Received 19/12/15

ON SOME INTEGRAL INEQUALITIES FOR $(k,H)-$RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

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Abstract. In this study, giving the definition of fractional integral, which are with the help of synchronous and monotonic function, some fractional integral inequalities have established.

1. Introduction

Integral inequalities play a fundamental role in the theory of differential equations, functional analysis and applied sciences. Important development in this theory has been achieved for the last two decades. For these, see [6]-[11] and the references therein. Moreover, the study of fractional type inequalities is also of vital importance. Also see [1]-[5] for further information and applications.

The researchers have studied Fractional Calculus since seventeenth century. From this date, mathematicians as well as biologists, chemists, economists, engineers and physicists have found this new theory very attractive. Many different derivatives were introduced.

2. Fractional Integrals

Now we will give fundamental definitions and notations for fractional integrals.

Definition 1. *Let $a,b\in\mathbb{R}$, $a<b$, and $\alpha>0$. For $f\in L_1(a,b)$*

$$\left(J_{a^+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ \alpha>0,\ x>a \tag{2.1}$$

and

$$\left(J_{b^-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ b>0,\ b>x. \tag{2.2}$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively [12]-[17].

This integrals is motivated by the well known Cauchy formula:

$$\int_a^x d\tau_1\int_a^{\tau_1}d\tau_2...\int_a^{\tau_{n-1}}f(\tau_n)d\tau_n=\frac{1}{\Gamma(n)}\int_a^x(x-\tau)^{n-1}f(\tau)d\tau. \tag{2.3}$$

Key words and phrases. Riemann-Liouville Fractional Integral, Integral Inequalities.
2010 Mathematics Subject Classification. 26A33, 26D15, 41A55.

Definition 2. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha)>0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by* [17]

(2.4)
$$\left(J^{\alpha}_{a^+,h}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[h(x)-h(t)\right]^{\alpha-1}h^{'}(t)\,f(t)\,dt,\ \ x\geq a,\ \ \Re(\alpha)>0$$

and

(2.5)
$$\left(J^{\alpha}_{b^-,h}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[h(t)-h(x)\right]^{\alpha-1}h^{'}(t)\,f(t)\,dt,\ \ x\leq b,\ \ \Re(\alpha)>0.$$

For (2.4) and (2.5)

$$\left(J^{\alpha}_{a^+,h}f\right)(a)=\left(J^{\alpha}_{b^-,h}f\right)(b)=0.$$

If we take $h(x)=x$ in (2.4) and (2.5) integral formulas, we will obtain

$$J^{\alpha}_{a^+,h}=J^{\alpha}_{a^+}\ \text{ and }\ J^{\alpha}_{b^-,h}=J^{\alpha}_{b^-}.$$

Also if we choose $h(x)=\dfrac{x^{\rho+1}}{\rho+1}$ for $\rho\geq 0$, then the equalities (2.4) and (2.5) will be

$$(J^{\alpha}_{a^+,\rho}f)(x)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x(x^{\rho+1}-t^{\rho+1})^{\alpha-1}t^{\rho}f(t)dt,\ \ x>a \tag{2.6}$$

and

$$(J^{\alpha}_{b^-,\rho}f)(x)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b(t^{\rho+1}-x^{\rho+1})^{\alpha-1}t^{\rho}f(t)dt,\ \ x<b \tag{2.7}$$

respectively. This kind of generalized fractional integrals are studied in [12], [13], [16], [18].

In [16], Katugampola gave a new fractional integration which generalized Riemann-Liouville fractional integrals. This (2.6) and (2.7) generalizations is based on the following equality,

(2.8)
$$\int_a^x\tau_1^{\rho}d\tau_1\int_a^{\tau_1}\tau_2^{\rho}d\tau_2...\int_a^{\tau_{n-1}}\tau_n^{\rho}f(\tau_n)d\tau_n=\frac{(\rho+1)^{1-n}}{(n-1)!}\int_a^x(x^{\rho+1}-\tau^{\rho+1})^{n-1}\tau^{\rho}f(\tau)d\tau.$$

For $a=0$ in (2.4), we can write

$$\left(J^{\alpha}_{0^+,h}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(h(x)-h(t))^{\alpha-1}h^{'}(t)\,f(t)dt,\ \ x>0 \tag{2.9}$$

$$\left(J^{0}_{0^+,h}f\right)(x)=f(x).$$

Semi group and commutative properties of (2.9) integral operator is the following

$$J^{\alpha}_{a^+,h}J^{\beta}_{a^+,h}f(x)=J^{\alpha+\beta}_{a^+,h}\ f(x),\ \ \alpha\geq 0,\ \ \beta\geq 0$$

and

$$J^{\alpha}_{a^+,h}J^{\beta}_{a^+,h}f(x)=J^{\beta}_{a^+,h}J^{\alpha}_{a^+,h}f(x).$$

To show the being unit operator property of (2.9) integral operator, we choose h function specially as $f(x) = h(x)$ we obtain the following equality

$$
\begin{aligned}
\left(J^{\alpha}_{0^+,h}h\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (h(x) - h(t))^{\alpha-1} h(t) h'(t)\, dt \\
&= \frac{(h(x) - h(0))^{\alpha}}{\Gamma(\alpha+2)} \left[h(x) + \alpha h(0)\right].
\end{aligned}
\tag{2.10}
$$

Let $\alpha = 0$ in (2.10), then we have

$$\left(J^{0}_{0^+,h}h\right)(x) = h(x).$$

In (2.9), let $f(x) = x^{\mu}$, $f(x) = 1$ and $h(x) = \frac{x^{\rho+1}}{\rho+1}$ for $\alpha > 0$; $\rho \geq 0$, $\mu > -1$, $t > 0$, then we have

$$J^{\alpha}_{0^+,h}(x^{\mu}) = \frac{(\rho+1)^{-\alpha}\Gamma(\frac{\rho+\mu+1}{\rho+1})}{\Gamma(\alpha + \frac{\rho+\mu+1}{\rho+1})} t^{\alpha(\rho+1)+\mu}$$

and

$$J^{\alpha}_{0^+,h}(1) = \frac{(\rho+1)^{-\alpha}}{\Gamma(\alpha+1)} t^{\alpha(\rho+1)}.$$

Definition 3. *Let $\alpha > 0$ and $x > 0$, defined by* [14], [19]

$$\left({}_kJ^{\alpha} f\right)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt. \tag{2.11}$$

Where $k-$gamma function is defined by

$$\Gamma_k(x) = \int_0^{\infty} t^{\frac{x}{k}-1} e^{-\frac{t^k}{k}}\, dt,\ x > 0.$$

and

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\, dt.$$

Also

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \text{ and } B_k(x,y) = \frac{1}{k}B_k(\frac{x}{k},\frac{y}{k}).$$

Definition 4. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha) > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by*

$$\left({}_kJ^{\alpha}_{a^+,h}f\right)(x) := \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left[h(x) - h(t)\right]^{\frac{\alpha}{k}-1} h'(t) f(t)\, dt,\ k > 0,\ \Re(\alpha) > 0 \tag{2.12}$$

and

$$\left({}_kJ^{\alpha}_{b^-,h}f\right)(x) := \frac{1}{k\Gamma_k(\alpha)}\int_x^b \left[h(t) - h(x)\right]^{\frac{\alpha}{k}-1} h'(t) f(t)\, dt,\ k > 0,\ \Re(\alpha) > 0. \tag{2.13}$$

If we take $h(x) = x$ in (2.12) and (2.13) integral formulas, we will obtain

$$({}_kJ^{\alpha}_{a^+}f)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt,\ x > a$$

$$({}_kJ^{\alpha}_{b^-}f)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt,\ b > x.$$

Note that when $k \to 1$, then it reduces to the classical Riemann-Liouville fractional integral.

Also if we choose $h(x) = \dfrac{x^{\rho+1}}{\rho+1}$ for $\rho \in \mathbb{R}/\{-1\}$, then the equalities (2.12) and (2.13) will be

$$({}^{\rho}_kJ^{\alpha}_{a^+}f)(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{\rho+1}-t^{\rho+1})^{\frac{\alpha}{k}-1}t^{\rho}f(t)dt,\ x > a \tag{2.14}$$

and

$$({}^{\rho}_kJ^{\alpha}_{b^-}f)(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b (t^{k+1}-x^{k+1})^{\alpha-1}t^{k}f(t)dt,\ x < b \tag{2.15}$$

respectively. This kind of generalized fractional integrals are studied in [20].

For $a = 0$ in (2.12), we can write

$$\left({}_kJ^{\alpha}_{0^+,h}f\right)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f(t)dt,\ x > 0 \tag{2.16}$$

$$\left({}_kJ^{0}_{0^+,h}f\right)(x) = f(x).$$

Semi group and commutative properties of (2.16) integral operator is the following

$$\left[\left({}_kJ^{\alpha}_{a^+,h}\right)\left({}_kJ^{\beta}_{a^+,h}\right)\right]f(x) = J^{\alpha+\beta}_{a^+,h}f(x),\ \alpha \geq 0,\ \beta \geq 0$$

and

$$\left[\left({}_kJ^{\alpha}_{a^+,h}\right)\left({}_kJ^{\beta}_{a^+,h}\right)\right]f(x) = \left({}_kJ^{\beta}_{a^+,h}\right)\left({}_kJ^{\alpha}_{a^+,h}\right)f(x).$$

To show the being unit operator property of (2.16) integral operator, we choose h function specially as $f(x) = h(x)$ we obtain the following equality

$$\begin{aligned}\left({}_kJ^{\alpha}_{0^+,h}h\right)(x) &= \frac{1}{k\Gamma_k(\alpha)}\int_0^x (h(x)-h(t))^{\frac{\alpha}{k}-1}h(t)h'(t)\,dt \\ &= \frac{(h(x)-h(0))^{\frac{\alpha}{k}}}{\Gamma(\alpha+k+1)}\left[h(x)+\alpha h(0)\right].\end{aligned} \tag{2.17}$$

For $\alpha = 0$ and $k = 1$ in (2.17), we have

$$\left(J^{0}_{0^+,h}h\right)(x) = h(x).$$

The main aim of this work is to establish a new fractional integral inequality for (k,h) −Riemann-Liouville fractional integral. Using the technique of [20] a key role in our study.

3. Main Results

Theorem 1. *Let f and g are two synchronous functions on $[0,\infty]$. Then for $t>0$, $\alpha>0$;*

$$ {}_kJ^{\alpha}_{a^+,h}(fg)(t) \geq \frac{\Gamma_k(\alpha+k)}{(h(x)-h(a))^{\frac{\alpha}{k}}}\left({}_kJ^{\alpha}_{a^+,h}\right)f(t)\left({}_kJ^{\alpha}_{a^+,h}\right)g(t). \tag{3.1}$$

Proof. For f and g synchronous functions, we have

$$ (f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0. \tag{3.2}$$

From (3.2) it can be written as following

$$ f(\tau)g(\tau)+f(\rho)g(\rho) \geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{3.3}$$

If we multiply two sides of the (3.3) with $\frac{(h(t)-h(\tau))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}(\tau)$, $\tau\in(a,t)$, we obtain

$$\begin{aligned} &\frac{(h(t)-h(\tau))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}(\tau)f(\tau)g(\tau) \\ &+\frac{(h(t)-h(\tau))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}(\tau)f(\rho)g(\rho) \\ &\geq \frac{(h(t)-h(\tau))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}(\tau)f(\tau)g(\rho) \\ &+\frac{(h(t)-h(\tau))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}(\tau)f(\rho)g(\tau). \end{aligned} \tag{3.4}$$

Integrating (3.4) inequality on (a,t), then

$$\begin{aligned} &\frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)f(\tau)g(\tau)d\tau \\ &+\frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)f(\rho)g(\rho)d\tau \\ &\geq \frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)f(\tau)g(\rho)d\tau \\ &+\frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)f(\rho)g(\tau)d\tau. \end{aligned} \tag{3.5}$$

Therefore

$$\begin{aligned} &{}_kJ^{\alpha}_{a^+,h}(fg)(t)+f(\rho)g(\rho)\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)d\tau \\ &\geq g(\rho)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)f(\tau)d\tau \\ &+f(\rho)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\tau))^{\frac{\alpha}{k}-1}h^{'}(\tau)g(\tau)d\tau \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & \left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)+f(\rho)g(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(1) \\ \geq\quad & g(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(f)(t)+f(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(g)(t). \end{aligned} \tag{3.7}$$

Now multiplying two sides of (3.7) with $\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)$, $\rho\in(a,t)$, we have

$$\begin{aligned} & \frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)\,J^{\alpha}_{a^+,h}(fg)(t) \\ & +\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)\,f(\rho)g(\rho)J^{\alpha}_{a^+,h}(1) \\ & \geq\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)\,g(\rho)J^{\alpha}_{a^+,h}f(t) \\ & +\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)\,f(\rho)J^{\alpha}_{a^+,h}(g)(t). \end{aligned} \tag{3.8}$$

By integrating to (3.9) on (a,t), then

$$\begin{aligned} & \left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)\int_a^t\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(\rho)\,d\rho \\ & +\frac{\left({}_kJ^{\alpha}_{a^+,h}(1)\right)}{k\Gamma_k(\alpha)}\int_a^t f(\rho)g(\rho)(h(t)-h(\rho))^{\frac{\alpha}{k}-1}h'(\rho)\,d\rho \\ & \geq\frac{\left({}_kJ^{\alpha}_{a^+,h}\right)f(t)}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\rho))^{\frac{\alpha}{k}-1}h'(\rho)\,g(\rho)d\rho \\ & +\frac{\left({}_kJ^{\alpha}_{a^+,h}\right)g(t)}{k\Gamma_k(\alpha)}\int_a^t(h(t)-h(\rho))^{\frac{\alpha}{k}-1}h'(\rho)\,f(\rho)d\rho. \end{aligned} \tag{3.9}$$

This inequality is can be written as the following at the same time

$$J^{\alpha}_{a^+,h}(fg)(t)\geq\frac{\Gamma_k(\alpha+k)}{(h(x)-h(a))^{\frac{\alpha}{k}}}J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}g(t). \tag{3.10}$$

So the proof is completed. □

Theorem 2. *Let f and g are two synchronous functions on $[a,b]$. Then for $t>a$, $\alpha>0$, $\beta>0$ and $k>0$,*

$$\begin{aligned} & \frac{(h(x)-h(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[\left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)+\left({}_kJ^{\beta}_{a^+,h}\right)(fg)(t)\right] \\ & \qquad\geq\left({}_kJ^{\alpha}_{a^+,h}\right)f(t)\left({}_kJ^{\beta}_{a^+,h}\right)g(t)+\left({}_kJ^{\alpha}_{a^+,h}\right)g(t)\left({}_kJ^{\beta}_{a^+,h}\right)f(t). \end{aligned}$$

Proof. Since the f and g are two synchronous functions on $[a,b]$ then for all $\tau,\rho\geq 0$. If we multiply two sides of (3.7) with $\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)$, then we obtain

$$
\begin{aligned}
&\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)\\
&+\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)f(\rho)g(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(1)\\
&\geq\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)g(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(f)(t)\\
&+\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)f(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(g)(t).
\end{aligned}
\tag{3.11}
$$

Integrating to (3.11) on (a,t), then

$$
\begin{aligned}
&\int_a^t\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)dt\\
&+\int_a^t\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)f(\rho)g(\rho)\left({}_kJ^{\alpha}_{a^+,h}(1)\right)dt\\
&\geq\int_a^t\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)g(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(f)(t)dt\\
&+\int_a^t\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h^{'}(\rho)f(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(g)(t)dt.
\end{aligned}
\tag{3.12}
$$

This is the proof of the theorem

$$
\begin{aligned}
&\left({}_kJ^{\beta}_{a^+,h}\right)(1)\left({}_kJ^{\alpha}_{a^+,h}\right)(fg)(t)+\left({}_kJ^{\alpha}_{a^+,h}\right)(1)\left({}_kJ^{\beta}_{a^+,h}\right)(fg)(t)\\
&\geq\left({}_kJ^{\alpha}_{a^+,h}\right)f(t)\left({}_kJ^{\beta}_{a^+,h}\right)g(t)+\left({}_kJ^{\alpha}_{a^+,h}\right)g(t)\left({}_kJ^{\beta}_{a^+,h}\right)f(t).
\end{aligned}
\tag{3.13}
$$

□

Remark 1. *It is obvious that if we take $\alpha=\beta$ in this theorem we will obtain Theorem 1.*

Theorem 3. *Let f,g and h be tree monotonic functions defined on $[0,\infty)$ satisfying the following inequality*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))\geq 0$$

for all $\rho,\tau\in[a,t]$, *then for all* $t>a\geq 0,\ \alpha>0,\ \beta>0$, *the following inequalities for* $(k,H)-$*fractional integrals hold:*

$$\begin{aligned}
&\left[{}_kJ^{\alpha}_{a^+,h}\left(fgh\right)\left(t\right)\right]\left({}_kJ^{\beta}_{a^+,h}\right)(1)-\left({}_kJ^{\alpha}_{a^+,h}\right)(1)\left[{}_kJ^{\beta}_{a^+,h}\left(fgh\right)\left(t\right)\right]\\
&\geq\left[{}_kJ^{\alpha}_{a^+,h}\left(fh\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(g\right)\left(t\right)\right]+\left[{}_kJ^{\alpha}_{a^+,h}\left(gh\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(f\right)\left(t\right)\right]\\
&-\left[{}_kJ^{\alpha}_{a^+,h}\left(h\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(fg\right)\left(t\right)\right]+\left[{}_kJ^{\alpha}_{a^+,h}\left(fg\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(h\right)\left(t\right)\right]\\
&+\left[{}_kJ^{\alpha}_{a^+,h}\left(f\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(gh\right)\left(t\right)\right]-\left[{}_kJ^{\alpha}_{a^+,h}\left(g\right)\left(t\right)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(fh\right)\left(t\right)\right].
\end{aligned}\tag{3.14}$$

Proof. Since the functions $f,\ g$ and h monotonic functions on $[0,\infty)$, then for all $\tau,\rho\geq 0$, we have

$$\left(f\left(\tau\right)-f(\rho)\right)\left(g\left(\tau\right)-g(\rho)\right)\left(h\left(\tau\right)-h\left(\rho\right)\right)\geq 0.\tag{3.15}$$

From (3.15) it can be written as following

$$\begin{aligned}
&f\left(\tau\right)g\left(\tau\right)h\left(\tau\right)-f\left(\rho\right)g\left(\rho\right)h\left(\rho\right)-f\left(\tau\right)g\left(\rho\right)h\left(\tau\right)-f\left(\rho\right)g\left(\tau\right)h\left(\tau\right)\\
&+f\left(\rho\right)g\left(\rho\right)h\left(\tau\right)-f\left(\tau\right)g\left(\tau\right)h\left(\rho\right)-f\left(\tau\right)g\left(\rho\right)h\left(\rho\right)+f\left(\rho\right)g\left(\tau\right)h\left(\rho\right)\geq 0.
\end{aligned}\tag{3.16}$$

If we multiply two sides of the (3.16) with $\dfrac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right),\ \tau\in(a,t)$, we obtain

$$\begin{aligned}
&\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)f\left(\tau\right)g\left(\tau\right)h\left(\tau\right)dt\\
&-f\left(\rho\right)g\left(\rho\right)h\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)dt\\
&\geq g\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)f\left(\tau\right)h\left(\tau\right)dt\\
&+f\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)g\left(\tau\right)h\left(\tau\right)dt\\
&-f\left(\rho\right)g\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)h\left(\tau\right)dt\\
&+h\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)f\left(\tau\right)g\left(\tau\right)dt\\
&+g\left(\rho\right)h\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)f\left(\tau\right)dt\\
&-f\left(\rho\right)h\left(\rho\right)\int\limits_a^t\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h^{'}\left(\tau\right)g\left(\tau\right)dt.
\end{aligned}\tag{3.17}$$

Therefore

$$
\begin{aligned}
& {}_kJ^{\alpha}_{a^+,h}\left(fgh\right)(t) - f(\rho)\, g(\rho)\, h(\rho)\left({}_kJ^{\alpha}_{a^+,h}\right)(1) \\
& \geq g(\rho)_k\, J^{\alpha}_{a^+,h}\left(fh\right)(t) + f(\rho)_k\, J^{\alpha}_{a^+,h}\left(gh\right)(t) \\
& -f(\rho)\, g(\rho)_k\, J^{\alpha}_{a^+,h}\left(h\right)(t) + h(\rho)_k\, J^{\alpha}_{a^+,h}\left(fg\right)(t) \\
& +g(\rho)\, h(\rho)_k\, J^{\alpha}_{a^+,h}\left(f\right)(t) - f(\rho)\, h(\rho)_k\, J^{\alpha}_{a^+,h}\left(g\right)(t).
\end{aligned} \tag{3.18}
$$

Now multiplying two sides of (3.18) with $\dfrac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} h'(\rho)$, $\rho \in (a,t)$, we have

$$
\begin{aligned}
& \left[{}_kJ^{\alpha}_{a^+,h}\left(fgh\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} h'(\rho)\, d\rho \\
& - \left({}_kJ^{\alpha}_{a^+,h}\right)(1) \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho)\, g(\rho)\, h(\rho)\, h'(\rho)\, d\rho \\
& \geq \left[{}_kJ^{\alpha}_{a^+,h}\left(fh\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} g(\rho)\, h'(\rho)\, d\rho \\
& + \left[{}_kJ^{\alpha}_{a^+,h}\left(gh\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho)\, h'(\rho)\, d\rho \\
& - \left[{}_kJ^{\alpha}_{a^+,h}\left(h\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho)\, g(\rho)\, h'(\rho)\, d\rho \\
& + \left[{}_kJ^{\alpha}_{a^+,h}\left(fg\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} h(\rho)\, h'(\rho)\, d\rho \\
& + \left[{}_kJ^{\alpha}_{a^+,h}\left(f\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} g(\rho)\, h(\rho)\, h'(\rho)\, d\rho \\
& - \left[{}_kJ^{\alpha}_{a^+,h}\left(g\right)(t)\right] \int_a^t \frac{\left(h(t)-h(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)} f(\rho)\, h(\rho)\, h'(\rho)\, d\rho.
\end{aligned} \tag{3.19}
$$

This is the proof of the theorem,

$$
\begin{aligned}
& \left[{}_kJ^{\alpha}_{a^+,h}\left(fgh\right)(t)\right]\left({}_kJ^{\beta}_{a^+,h}\right)(1) - \left({}_kJ^{\alpha}_{a^+,h}\right)(1)\left[{}_kJ^{\beta}_{a^+,h}\left(fgh\right)(t)\right] \\
& \geq \left[{}_kJ^{\alpha}_{a^+,h}\left(fh\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(g\right)(t)\right] + \left[{}_kJ^{\alpha}_{a^+,h}\left(gh\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(f\right)(t)\right] \\
& - \left[{}_kJ^{\alpha}_{a^+,h}\left(h\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(fg\right)(t)\right] + \left[{}_kJ^{\alpha}_{a^+,h}\left(fg\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(h\right)(t)\right] \\
& + \left[{}_kJ^{\alpha}_{a^+,h}\left(f\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(gh\right)(t)\right] - \left[{}_kJ^{\alpha}_{a^+,h}\left(g\right)(t)\right]\left[{}_kJ^{\beta}_{a^+,h}\left(fh\right)(t)\right].
\end{aligned}
$$

□

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