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# Affine and Functional Form of Jensen's Inequalitiy for 3-convex Functions at a Point

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ABSTRACT

In this paper, we give the refinement of an extension of Jensen's inequality to affine combinations. Furthermore, we present the functional form of Jensen's inequality for continuous 3-convex functions of one variable at a point.

## 1. Introduction

Let $\mathscr{X}$ be a real linear space. A set $\mathscr{A} \subset \mathscr{X}$ is affine if it contain all binomial affine combinations $\alpha a + \beta b$ of points $a, b \in \mathscr{A}$ and coefficient $\alpha, \beta \in \mathbb{R}$ of sum $\alpha + \beta = 1$. The affine hull of a set $\mathscr{I} \subset \mathscr{X}$ as the smallest affine set that contains $\mathscr{I}$ is denoted with aff$\mathscr{I}$. A function $f : \mathscr{A} \to \mathbb{R}$ is affine if the equality

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$$

holds for all binomial affine combinations of points of $\mathscr{A}$.
A set $\mathscr{Y} \subset \mathscr{X}$ is convex if it contains all binomial convex combinations $\alpha a + \beta b$ of points $a, b \in \mathscr{Y}$ and non-negative coefficient $\alpha, \beta \in \mathbb{R}$ of sum $\alpha + \beta = 1$. The convex hull of of a set $\mathscr{I} \subset \mathscr{X}$ as the smallest convex set that contains $\mathscr{I}$ is denoted with conv$\mathscr{I}$. A function $f : \mathscr{Y} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$$

holds for all binomial convex combinations of points of $\mathscr{Y}$.
Let $\Omega$ be a non-empty set, and let $\mathbb{X}$ be a subspace of linear space of all real functions on the domain $\Omega$. Also, assume that the unit function defined by $I(x) = 1$ for every $x \in \Omega$ belongs to $\mathbb{X}$. Let $\mathscr{I} \subset \mathbb{R}$ be an interval, and let $\mathbb{X}_{\mathscr{I}} \subset \mathbb{X}$ be asubset containing all functions with image in $\mathscr{I}$. If $\alpha g + \beta h$ is a convex combination of functions $g, h \in \mathbb{X}_{\mathscr{I}}$, then the convex combination $\alpha g(x) + \beta h(x)$ is in $\mathscr{I}$ for every $x \in \Omega$, which indicates that functions set $\mathbb{X}_{\mathscr{I}}$ is convex.
A linear functional $L : \mathbb{X} \to \mathbb{R}$ is positive (non-negative) if $L(g) \geq 0$ for every non-negative function $g \in \mathbb{X}$, and $L$ is unital(normalized) if $L(1) = 1$. If $g \in \mathbb{X}$, then every unital positive functional $L$, the number $L(g)$ is in the closed interval of real numbers containing the image of the function $g$.
In 2015, Z. Pavić [3] gave the extension of Jensen's inequality to affine combinations in the following form

**Theorem 1.** *Let $\alpha_i, \beta_j, \gamma_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^{n} \alpha_i$, $\beta = \sum_{j=1}^{m} \beta_j$, $\gamma = \sum_{k=1}^{l} \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0,1]$. Let $a_i, b_j, c_k \in \mathbb{R}$ be points such that $c_k \in conv\{a, b\}$, where*

$$a = \frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i a_i \ , \ b = \frac{1}{\beta} \sum_{j=1}^{m} \beta_j b_j. \tag{1.1}$$

*Then the affine combination*

$$\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k \tag{1.2}$$

*belongs to conv{a, b}, and for every convex functions* $f:conv\{a_i,b_j\}\to\mathbb{R}$ *satisfies the inequality*

$$f\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)\leq\sum_{i=1}^{n}\alpha_i f(a_i)+\sum_{j=1}^{m}\beta_j f(b_j)-\sum_{k=1}^{l}\gamma_k f(c_k). \tag{1.3}$$

In 2014, Z. Pavić [4] also gave the functional form of jensen's inequality for the continuous convex functions of one variable in the following form

**Theorem 2.** *Let* $\mathscr{I}\subset\mathbb{R}$ *be a closed interval, let* $[a,b]\subset\mathscr{I}$*, let function* $g\in\mathbb{X}_{[a,b]}$ *and function* $h\in\mathbb{X}_{\mathscr{I}\setminus(a,b)}$*. Let* $f:\mathscr{I}\to\mathbb{R}$ *be a continuous convex function such that* $f(g),f(h)\in\mathbb{X}$*. If a pair of unital positive linear functionals* $L,H:\mathbb{X}\to\mathbb{R}$ *satisfies*

$$L(g)=H(h), \tag{1.4}$$

*then*

$$L(f(g))\leq H(f(h)). \tag{1.5}$$

Furthermore, Z. Pavić [4] also gave some consequent results in the form of corollaries and using these corollaries, he gave another important result as follow

**Corollary 1.** *Let* $\mathscr{I}\subset\mathbb{R}$ *be a closed interval, let function* $g\in\mathbb{X}_{[a,b]}$*. Let* $f:\mathscr{I}\to\mathbb{R}$ *be a continuous convex function such that* $f(g)\in\mathbb{X}$*. If a unital positive linear functional* $L:\mathbb{X}\to\mathbb{R}$ *satisfies the implication* $(1.4)\Rightarrow(1.5)$ *of Theorem* 2 *for* $L=H$*, then*

$$f(L(g))\leq L(f(g)). \tag{1.6}$$

**Corollary 2.** *Let* $[a_1,b_1]\subseteq...\subseteq[a_{n-1},b_{n-1}]\subseteq\mathscr{I}$*. Let function* $g_1\in\mathbb{X}_{[a_1,b_1]}$*, let functions* $g_k\in\mathbb{X}_{[a_k,b_k]\setminus(a_{k-1},b_{k-1})}$ *for* $k=2,...,n-1$*, and let function* $g_n\in\mathbb{X}_{\mathscr{I}\setminus(a_{n-1},b_{n-1})}$*. Let* $f:\mathscr{I}\to\mathbb{R}$ *be a continuous convex function such that* $f(g_i)\in\mathbb{X}$*.*
*If an n-tuple of unital positive linear functionals* $L_i:\mathbb{X}\to\mathbb{R}$ *satisfies*

$$L_i(g_i)=L_{i+1}(g_{i+1})\ \ for\ i=1,...,n-1, \tag{1.7}$$

*then*

$$L_i(f(g_i))\leq L_{i+1}((g_{i+1}))\ \ for\ i=1,...,n-1. \tag{1.8}$$

**Corollary 3.** *Let* $\mathscr{I}\subset\mathbb{R}$ *be a closed interval, and let functions* $g_1,...,g_n\in\mathbb{X}_{\mathscr{I}}$*. Let* $f:\mathscr{I}\to\mathbb{R}$ *be a continuous convex function such that* $f(g_i)\in\mathbb{X}$*.*
*Then every n-tuple of positive linear functionals* $L_i:\mathbb{X}\to\mathbb{R}$ *with* $\sum_{i=1}^{n}L_i(1)=1$ *satisfies the inclusion*

$$\sum_{i=1}^{n}L_i(g_i)\in\mathscr{I} \tag{1.9}$$

*and the inequality*

$$f\big(\sum_{i=1}^{n}L_i(g_i)\big)\leq\sum_{i=1}^{n}L_i(f(g_i)). \tag{1.10}$$

**Theorem 3.** *Let* $\mathscr{I}\subset\mathbb{R}$ *be a closed interval, and let* $[a,b]\subset\mathscr{I}$*. Let functions* $g_1,...,g_n\in\mathbb{X}_{[a,b]}$ *and* $h_1,...,h_m\in\mathbb{X}_{\mathscr{I}\setminus(a,b)}$*. Let* $f:\mathscr{I}\to\mathbb{R}$ *be a continuous convex function such that* $f(g_i),f(h_j)\in\mathbb{X}$*.*
*If a pair of n-tuple of positive linear functionals* $L_i,H_j:\mathbb{X}\to\mathbb{R}$ *with* $\sum_{i=1}^{n}L_i(1)=\sum_{j=1}^{m}H_j(1)=1$ *satisfies*

$$\sum_{i=1}^{n}L_i(g_i)=\sum_{j=1}^{m}H_j(h_j), \tag{1.11}$$

*then*

$$\sum_{i=1}^{n} L_i(f(g_i)) \leq \sum_{j=1}^{m} H_j(f(h_j)). \tag{1.12}$$

## 2. Main Results

In [2], I. A. Baloch, J. Pečarič, M. Praljak defined a new class of functions which is defined as follow

**Definition 1.** Let $c \in I^\circ$, where $I$ is an arbitrary interval(open, closed or semi-open in either direction) in $\mathbb{R}$ and $I^\circ$ is its interior. We say that $f : I \to \mathbb{R}$ is 3-convex function in point $c$ (respectively 3-concave function in point $c$) if there exists a constant $A$ such that the function $F(x) = f(x) - \frac{A}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$. A $f$ is 3-concave function in point $c$ if $-f$ is 3-convex function in point $c$.

A property that explains the name of the class is the fact that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval (see [2]). Note that $K_1^c(I)$ and $K_2^c(I)$) denote the class of all 3-convex functions in point $c$ and the class of all 3-concave functions in point $c$ respectively.

**Theorem 4.** *Let $\alpha_i, \beta_j, \gamma_k \geq 0$ and $\lambda_i, \mu_j, \nu_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^{n} \alpha_i$, $\beta = \sum_{j=1}^{m} \beta_j$, $\gamma = \sum_{k=1}^{l} \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0,1]$; $\lambda = \sum_{i=1}^{n} \lambda_i$, $\mu = \sum_{j=1}^{m} \mu_j$, $\nu = \sum_{k=1}^{l} \nu_k$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0,1]$. Let $a_i, b_j, c_k \in [a,c]$ be points such that $c_k \in conv\{a_i, b_j\}$ and $r_i, s_j, t_k \in [c,b]$ be points such that $t_k \in conv\{r_i, s_j\}$ , where*

$$a = \frac{1}{\alpha}\sum_{i=1}^{n} \alpha_i a_i \ , \ b = \frac{1}{\beta}\sum_{j=1}^{m} \beta_j b_j \ , \ r = \frac{1}{\lambda}\sum_{i=1}^{n} \lambda_i r_i \ , \ s = \frac{1}{\mu}\sum_{j=1}^{m} \mu_j s_j.$$

*Now, if*

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i (a_i)^2 + \sum_{j=1}^{m} \beta_j (b_j)^2 - \sum_{k=1}^{l} \gamma_k (c_k)^2 - \big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big)^2 \\ &= \sum_{i=1}^{n} \lambda_i (r_i)^2 + \sum_{j=1}^{m} \mu_j (s_j)^2 - \sum_{k=1}^{l} \nu_k (t_k)^2 - \big(\sum_{i=1}^{n} \lambda_i r_i + \sum_{j=1}^{m} \mu_j s_j - \sum_{k=1}^{l} \nu_k t_k\big)^2 \end{aligned} \tag{2.1}$$

*and also there exists $c \in I^\circ$ ($I = [a,b]$) such that*

$$\max\{\max_i\{a_i\}, \max_j\{b_j\}, \max_k\{c_k\}\} \ \leq \ c \ \leq \ \min\{\min_i\{r_i\}, \min_j\{s_j\}, \min_k\{t_k\}\}. \tag{2.2}$$

*Then for every $f \in K_1^c(I)$, the following inequality holds*

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i f(a_i) + \sum_{j=1}^{m} \beta_j f(b_j) - \sum_{k=1}^{l} \gamma_k f(c_k) - f\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ &\leq \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \end{aligned} \tag{2.3}$$

*Proof.* Since $f \in K_1^c(I)$, then there exists a constant $A$ such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $I \cap (-\infty, c]$ and for $a_i, b_j, c_k \in [a,c]$ be points such that $c_k \in conv\{a_i, b_j\}$, so by using inequality (1.3) we have

$$\begin{aligned} 0 \ &\geq \ \sum_{i=1}^{n} \alpha_i F(a_i) + \sum_{j=1}^{m} \beta_j F(b_j) - \sum_{k=1}^{l} \gamma_k F(c_k) - F\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ &= \ \sum_{i=1}^{n} \alpha_i f(a_i) + \sum_{j=1}^{m} \beta_j f(b_j) - \sum_{k=1}^{l} \gamma_k f(c_k) - f\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ &- \ \frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (a_i)^2 + \sum_{j=1}^{m} \beta_j (b_j)^2 - \sum_{k=1}^{l} \gamma_k (c_k)^2 - \big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big)^2\big\} \end{aligned}$$

Also, since $f \in K_1^c(I)$ is convex on $I \cap [c,\infty)$, hence for $r_i, s_j, t_k \in [c,b]$ be points such that $t_k \in conv\{r_i, s_j\}$, so by using inequality (1.3) we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} \alpha_i F(r_i) + \sum_{j=1}^{m} \beta_j F(s_j) - \sum_{k=1}^{l} \gamma_k F(t_k) - F\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \\ &= \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \\ &- \frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (r_i)^2 + \sum_{j=1}^{m} \beta_j (s_j)^2 - \sum_{k=1}^{l} \gamma_k (t_k)^2 - \big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big)^2\big\} \end{aligned}$$

From above, we have

$$\begin{gathered} \sum_{i=1}^{n} \alpha_i f(a_i) + \sum_{j=1}^{m} \beta_j f(b_j) - \sum_{k=1}^{l} \gamma_k f(c_k) - f\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ -\frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (a_i)^2 + \sum_{j=1}^{m} \beta_j (b_j)^2 - \sum_{k=1}^{l} \gamma_k (c_k)^2 - \big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big)^2\big\} \\ \leq 0 \leq \\ \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \\ -\frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (r_i)^2 + \sum_{j=1}^{m} \beta_j (s_j)^2 - \sum_{k=1}^{l} \gamma_k (t_k)^2 - \big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big)^2\big\} \qquad (2.4) \end{gathered}$$

So

$$\begin{gathered} \sum_{i=1}^{n} \alpha_i f(a_i) + \sum_{j=1}^{m} \beta_j f(b_j) - \sum_{k=1}^{l} \gamma_k f(c_k) - f\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ -\frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (a_i)^2 + \sum_{j=1}^{m} \beta_j (b_j)^2 - \sum_{k=1}^{l} \gamma_k (c_k)^2 - \big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big)^2\big\} \\ \leq \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \\ -\frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (r_i)^2 + \sum_{j=1}^{m} \beta_j (s_j)^2 - \sum_{k=1}^{l} \gamma_k (t_k)^2 - \big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big)^2\big\} \end{gathered}$$

by using (2.1), we get (2.3). ∎

**Remark 1.** *From the proof of Theorem* 4*, we have*

$$\begin{gathered} \sum_{i=1}^{n} \alpha_i f(a_i) + \sum_{j=1}^{m} \beta_j f(b_j) - \sum_{k=1}^{l} \gamma_k f(c_k) - f\big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big) \\ \leq \frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (a_i)^2 + \sum_{j=1}^{m} \beta_j (b_j)^2 - \sum_{k=1}^{l} \gamma_k (c_k)^2 - \big(\sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j - \sum_{k=1}^{l} \gamma_k c_k\big)^2\big\} \qquad (2.5) \end{gathered}$$

*and*

$$\begin{gathered} \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big) \\ \geq \frac{A}{2}\big\{\sum_{i=1}^{n} \alpha_i (r_i)^2 + \sum_{j=1}^{m} \beta_j (s_j)^2 - \sum_{k=1}^{l} \gamma_k (t_k)^2 - \big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\big)^2\big\} \qquad (2.6) \end{gathered}$$

*So under assumption* (2.1)*, we can get a improvement of* (2.3) *as follows*

$$\sum_{i=1}^{n}\alpha_i f(a_i)+\sum_{j=1}^{m}\beta_j f(b_j)-\sum_{k=1}^{l}\gamma_k f(c_k)-f\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)$$

$$\leq \frac{A}{2}\Big\{\sum_{i=1}^{n}\alpha_i (a_i)^2+\sum_{j=1}^{m}\beta_j (b_j)^2-\sum_{k=1}^{l}\gamma_k (c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2\Big\}$$

$$\Big(=\frac{A}{2}\Big\{\sum_{i=1}^{n}\alpha_i (r_i)^2+\sum_{j=1}^{m}\beta_j (s_j)^2-\sum_{k=1}^{l}\gamma_k (t_k)^2-\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big)^2\Big\}\Big)$$

$$\leq \sum_{i=1}^{n}\alpha_i f(r_i)+\sum_{j=1}^{m}\beta_j f(s_j)-\sum_{k=1}^{l}\gamma_k f(t_k)-f\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big) \tag{2.7}$$

Assume that $\tilde{a}=\max_i\{a_i\},\tilde{b}=\max_j\{b_j\},\tilde{c}=\max_k\{c_k\}$ and $\tilde{r}=\min_i\{r_i\},\tilde{s}=\min_j\{s_j\},\tilde{t}=\min_k\{t_k\}$. Also, let $\tilde{\tilde{a}}=\max\{\tilde{a},\tilde{b},\tilde{c}\}$ and $\tilde{\tilde{r}}=\min\{\tilde{r},\tilde{s},\tilde{t}\}$ Now, we give the next result which weakens the assumption (2.1) such that inequality (2.5) also holds.

**Theorem 5.** *Let $\alpha_i,\beta_j,\gamma_k\geq 0$ and $\lambda_i,\mu_j,\nu_k\geq 0$ be coefficients such that their sum $\alpha=\sum_{i=1}^{n}\alpha_i$, $\beta=\sum_{j=1}^{m}\beta_j$, $\gamma=\sum_{k=1}^{l}\gamma_k$ satisfy $\alpha+\beta-\gamma=1$ and $\alpha,\beta\in(0,1]$; $\lambda=\sum_{i=1}^{n}\lambda_i$, $\mu=\sum_{j=1}^{m}\mu_j$, $\nu=\sum_{k=1}^{l}\nu_k$ satisfy $\lambda+\mu-\nu=1$ and $\lambda,\mu\in(0,1]$. Let $a_i,b_j,c_k\in[a,c]$ be points such that $c_k\in conv\{a_i,b_j\}$ and $r_i,s_j,t_k\in[c,b]$ be points such that $t_k\in conv\{r_i,s_j\}$ , where*

$$a=\frac{1}{\alpha}\sum_{i=1}^{n}\alpha_i a_i\ ,\ b=\frac{1}{\beta}\sum_{j=1}^{m}\beta_j b_j\ ,\ r=\frac{1}{\lambda}\sum_{i=1}^{n}\lambda_i r_i\ ,\ s=\frac{1}{\mu}\sum_{j=1}^{m}\mu_j s_j.$$

*such that*

$$\tilde{\tilde{a}}\leq\tilde{\tilde{r}} \tag{2.8}$$

*and $f\in K_1^c(I)$ for some $c\in[\tilde{\tilde{a}},\tilde{\tilde{r}}]$. Then if*
*(a)*

$$f''_{-}(\tilde{\tilde{a}})\geq 0$$

*and*

$$\sum_{i=1}^{n}\alpha_i (a_i)^2+\sum_{j=1}^{m}\beta_j (b_j)^2-\sum_{k=1}^{l}\gamma_k (c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2$$

$$\leq \sum_{i=1}^{n}\lambda_i (r_i)^2+\sum_{j=1}^{m}\mu_j (s_j)^2-\sum_{k=1}^{l}\nu_k (t_k)^2-\big(\sum_{i=1}^{n}\lambda_i r_i+\sum_{j=1}^{m}\mu_j s_j-\sum_{k=1}^{l}\nu_k t_k\big)^2$$

*or*
*(b)*

$$f''_{+}(\tilde{\tilde{r}})\leq 0$$

*and*

$$\sum_{i=1}^{n}\alpha_i (a_i)^2+\sum_{j=1}^{m}\beta_j (b_j)^2-\sum_{k=1}^{l}\gamma_k (c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2$$

$$\geq \sum_{i=1}^{n}\lambda_i (r_i)^2+\sum_{j=1}^{m}\mu_j (s_j)^2-\sum_{k=1}^{l}\nu_k (t_k)^2-\big(\sum_{i=1}^{n}\lambda_i r_i+\sum_{j=1}^{m}\mu_j s_j-\sum_{k=1}^{l}\nu_k t_k\big)^2$$

*or*
*(c)*

$$f''_{-}(\tilde{\tilde{a}})<0<f''_{+}(\tilde{\tilde{r}})\ \text{ and }\ f \text{ is } 3-convex,$$

*then* (2.3) *holds.*

*Proof.* The idea of proof is similar to proof of Theorem 4. Hence, by proceeding as in the proof of Theorem 4. From the inequality 2.4, we have

$$\frac{A}{2}\Big[\sum_{i=1}^{n}\alpha_i(r_i)^2+\sum_{j=1}^{m}\beta_j(s_j)^2-\sum_{k=1}^{l}\gamma_k(t_k)^2-\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big)^2$$

$$-\big\{\sum_{i=1}^{n}\alpha_i(a_i)^2+\sum_{j=1}^{m}\beta_j(b_j)^2-\sum_{k=1}^{l}\gamma_k(c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2\big\}\Big]$$

$$\leq\sum_{i=1}^{n}\alpha_i f(r_i)+\sum_{j=1}^{m}\beta_j f(s_j)-\sum_{k=1}^{l}\gamma_k f(t_k)-f\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big)$$

$$-\big\{\sum_{i=1}^{n}\alpha_i f(a_i)+\sum_{j=1}^{m}\beta_j f(b_j)-\sum_{k=1}^{l}\gamma_k f(c_k)-f\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)\big\}$$

Now, due to the concavity of $F$ on $[a,c]$ and convexity of $F$ on $[c,b]$, for every distinct points $a_j\in[a,\tilde{\tilde{a}}]$ and $r_j\in[\tilde{\tilde{r}},b]$, $j=1,2,3$, we have

$$[a_1,a_2,a_3]f\leq A\leq[r_1,r_2,r_3]f$$

Letting $a_j\nearrow\tilde{\tilde{a}}$ and $r_j\searrow\tilde{\tilde{r}}$, we get (if exists)

$$f''_-(\tilde{\tilde{a}})\leq A\leq f''_+(\tilde{\tilde{r}})$$

Therefore, if assumptions (a) or (b) holds, then

$$\frac{A}{2}\Big[\sum_{i=1}^{n}\alpha_i(r_i)^2+\sum_{j=1}^{m}\beta_j(s_j)^2-\sum_{k=1}^{l}\gamma_k(t_k)^2-\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big)^2$$

$$-\big\{\sum_{i=1}^{n}\alpha_i(a_i)^2+\sum_{j=1}^{m}\beta_j(b_j)^2-\sum_{k=1}^{l}\gamma_k(c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2\big\}\Big]$$

is positive and we conclude the result. If the assumption (c) holds, the $f''_-$ is left continuous, $f''_+$ is right continuous, they are both non-decreasing and $f''_-\leq f''_+$. Therefore, there exists $\tilde{c}\in[\tilde{a},\tilde{r}]$ such that $f\in K_1^c(I)$ with associated constant $\tilde{A}=0$ and again, we can deduce the result. ∎

**Remark 2.** *Again from the proof of Theorem* 5*, we obtain the inequalities* (2.5) *and* (2.6)*. Now, under assumption (a), (b) or (c) of Theorem* 5*, A is positive or negative or zero respectively due to argument discussed in the proof. Therefore, we get a better improvement of* (2.3) *then* (2.7)*. in this case as follow*

$$\sum_{i=1}^{n}\alpha_i f(a_i)+\sum_{j=1}^{m}\beta_j f(b_j)-\sum_{k=1}^{l}\gamma_k f(c_k)-f\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)$$

$$\leq\frac{A}{2}\big\{\sum_{i=1}^{n}\alpha_i(a_i)^2+\sum_{j=1}^{m}\beta_j(b_j)^2-\sum_{k=1}^{l}\gamma_k(c_k)^2-\big(\sum_{i=1}^{n}\alpha_i a_i+\sum_{j=1}^{m}\beta_j b_j-\sum_{k=1}^{l}\gamma_k c_k\big)^2\big\}$$

$$\leq\frac{A}{2}\big\{\sum_{i=1}^{n}\alpha_i(r_i)^2+\sum_{j=1}^{m}\beta_j(s_j)^2-\sum_{k=1}^{l}\gamma_k(t_k)^2-\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big)^2\big\}$$

$$\leq\sum_{i=1}^{n}\alpha_i f(r_i)+\sum_{j=1}^{m}\beta_j f(s_j)-\sum_{k=1}^{l}\gamma_k f(t_k)-f\big(\sum_{i=1}^{n}\alpha_i r_i+\sum_{j=1}^{m}\beta_j s_j-\sum_{k=1}^{l}\gamma_k t_k\big) \tag{2.9}$$

Under the assumption of Theorem 4 with $f\in K_2^c(I)$, the reverse of inequality (2.3) holds. Now, we give only the statement of Theorem with weaker condition under which the reverse of inequality (2.3) also holds for $f\in K_2^c(I)$.

**Theorem 6.** *Let $\alpha_i,\beta_j,\gamma_k \geq 0$ and $\lambda_i,\mu_j,\nu_k \geq 0$ be coefficients such that their sum $\alpha=\sum_{i=1}^n \alpha_i$, $\beta=\sum_{j=1}^m \beta_j$, $\gamma=\sum_{k=1}^l \gamma_k$ satisfy $\alpha+\beta-\gamma=1$ and $\alpha,\beta\in(0,1]$; $\lambda=\sum_{i=1}^n \lambda_i$, $\mu=\sum_{j=1}^m \mu_j$, $\nu=\sum_{k=1}^l \nu_k$ satisfy $\lambda+\mu-\nu=1$ and $\lambda,\mu\in(0,1]$. Let $a_i,b_j,c_k\in[a,c]$ be points such that $c_k\in conv\{a_i,b_j\}$ and $r_i,s_j,t_k\in[c,b]$ be points such that $t_k\in conv\{r_i,s_j\}$ , where*

$$a=\frac{1}{\alpha}\sum_{i=1}^n \alpha_i a_i\ ,\ b=\frac{1}{\beta}\sum_{j=1}^m \beta_j b_j\ ,\ r=\frac{1}{\alpha}\sum_{i=1}^n \lambda_i r_i\ ,\ s=\frac{1}{\mu}\sum_{j=1}^m \mu_j s_j.$$

*such that*

$$\tilde{\tilde{a}} \leq \tilde{\tilde{r}} \tag{2.10}$$

*and $f\in K_2^c(I)$ for some $c\in[\tilde{\tilde{a}},\tilde{\tilde{r}}]$. Then if*
*(a)*

$$f''_-(\tilde{\tilde{a}}) \leq 0$$

*and*

$$\begin{aligned}&\sum_{i=1}^n \alpha_i(a_i)^2+\sum_{j=1}^m \beta_j(b_j)^2-\sum_{k=1}^l \gamma_k(c_k)^2-\big(\sum_{i=1}^n \alpha_i a_i+\sum_{j=1}^m \beta_j b_j-\sum_{k=1}^l \gamma_k c_k\big)^2\\&\geq \sum_{i=1}^n \lambda_i(r_i)^2+\sum_{j=1}^m \mu_j(s_j)^2-\sum_{k=1}^l \nu_k(t_k)^2-\big(\sum_{i=1}^n \lambda_i r_i+\sum_{j=1}^m \mu_j s_j-\sum_{k=1}^l \nu_k t_k\big)^2\end{aligned}$$

*or*
*(b)*

$$f''_+(\tilde{\tilde{r}}) \geq 0$$

*and*

$$\begin{aligned}&\sum_{i=1}^n \alpha_i(a_i)^2+\sum_{j=1}^m \beta_j(b_j)^2-\sum_{k=1}^l \gamma_k(c_k)^2-\big(\sum_{i=1}^n \alpha_i a_i+\sum_{j=1}^m \beta_j b_j-\sum_{k=1}^l \gamma_k c_k\big)^2\\&\leq \sum_{i=1}^n \lambda_i(r_i)^2+\sum_{j=1}^m \mu_j(s_j)^2-\sum_{k=1}^l \nu_k(t_k)^2-\big(\sum_{i=1}^n \lambda_i r_i+\sum_{j=1}^m \mu_j s_j-\sum_{k=1}^l \nu_k t_k\big)^2\end{aligned}$$

*or*
*(c)*

$$f''_-(\tilde{\tilde{a}}) < 0 < f''_+(\tilde{\tilde{r}}) \ \text{ and } \ f \text{ is } 3-concave,$$

*then reverse of* (2.3) *holds.*

**Remark 3.** *From the proof of the Theorem* 6*, we obtain the reverse of inequalities* (2.5) *and* (2.6)*. Now, due to the convexity of $F$ on $[a,c]$ and concavity of $F$ on $[c,b]$, for every distinct points $a_j\in[a,\tilde{\tilde{a}}]$ and $r_j\in[\tilde{\tilde{r}},b]$, $j=1,2,3$, we have*

$$[a_1,a_2,a_3]f \geq A \geq [r_1,r_2,r_3]f$$

*Letting $a_j \nearrow \tilde{\tilde{a}}$ and $r_j \searrow \tilde{\tilde{r}}$, we get (if exists)*

$$f''_-(\tilde{\tilde{a}}) \geq A \geq f''_+(\tilde{\tilde{r}})$$

*Now, under assumption (a), (b) or (c) of Theorem* 5*, A is negative or positive or zero respectively due to argument discussed above. Therefore, we get a better improvement in this case as follow*

$$\begin{aligned}&\sum_{i=1}^n \alpha_i f(a_i)+\sum_{j=1}^m \beta_j f(b_j)-\sum_{k=1}^l \gamma_k f(c_k)-f\big(\sum_{i=1}^n \alpha_i a_i+\sum_{j=1}^m \beta_j b_j-\sum_{k=1}^l \gamma_k c_k\big)\\&\geq \frac{A}{2}\big\{\sum_{i=1}^n \alpha_i(a_i)^2+\sum_{j=1}^m \beta_j(b_j)^2-\sum_{k=1}^l \gamma_k(c_k)^2-\big(\sum_{i=1}^n \alpha_i a_i+\sum_{j=1}^m \beta_j b_j-\sum_{k=1}^l \gamma_k c_k\big)^2\big\}\\&\geq \frac{A}{2}\big\{\sum_{i=1}^n \alpha_i(r_i)^2+\sum_{j=1}^m \beta_j(s_j)^2-\sum_{k=1}^l \gamma_k(t_k)^2-\big(\sum_{i=1}^n \alpha_i r_i+\sum_{j=1}^m \beta_j s_j-\sum_{k=1}^l \gamma_k t_k\big)^2\big\}\end{aligned}$$

$$\geq \sum_{i=1}^{n} \alpha_i f(r_i) + \sum_{j=1}^{m} \beta_j f(s_j) - \sum_{k=1}^{l} \gamma_k f(t_k) - f\Big(\sum_{i=1}^{n} \alpha_i r_i + \sum_{j=1}^{m} \beta_j s_j - \sum_{k=1}^{l} \gamma_k t_k\Big) \tag{2.11}$$

**Theorem 7.** *Let $\mathscr{I} \subset \mathbb{R}$ be a closed interval, let $[a,b] \subset \mathscr{I}$, let function $g_i \in \mathbb{X}_{[a,b]}$ and function $h_i \in \mathbb{X}_{\mathscr{I}\setminus(a,b)}$ for $i = 1,2$. Let $f \in K_1^c(\mathscr{I})$ be continuous function such that $f(g_i), f(h_i) \in \mathbb{X}$. If a pair of unital positive linear functionals $L, H : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L(g_i) = H(h_i) \ \text{ and } \ H(h_1^2) - L(g_1^2) = H(h_2^2) - L(g_2^2), \ \ i \ = \ 1,2, \tag{2.12}$$

*then inequality*

$$H(f(h_1)) - L(f(g_1)) \leq H(f(h_2)) - L(f(g_2)) \tag{2.13}$$

*holds.*

*Proof.* Since $f \in K_1^c(\mathscr{I})$, there exists a constant $A$ such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $\mathscr{I} \cap (-\infty, c]$, therefore by reverse of (1.5) for $F$ on $\mathscr{I} \cap (-\infty, c]$, we get

$$\begin{aligned} 0 &\geq H(F(h_1)) - L(F(g_1)) \\ &= H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \end{aligned}$$

Also, since $F(x) = f(x) - \frac{A}{2}x^2$ is convex on $\mathscr{I} \cap [c, \infty)$, therefore by (1.5) for $F$ on $\mathscr{I} \cap (-\infty, c]$, we get

$$\begin{aligned} 0 &\leq H(F(h_2)) - L(F(g_2)) \\ &= H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)) \end{aligned}$$

From above, we have

$$H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \leq 0 \leq H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)).$$

So

$$H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \leq H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)),$$

therefore, by the use of (2.12), we get (2.13). ∎

**Remark 4.** *From the proof of the Theorem 7, we have*

$$H(f(h_1)) - L(f(g_1)) \leq \frac{A}{2}(H(h_1^2) - L(g_1^2)) \tag{2.14}$$

*and*

$$H(f(h_2)) - L(f(g_2)) \geq \frac{A}{2}(H(h_2^2) - L(g_2^2)) \tag{2.15}$$

*So, under assumption (2.12), we can get a better improvement of (2.13) as follow*

$$H(f(h_1)) - L(f(g_1)) \leq \frac{A}{2}(H(h_1^2) - L(g_1^2))\Big(= \frac{A}{2}(H(h_2^2) - L(g_2^2))\Big) \leq H(f(h_2)) - L(f(g_2)) \tag{2.16}$$

**Corollary 4.** *Let $\mathscr{I} \subset \mathbb{R}$ be a closed interval, let $[a,b] \subset \mathscr{I}$, let function $g_i \in \mathbb{X}_{[a,b]}$ for $i = 1,2$. Let $f \in K_1^c(\mathscr{I})$ be continuous function such that $f(g_i) \in \mathbb{X}$. If a unital positive linear functionals $L : \mathbb{X} \to \mathbb{R}$ satisfies implication (2.12) $\Rightarrow$ (2.13) for $L = H$ such that*

$$L(g_1^2) - (L(g_1))^2 = L(g_2^2) - (L(g_2))^2 \tag{2.17}$$

*then following inequality holds*

$$L(f(g_1)) - f(L(g_1)) \leq L(f(g_2)) - f(L(g_2)) \tag{2.18}$$

**Corollary 5.** *Let $[a_1,b_1] \subseteq \dots \subseteq [a_{n-1},b_{n-1}] \subseteq \mathscr{I}$. Let function $g_1,h_1 \in \mathbb{X}_{[a_1,b_1]}$, let $g_k,h_k \in \mathbb{X}_{[a_k,b_k]\setminus(a_{k-1},b_{k-1})}$ for $k=2,\dots,n-1$, and let function $g_n,h_n \in \mathbb{X}_{\mathscr{I}\setminus(a_{n-1},b_{n-1})}$. Let $f \in K_1^c(\mathscr{I})$ be continuous function such that $f(g_i) \in \mathbb{X}$.*
*If an n-tuple of unital positive linear functionals $L_i : \mathbb{X} \to \mathbb{R}$ satisfies*

$$L_i(g_i) = L_{i+1}(g_{i+1}) \text{ and } L_i(h_i) = L_{i+1}(h_{i+1}) \ for\ i = 1,\dots,n-1, \tag{2.19}$$

*such that*

$$L_{i+1}(g_{i+1}^2) - L_i(g_i^2) = L_{i+1}(h_{i+1}^2) - L_i(h_i^2), \tag{2.20}$$

*then*

$$L_{i+1}f(g_{i+1}) - L_i f(g_i) \le L_{i+1}f(h_{i+1}) - L_i f(h_i) \ for\ i = 1,\dots,n-1. \tag{2.21}$$

**Corollary 6.** *Let $\mathscr{I} \subset \mathbb{R}$ be a closed interval, and let functions $g_i,h_i \in \mathbb{X}_{\mathscr{I}}$ for $i=1,\dots,n$. Let $f \in K_1^c(\mathscr{I})$ be continuous function such that $f(g_i), f(h_i) \in \mathbb{X}$.*
*Then every n-tuple of positive linear functionals $L_i : \mathbb{X} \to \mathbb{R}$ with $\sum_{i=1}^n L_i(1) = 1$ such that*

$$\sum_{1=1}^n L_i((g_i)^2) - \big(\sum_{1=1}^n L_i(g_i)\big)^2 = \sum_{1=1}^n L_i((h_i)^2) - \big(\sum_{1=1}^n L_i(h_i)\big)^2 \tag{2.22}$$

*satisfies the inclusion*

$$\sum_{1=1}^n L_i(g_i), \sum_{1=1}^n L_i(g_i) \in \mathscr{I} \tag{2.23}$$

*and the inequality*

$$\sum_{1=1}^n L_i(f(g_i)) - f\big(\sum_{1=1}^n L_i(g_i)\big) \le \sum_{1=1}^n L_i(f(h_i)) - f\big(\sum_{1=1}^n L_i(h_i)\big) \tag{2.24}$$

**Theorem 8.** *Let $\mathscr{I} \subset \mathbb{R}$ be a closed interval, let $[a,b] \subset \mathscr{I}$, let function $g_i, g_i^* \in \mathbb{X}_{[a,b]}$ for $i=1,\dots,n$ and $h_i, h_i^* \in \mathbb{X}_{\mathscr{I}\setminus(a,b)}$ for $j=1,\dots,m$. Let $f \in K_1^c(\mathscr{I})$ be continuous function such that $f(g_i), f(g_i^*), f(h_i), f(h_i^*) \in \mathbb{X}$. If two pair of n-tuple of positive linear functionals $L_i, L_i^*, H_j, H_j^* : \mathbb{X} \to \mathbb{R}$ with $\sum_{i=1}^n L_i(1) = \sum_{1=1}^n L_i^*(1) = \sum_{j=1}^m H_j(1) = \sum_{j=1}^m H_j^*(1) = 1$ satify*

$$\sum_{j=1}^m H_j(h_j) = \sum_{i=1}^n L_i(g_i) \text{ and } \sum_{j=1}^m H_j^*(h_j^*) = \sum_{i=1}^n L_i^*(g_i^*) \tag{2.25}$$

*and*

$$\sum_{j=1}^m H_j((h_j)^2) - \sum_{i=1}^n L_i((g_i)^2) = \sum_{j=1}^m H_j^*((h_j^*)^2) - \sum_{i=1}^n L_i^*((g_i^*)^2). \tag{2.26}$$

*Then*

$$\sum_{j=1}^m H_j f(h_j) - \sum_{i=1}^n L_i f(g_i) \le \sum_{j=1}^m H_j^* f(h_j^*) - \sum_{i=1}^n L_i^* f(g_i^* s) \tag{2.27}$$

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