\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_x^b (t-b)^{\alpha}(dt)^{\alpha}\\ &= 2^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b-a)^{\alpha}\left(x-\frac{a+b}{2}\right)^{\alpha}\end{aligned}$$

and

$$(3.9)\qquad {}_aI_b^{\alpha}f^{(\alpha)}(t)=f(b)-f(a).$$

Putting (3.6), (3.8) and (3.9) in (3.7) and mutiplying the resuting inequality by $\frac{\Gamma(1+\alpha)}{(b-a)^{2\alpha}}$, we obtain desired result. □

Theorem 8. *Let X be a random variable whose probability density function $f:[a,b]\to R^{\alpha}$ is local fractional continuous with $c\in R$ and $\varphi\leq f^{(\alpha)}(x)\leq\Phi$ for all*

$x \in [a,b]$, $a < b$. Then, for any positive integer r, we have te inequality

$$
\begin{aligned}
(3.10)\qquad & M_r^{\alpha}(c) \\
\leq\ & \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left(\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}+\frac{(b-a)^{\alpha}}{4^{\alpha}\Gamma(1+\alpha)}(\Phi-\varphi)\right. \\
& \left.-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^{\alpha}}(a+b)^{\alpha}\right)\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right] \\
& +2^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^{\alpha}} \\
& \times\left(\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[(b-c)^{(r+1)\alpha}b^{\alpha}-(a-c)^{(r+1)\alpha}a^{\alpha}\right]\right. \\
& \left.-\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\left[(b-c)^{(r+2)\alpha}-(a-c)^{(r+2)\alpha}\right]\right).
\end{aligned}
$$

Proof. From (3.5), we have the for all $x \in [a,b]$

$$
\begin{aligned}
(3.11) f(x)\ \leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)+2^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(x-\frac{a+b}{2}\right)^{\alpha}\frac{f(b)-f(a)}{(b-a)^{\alpha}} \\
& +\frac{(b-a)^{\alpha}}{4^{\alpha}\Gamma(1+\alpha)}(\Phi-\varphi).
\end{aligned}
$$

Multiplying both sides of (3.11) by $(x-c)^{r\alpha}$ and integrating the resulting inequality with respect to with respect to x from a to b, since ${}_aI_b^{\alpha}f(t)=1$, we have

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)}\int_a^b (x-c)^{r\alpha}f(x)\,(dx)^{\alpha} \\
\leq\ & \left[\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}+\frac{(b-a)^{\alpha}}{4^{\alpha}\Gamma(1+\alpha)}(\Phi-\varphi)\right]\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-c)^{r\alpha}(dx)^{\alpha} \\
& +2^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b\left(x-\frac{a+b}{2}\right)^{\alpha}(x-c)^{r\alpha}(dx)^{\alpha}
\end{aligned}
$$

or,

$$
\begin{aligned}
&M_r^\alpha(c) \\
\leq\ & \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)}(\Phi-\varphi)\right. \\
&\left.-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}(a+b)^\alpha\right]\left[(b-c)^{(r+1)\alpha}-(a-c)^{(r+1)\alpha}\right] \\
&+2^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}\left[\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[(b-c)^{(r+1)\alpha}b^\alpha-(a-c)^{(r+1)\alpha}a^\alpha\right]\right. \\
&\left.-\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\left[(b-c)^{(r+2)\alpha}-(a-c)^{(r+2)\alpha}\right]\right]
\end{aligned}
$$

which completes the proof. □

Corollary 3. *Under assumptions of Theorem 8 with* $c=0$*, we have*

$$
\begin{aligned}
&M_r^\alpha(0) \\
\leq\ & \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\left[\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)}(\Phi-\varphi)\right]\left[b^{(r+1)\alpha}-a^{(r+1)\alpha}\right] \\
&-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}\left[\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}(a+b)^\alpha\left[b^{(r+1)\alpha}-a^{(r+1)\alpha}\right]\right. \\
&\left.-2^\alpha\left(\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}-\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+2)\alpha)}\right)\left[b^{(r+2)\alpha}-a^{(r+2)\alpha}\right]\right].
\end{aligned}
\tag{3.12}
$$

If we $r=1$ *in (3.12), then we get*

$$
\begin{aligned}
\mu_\alpha \leq\ & \frac{\Gamma^2(1+\alpha)(a+b)^\alpha}{\Gamma(1+2\alpha)}+\frac{(b-a)^{2\alpha}(a+b)^\alpha}{4^\alpha\Gamma(1+2\alpha)}(\Phi-\varphi) \\
&-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(a+b)^{2\alpha}(b-a)^\alpha\right. \\
&\left.-2^\alpha\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right)\left(b^{3\alpha}-a^{3\alpha}\right)\right].
\end{aligned}
$$

Corollary 4. *Under assumptions of Theorem 8 with* $r=2$ *and* $c=\mu_\alpha$*, we have*

$$
\begin{aligned}
&\sigma_\alpha^2 \leq \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)}(\Phi-\varphi)\right. \\
&\left.-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}(a+b)^\alpha\right]\left[(b-\mu_\alpha)^{3\alpha}-(a-\mu_\alpha)^{3\alpha}\right] \\
&+2^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{f(b)-f(a)}{(b-a)^\alpha}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[(b-\mu_\alpha)^{(r+1)\alpha}b^\alpha-(a-\mu_\alpha)^{(r+1)\alpha}a^\alpha\right]\right. \\
&\left.-\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\left[(b-\mu_\alpha)^{4\alpha}-(a-\mu_\alpha)^{4\alpha}\right]\right]
\end{aligned}
$$

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