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On the G. Bennett's inequality

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ABSTRACT

The aim of this paper is to establish similar results due to G. Bennett[2] and C. P. Niculescu[4] for newly defined class.

1. Introduction

A vast theory evolved in the study of convex functions, may be readily appealed to most significant topics in real analysis and economics. In past recent years, a speedy development has encontered in the theory of convex functions. This can be endorsed to several causes, two of which are as follows: First, so many sections in modern analysis directly or indirectly covers application of convex functions. Second, convex functions have huge impact on the theory of inequalities and several important inequalities are consequences of the application of convex functions (see [5]).
In [2], G. Bennett presented some out-turns of an inequality which illustrate about the behavior of convex functions with respect to a mass distribution. Later on, C. P. Niculescu established an abstract version of this inequality([4]), which is shown in the following theorem

Theorem 1. [4] *Let $\mathscr{I}$ is an interval carrying a +ve Borel measure ℓ and $\mathscr{A}$; $\mathscr{B}$; $\mathscr{C}$ are three disjoint compact subintervals of $\mathscr{I}$ of +ve measure. Then*

$$\ell(\mathscr{B}) = \ell(\mathscr{A}) + \ell(\mathscr{C}) \tag{1.1}$$

and

$$\int_{\mathscr{B}} \alpha d\ell(\alpha) = \int_{\mathscr{A}} \alpha d\ell(\alpha) + \int_{\mathscr{C}} \alpha d\ell(\alpha); \tag{1.2}$$

give a necessary and sufficient condition under which the inequality

$$\int_{\mathscr{B}} f(\alpha) d\ell(\alpha) \leq \int_{\mathscr{A}} f(\alpha) d\ell(\alpha) + \int_{\mathscr{C}} f(\alpha) d\ell(\alpha). \tag{1.3}$$

is valid for every convex function $f : \mathscr{I} \to \mathbb{R}$.

Definition 1. ([4] and also see [4], p. 179, for details). Any real Borel measure ℓ on an interval $\mathscr{I}$ such that $\ell(\mathscr{I}) > 0$ and

$$\int_{\mathscr{I}} f(\alpha) d\ell(\alpha) \geq 0 \ \ for\ every\ nonnegative\ conve\alpha\ function\ f : \mathscr{I} \to \mathbb{R}.$$

is called a Steffensen-Popoviciu measure.

A brief description of this concept is given in the following result, independently due to T. Popoviciu [6] and A. M. Fink [3]:

Lemma 1. *Suppose that ℓ be a real Borel measure on an interval $\mathscr{I}$ with $\ell(\mathscr{I}) > 0$. Then ℓ is a Steffensen-Popoviciu measure iff the following condition of endpoints positivity,*

$$\int_{\mathscr{I} \cap (-\infty, t]} (t - \alpha) d\ell(\alpha) \geq 0 \ \ and \ \ \int_{\mathscr{I} \cap [t, \infty)} (\alpha - t) d\ell(\alpha) \geq 0$$

holds for every $t \in [a, b]$.

Theorem 2. *([4] and also see [4], p. 184-185, for details) Let ℓ is a Steffensen- Popoviciu measure on an interval $\mathscr{I}$. Then the inequality*

$$f(b_\ell) \leq \frac{1}{\ell(\mathscr{I})}\int_{\mathscr{I}} f(\alpha)d\ell(\alpha)$$

holds for every continuous convex function f on $\mathscr{I}$, here $b_\ell = \frac{1}{\ell(\mathscr{I})}\int_{\mathscr{I}} \alpha d\ell(\alpha)$ represents the barycenter of ℓ.

Definition 2. [4] Any real Borel measure ℓ on an interval $\mathscr{I}$ such that $\ell(\mathscr{I}) > 0$ and

$$\int_{\mathscr{I}} f(\alpha)d\ell(\alpha) \geq 0 \ \ for\ every\ nonnegative\ concave\ function\ f : \mathscr{I} \to \mathbb{R}.$$

is called a dual Steffensen-Popoviciu measure.

Theorem 3. *Theorem 1 even works if ℓ is a real Borel measure on $\mathscr{I}$ and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are three disjoint subintervals of $\mathscr{I}$ such that the restriction of ℓ to each of the intervals $\mathscr{A}$ and $\mathscr{C}$ is a Steffensen-Popoviciu measure and the restriction of ℓ to $\mathscr{B}$ is a dual Steffensen-Popoviciu measure.*

Now, we consider the inequality of G. Bennett for new class of functions in following section.

2. MAIN RESULTS

In [1], I. A. Baloch, J. Pečarič, M. Praljak defined a new class of functions which is defined as follow

Definition 3. Let $c \in \mathscr{I}^\circ$, where $\mathscr{I}$ is an open, closed or semi-open interval in either direction in $\mathbb{R}$ and $\mathscr{I}^\circ$ is its interior. We say that $f : \mathscr{I} \to \mathbb{R}$ is 3-convex function in point c (respectively 3-concave function at point c) if there exists a constant K such that the function $F(\alpha) = f(\alpha) - \frac{K}{2}\alpha^2$ is concave (resp. convex) on $\mathscr{I} \cap (-\infty, c]$ and convex (resp. concave) on $\mathscr{I} \cap [c, \infty)$. A f is 3-concave function in point c if $-f$ is 3-convex function in point c.

A property that explains the name of the class is the fact that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval (see [1]).

Theorem 4. *Let $\mathscr{I}$ is an interval bearing a +ve Borel measure ℓ and $\mathscr{A}$; $\mathscr{B}$; $\mathscr{C}$; $\mathscr{P}$; $\mathscr{Q}$; $\mathscr{R}$ are six disjoint compact subintervals of $\mathscr{I}$ of +ve measure such that*

$$\ell(\mathscr{B}) = \ell(\mathscr{A}) + \ell(\mathscr{C}) \ ; \ \ell(\mathscr{Q}) = \ell(\mathscr{P}) + \ell(\mathscr{R}) \tag{2.1}$$

and

$$\int_{\mathscr{B}} \alpha d\ell(\alpha) = \int_{\mathscr{A}} \alpha d\ell(\alpha) + \int_{\mathscr{C}} \alpha d\ell(\alpha) \ ; \ \int_{\mathscr{Q}} \beta d\ell(\beta) = \int_{\mathscr{P}} \beta d\ell(\beta) + \int_{\mathscr{R}} \beta d\ell(\beta). \tag{2.2}$$

and also $c \in I^\circ$ is such that

$$\max\{right\ end\ points\ of\ interval\ \mathscr{A},\mathscr{B},\mathscr{C}\} \leq c \leq \min\{left\ end\ points\ of\ interval\ \mathscr{P},\mathscr{Q},\mathscr{R}\} \tag{2.3}$$

Now, if

$$\int_{\mathscr{A}} \alpha^2 d\ell(t) + \int_{\mathscr{C}} \alpha^2 d\ell(\alpha) - \int_{\mathscr{B}} \alpha^2 d\ell(\alpha) = \int_{\mathscr{P}} \beta^2 d\ell(\beta) + \int_{\mathscr{R}} \beta^2 d\ell(\beta) - \int_{\mathscr{Q}} \beta^2 d\ell(\beta), \tag{2.4}$$

then following inequality

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha) + \int_{\mathscr{C}} f(\alpha)d\ell(\alpha) - \int_{\mathscr{B}} f(\alpha)d\ell(\alpha) \leq \int_{\mathscr{P}} f(\beta)d\ell(\beta) + \int_{\mathscr{R}} f(\beta)d\ell(\beta) - \int_{\mathscr{Q}} f(\beta)d\ell(\beta) \tag{2.5}$$

holds for every 3-convex function $f : \mathscr{I} \to \mathbb{R}$ at a point c.

Proof. Since f is 3-convex function at point $c \in \mathscr{I}^\circ$, then we have a constant K such that $F(\alpha) = f(\alpha) - \frac{K}{2}\alpha^2$ is concave on $\mathscr{I} \cap (-\infty, c]$. Therefore, for $\mathscr{A}$; $\mathscr{B}$; $\mathscr{C}$, three disjoint compact subintervals of $\mathscr{I} \cap [c, \infty)$ of +ve measure, so by reverse of the inequality (1.3), we have

$$\begin{aligned} 0 &\geq \int_{\mathscr{A}} F(\alpha)d\ell(\alpha) + \int_{\mathscr{C}} F(\alpha)d\ell(\alpha) - \int_{\mathscr{B}} F(\alpha)d\ell(\alpha) \\ &= \int_{\mathscr{A}} f(\alpha)d\ell(\alpha) + \int_{\mathscr{C}} f(\alpha)d\ell(\alpha) - \int_{\mathscr{B}} f(\alpha)d\ell(\alpha) - \frac{K}{2}\Big(\int_{\mathscr{A}} \alpha^2 d\ell(\alpha) + \int_{\mathscr{C}} \alpha^2 d\ell(\alpha) - \int_{\mathscr{B}} \alpha^2 d\ell(\alpha)\Big) \end{aligned}$$

Also, by using the fact that $F(\beta)=f(\beta)-\frac{K}{2}\beta^2$ is convex on $\mathscr{I}\cap[c,\infty)$. Therefore, for $\mathscr{P}$; $\mathscr{Q}$; $\mathscr{R}$, three disjoint compact subintervals of $\mathscr{I}\cap[c,\infty)$ of +ve measure, so by use of the inequality (1.3), we have

$$\begin{aligned} 0 &\leq \int_{\mathscr{P}} F(\beta)d\ell(\beta)+\int_{\mathscr{R}} F(\beta)d\ell(\beta)-\int_{\mathscr{Q}} F(\beta)d\ell(\beta) \\ &= \int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta)-\frac{K}{2}\Big(\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)\Big) \end{aligned}$$

From above, we have

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha)-\frac{K}{2}\Big(\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\Big)$$

$$\leq 0 \leq$$

$$\int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta)-\frac{K}{2}\Big(\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)\Big)$$

So,

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha)-\frac{K}{2}\Big(\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\Big)$$

$$\leq \int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta)-\frac{K}{2}\Big(\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)\Big) \tag{2.6}$$

By using (2.4), we get (2.5). ∎

Remark 1. *From the proof of the Theorem* 4*, we have*

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha)\leq\frac{K}{2}\Big(\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\Big) \tag{2.7}$$

and

$$\int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta)\geq\frac{K}{2}\Big(\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)\Big) \tag{2.8}$$

So under assumption (2.4)*, we can get a improvement of* (2.5) *as follow*

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha)\leq\frac{K}{2}\Big(\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\Big)$$

$$\Big\{=\frac{K}{2}\Big(\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)\Big)\Big\}\leq\int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta) \tag{2.9}$$

Now, we give next result which weakens the assumption (2.4) such that inequality (2.5) holds under this new condition.

Theorem 5. *Suppose that $\mathscr{I}$ is an interval bearing a +ve Borel measure ℓ and $\mathscr{A}$; $\mathscr{B}$; $\mathscr{C}$; $\mathscr{P}$; $\mathscr{Q}$; $\mathscr{R}$ are six disjoint compact subintervals of $\mathscr{I}$ of +ve measure such that* (2.1) *and* (2.2) *hold with*

$$a=\max\{right\ end\ points\ of\ interval\ \mathscr{A},\mathscr{B},\mathscr{C}\}\leq\min\{left\ end\ points\ of\ interval\ \mathscr{P},\mathscr{Q},\mathscr{R}\}=b \tag{2.10}$$

and $f\to\mathbb{R}$ is 3*-convex at a point c for some $c\in[a,b]$. Then if*
(a)

$$f''_{-}(a)\ \geq\ 0$$

and

$$\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\leq\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)$$

or
(b)

$$f''_{+}(b)\ \leq\ 0$$

and

$$\int_{\mathscr{A}}\alpha^2d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2d\ell(\alpha)\geq\int_{\mathscr{P}}\beta^2d\ell(\beta)+\int_{\mathscr{R}}\beta^2d\ell(\beta)-\int_{\mathscr{Q}}\beta^2d\ell(\beta)$$

or
(c)

$$f''_-(a) \;<\; 0 \;<\; f''_+(b) \;\; and \;\; f \; is \; 3-convex,$$

then (2.5) *holds.*

Proof. The idea of proof is similar to proof of Theorem 4. Hence, by proceeding as in the proof of Theorem 4. From the inequality 2.6, we have

$$\frac{K}{2}\big[\big(\int_{\mathscr{P}}\beta^2 d\ell(\beta)+\int_{\mathscr{R}}\beta^2 d\ell(\beta)-\int_{\mathscr{Q}}\beta^2 d\ell(\beta)\big)-\big(\int_{\mathscr{A}}\alpha^2 d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2 d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2 d\ell(\alpha)\big)\big]$$

$$\leq \int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta)-\big(\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha)\big)$$

Now, due to the concavity of F on $\mathscr{I}\cap(-\infty,c]$ and convexity of F on $\mathscr{I}\cap[c,\infty)$, so for every distinct points $\alpha_j \in \mathscr{I}\cap(-\infty,a]$ and $y_j \in \mathscr{I}\cap[b,\infty)$, $j=1,2,3$, we have

$$[\alpha_1,\alpha_2,\alpha_3]f \leq K \leq [\beta_1,\beta_2,\beta_3]f$$

Letting $\alpha_j \nearrow a$ and $\beta_j \searrow b$, we get (if exists)

$$f''_-(a) \leq K \leq f''_+(b)$$

Therefore, if assumptions (a) or (b) holds, then

$$\frac{K}{2}\big[\big(\int_{\mathscr{P}}\beta^2 d\ell(\beta)+\int_{\mathscr{R}}\beta^2 d\ell(\beta)-\int_{\mathscr{Q}}\beta^2 d\ell(\beta)\big)-\big(\int_{\mathscr{A}}\alpha^2 d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2 d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2 d\ell(\alpha)\big)\big]$$

is positive and we conclude the result. If the assumption (c) holds, the f''_- is left continuous, f''_+ is right continuous, they are both non-decreasing and $f''_- \leq f''_+$. Therefore, there exists $\tilde{c}\in[a,b]$ such that f with associated constant $\tilde{K}=0$ and again, we can deduce the result. ∎

Remark 2. *Again from the proof of Theorem* 5*, we obtain the inequalities* (2.7) *and* (2.8)*. Now, under assumption (a), (b) or (c) of Theorem* 5*, K is positive or negative or zero respectively due to argument discussed in the proof. Therefore, we get a better improvement of* (2.5) *then* (2.9) *in this case as follow*

$$\int_{\mathscr{A}} f(\alpha)d\ell(\alpha)+\int_{\mathscr{C}} f(\alpha)d\ell(\alpha)-\int_{\mathscr{B}} f(\alpha)d\ell(\alpha) \leq \frac{K}{2}\big(\int_{\mathscr{A}}\alpha^2 d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2 d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2 d\ell(\alpha)\big)$$

$$\leq \frac{K}{2}\big(\int_{\mathscr{P}}\beta^2 d\ell(\beta)+\int_{\mathscr{R}}\beta^2 d\ell(\beta)-\int_{\mathscr{Q}}\beta^2 d\ell(\beta)\big) \leq \int_{\mathscr{P}} f(\beta)d\ell(\beta)+\int_{\mathscr{R}} f(\beta)d\ell(\beta)-\int_{\mathscr{Q}} f(\beta)d\ell(\beta) \tag{2.11}$$

Under the assumption of Theorem 4 with $f:\mathscr{I}\to\mathbb{R}$ is 3-concave at point $c\in\mathscr{I}^\circ$, the reverse of inequality (2.5) holds. Now, we give only the statement of the theorem with weaker condition which can be proved in similar way under which the reverse of inequality (2.5) holds for $f:\mathscr{I}\to\mathbb{R}$ is 3-concave at point $c\in\mathscr{I}^\circ$.

Theorem 6. *Suppose that $\mathscr{I}$ is an interval bearing a +ve Borel measure ℓ and $\mathscr{A}$; $\mathscr{B}$; $\mathscr{C}$; $\mathscr{P}$; $\mathscr{Q}$; $\mathscr{R}$ are six disjoint compact subintervals of $\mathscr{I}$ of +ve measure such that* (2.1) *and* (2.2) *hold with*

$$a=\max\{right\ end\ points\ of\ interval\ \mathscr{A},\mathscr{B},\mathscr{C}\} \leq \min\{left\ end\ points\ of\ interval\ \mathscr{P},\mathscr{Q},\mathscr{R}\}=b \tag{2.12}$$

and $f:\mathscr{I}\to\mathbb{R}$ is 3*-concave at a point c for some $c\in[a,b]$. Then if*
(a)

$$f''_-(a) \;\leq\; 0$$

and

$$\int_{\mathscr{A}}\alpha^2 d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2 d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2 d\ell(\alpha) \geq \int_{\mathscr{P}}\beta^2 d\ell(\beta)+\int_{\mathscr{R}}\beta^2 d\ell(\beta)-\int_{\mathscr{Q}}\beta^2 d\ell(\beta)$$

or
(b)

$$f''_+(b) \;\geq\; 0$$

and

$$\int_{\mathscr{A}}\alpha^2 d\ell(\alpha)+\int_{\mathscr{C}}\alpha^2 d\ell(\alpha)-\int_{\mathscr{B}}\alpha^2 d\ell(\alpha) \leq \int_{\mathscr{P}}\beta^2 d\ell(\beta)+\int_{\mathscr{R}}\beta^2 d\ell(\beta)-\int_{\mathscr{Q}}\beta^2 d\ell(\beta)$$

or
(c)

$$f''_-(a) \ < \ 0 \ < \ f''_+(b) \ \text{and} \ f \text{ is } 3-\text{concave},$$

then reverse of (2.5) *holds.*

Remark 3. *Similarly as in Remark* 2*, we obtain the reverse of inequalities* (2.7) *and* (2.8) *from the proof of Theorem* 6*. Now, due the convexity of F on* $\mathscr{I} \cap (-\infty, c]$ *and concavity of F on* $\mathscr{I} \cap [c, \infty)$*, so for every distinct points* $\tilde{\alpha}_j \in \mathscr{I} \cap (-\infty, a]$ *and* $\tilde{\beta}_j \in \mathscr{I} \cap [b, \infty)$*,* $j = 1, 2, 3.$*, we have*

$$[\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3]f \geq K \geq [\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3]f$$

Letting $\tilde{\alpha}_j \nearrow a$ *and* $\tilde{\beta}_j \searrow b$*, we get (if exists)*

$$f''_-(a) \geq K \geq f''_+(b)$$

Now, under assumption (a) or (b) or (c) of Theorem 5*, K is negative or positive or zero respectively due to argument discussed above. Therefore, we get a better improvement in this case as follow*

$$\int_{\mathscr{A}} f(\alpha) d\ell(\alpha) + \int_{\mathscr{C}} f(\alpha) d\ell(\alpha) - \int_{\mathscr{B}} f(\alpha) d\ell(\alpha) \geq \frac{K}{2} \big(\int_{\mathscr{A}} \alpha^2 d\ell(\alpha) + \int_{\mathscr{C}} \alpha^2 d\ell(\alpha) - \int_{\mathscr{B}} \alpha^2 d\ell(\alpha) \big)$$

$$\geq \frac{K}{2} \big(\int_{\mathscr{P}} \beta^2 d\ell(\beta) + \int_{\mathscr{R}} \beta^2 d\ell(\beta) - \int_{\mathscr{Q}} \beta^2 d\ell(\beta) \big) \geq \int_{\mathscr{P}} f(\beta) d\ell(\beta) + \int_{\mathscr{R}} f(\beta) d\ell(\beta) - \int_{\mathscr{Q}} f(\beta) d\ell(\beta) \tag{2.13}$$

Theorem 7. *Suppose that* $\mathscr{I}$ *is an interval bearing a Borel measure* ℓ *and* $\mathscr{A}$ *;* $\mathscr{B}$*;* $\mathscr{C}$*;* $\mathscr{P}$ *;* $\mathscr{Q}$*;* $\mathscr{R}$ *are six disjoint subintervals of* $\mathscr{I}$ *with the restriction of* ℓ *to each of the intervals* $\mathscr{A}, \mathscr{C}, \mathscr{P}$ *and* $\mathscr{R}$ *is a Steffensen-Popoviciu measure and the restriction of* ℓ *to* $\mathscr{B}$ *and* $\mathscr{Q}$ *is a dual Steffensen-Popoviciu measure such that*

$$\ell(\mathscr{B}) = \ell(\mathscr{A}) + \ell(\mathscr{C}) \ ; \ \ell(\mathscr{Q}) = \ell(\mathscr{P}) + \ell(\mathscr{R}) \tag{2.14}$$

and

$$\int_{\mathscr{B}} \alpha d\ell(\alpha) = \int_{\mathscr{A}} \alpha d\ell(\alpha) + \int_{\mathscr{C}} \alpha d\ell(\alpha) \ ; \ \int_{\mathscr{Q}} \beta d\ell(\beta) = \int_{\mathscr{P}} \beta d\ell(\beta) + \int_{\mathscr{R}} \beta d\ell(\beta). \tag{2.15}$$

and also $c \in \mathscr{I}^{\circ}$ *is such that*

$$\max\{right\ end\ points\ of\ interval\ \mathscr{A}, \mathscr{B}, \mathscr{I}\mathscr{C}\} \leq c \leq \min\{left\ end\ points\ of\ interval\ \mathscr{P}, \mathscr{Q}, \mathscr{R}\} \tag{2.16}$$

Now, if

$$\int_{\mathscr{A}} \alpha^2 d\ell(\alpha) + \int_{\mathscr{C}} \alpha^2 d\ell(\alpha) - \int_{\mathscr{B}} \alpha^2 d\ell(\alpha) = \int_{\mathscr{P}} \beta^2 d\ell(\beta) + \int_{\mathscr{R}} \beta^2 d\ell(\beta) - \int_{\mathscr{Q}} \beta^2 d\ell(\beta), \tag{2.17}$$

then following inequality

$$\int_{\mathscr{A}} f(\alpha) d\ell(\alpha) + \int_{\mathscr{C}} f(\alpha) d\ell(\alpha) - \int_{\mathscr{B}} f(\alpha) d\ell(\alpha) \leq \int_{\mathscr{P}} f(\beta) d\ell(\beta) + \int_{\mathscr{R}} f(\beta) d\ell(\beta) - \int_{\mathscr{Q}} f(\beta) d\ell(\beta) \tag{2.18}$$

holds for every $f : \mathscr{I} \to \mathbb{R}$ 3*-convex function at a point c.*

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