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SESQUILINEAR VERSION OF NUMERICAL RANGE AND NUMERICAL RADIUS

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ABSTRACT. In this paper by using the notion of sesquilinear form we introduce a new class of numerical range and numerical radius in normed space $\mathscr{V}$, also its various characterizations are given. We apply our results to get some inequalities.

1. Introduction

A related concept to our work is the notion of sesquilinear form. Sesquilinear forms and quadratic forms were studied extensively by various authors, who have developed a rich array of tools to study them; cf. [16, 19]. There is considerable amount of literature devoting to the study of sesquilinear form. We refer to [1, 17, 22] for a recent survey and references therein.

During the past decades several definitions of the numerical range in various settings have been introduced by many mathematicians. For instance, Marcus and Wang [14] introduced the rth permanental numerical range of operator A. In addition, Descloux in [3] defined the notion of essential numerical range of an operator with respect to a coercive sesquilinear form. In 1977, Marvin [15] and in 1984, independently, Tsing [23] introduce and characterize a new version of numerical range in a space $\mathbb{C}^n$ equipped with a sesquilinear form. Li in [13], generalized the work of Tsing and explored fundamental properties and consequences of numerical range in the framework sesquilinear form. We also refer to another interesting paper by Fox [9] of this type.

The concept of a sesquilinear form and quadratic form do not require the structure of an inner product space. They can be defined in any vector space. Obviously, inner product is a sesquilinear form but the converse is not true. The motivation of this paper is to introduce the notions of numerical range and numerical radius without the inner product structure. In fact, the result extends immediately to the case where the Hilbert space $\mathscr{H}$ and inner product $\langle\cdot,\cdot\rangle$, replaced by vector space $\mathscr{V}$ and sesquilinear form φ, respectively.

In Section 2 we invoke some fundamental facts about the sesquilinear forms in vector space that are used throughout the paper. Some famous inequalities due to Kittaneh, Dragomir and

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Sánador are given. In Section 3 of this paper, we introduce and study the numerical range and numerical radius by using sesquilinear form φ in normed space $\mathscr{V}$, which we call them φ-numerical range and φ-numerical radius respectively. Also some inequalities for φ-numerical radius are extended. For this purpose, we employ some classical inequalities for numerical radius in Hilbert space.

2. **Preliminary Results**

For the sake of completeness, we reproduce the following definitions and preliminary results, which will be needed in the sequel.

Definition 2.1. $\varphi : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$ where $\mathscr{V}$ is complex vector space, is a sesquilinear form if satisfying the following two conditions:

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,

(b) $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$,

for any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{V}$.

Definition 2.2. Let φ be a sesquilinear form on vector space $\mathscr{V}$,

(a) φ is called symmetric if $\varphi(x, y) = \overline{\varphi(y, x)}$ for all $x, y \in \mathscr{V}$.

(b) φ is called positive if $\varphi(x, x) \geq 0$ for all $x \in \mathscr{V}$.

(c) φ is called strictly if it is positive and $\varphi(x, x) > 0$ for all $x \in \mathscr{V}$.

(d) φ is called bounded if $|\varphi(x, y)| \leq M \|x\| \|y\|$ for some $M > 0$ and all $x, y \in \mathscr{V}$. Note that for a bounded sesquilinear form φ on $\mathscr{V}$ we have

$$|\varphi(x, y)| \leq \|\varphi\| \|x\| \|y\| , \tag{2.1}$$

for all $x, y \in \mathscr{V}$.

Remark 2.1. *For each positive sesquilinear form $\sqrt{\varphi(x, x)}$ is a semi norm on $\mathscr{V}$; since satisfied the axioms of a norm exept that the implication $\sqrt{\varphi(x, x)} = 0 \Rightarrow x = 0$ may not hold; see* [18, p. 52].

Note that the norm of $\mathscr{V}$, will be denoted by $\|\cdot\|_{\varphi}$.

Theorem 2.1. *For operator $A \in \mathcal{B}(\mathscr{V})$ there exist $B \in \mathcal{B}(\mathscr{V})$ such that for each x and y in $\mathscr{V}$*

$$\varphi(Ax, y) = \varphi(x, By) .$$

In this case, B is φ-adjoint of A and it is denoted by A^*. If $A = A^*$, then A is called self adjoint. For more information on related ideas and concepts we refer the reader to [21, p. 88-90].

Definition 2.3. An operator A is called φ-positive if it is self adjoint and $\varphi(Ax, x) \geq 0$ for all $x \in \mathscr{V}$.

The following lemma is known as Polarization identity for sesquilinear forms; see [2, Theorem 4.3.7].

Lemma 2.1. *Let φ be a sesquilinear form on $\mathscr{V}$, and let Φ be the quadratic form associated with φ. Then*

$$4\varphi(x,y) = \|x+y\|_\varphi^2 - \|x-y\|_\varphi^2 + i\|x+iy\|_\varphi^2 - i\|x-iy\|_\varphi^2. \tag{2.2}$$

The next lemma is known as the Cauchy-Schwarz inequality and follows from Lemma 2.1.

Lemma 2.2. *For any positive sesquilinear form φ on $\mathscr{V}$ we have*

$$|\varphi(x,y)| \le \sqrt{\varphi(x,x)}\sqrt{\varphi(y,y)}.$$

Lemma 2.3. *The Schwarz inequality for φ-positive operators asserts that A is a φ-positive operator in $\mathcal{B}(\mathscr{V})$, then*

$$|\varphi(Ax,y)|^2 \le \varphi(Ax,x)\,\varphi(Ay,y), \tag{2.3}$$

for all $x,\ y$ in $\mathscr{V}$.

The following result is known as the Mixed Schwarz Inequality, that obtained by Kittaneh in Hilbert space case; see [12, Lemma 1].

Proposition 2.1. *Let A, B and C be operators in $\mathcal{B}(\mathscr{V})$, where A and B are φ-positive. Then $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a φ-positive operator in $\mathcal{B}(\mathscr{V}\oplus\mathscr{V})$ if and only if*

$$|\varphi(Cx,y)|^2 \le \varphi(Ax,x)\,\varphi(By,y), \tag{2.4}$$

for all $x,\ y$ in $\mathscr{V}$.

Proof. First assume that $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a φ-positive operator in $\mathcal{B}(\mathscr{V}\oplus\mathscr{V})$. Then by (2.3) we have

$$\left|\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right)\right|^2 \le \varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right),$$

for all $x,\ y$ in $\mathscr{V}$. A direct simplification of above inequality now yields (2.4).

Convesely, assume that (2.3) holds, then for every x, y in $\mathscr{V}$,

$$\begin{aligned}\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) &= \varphi\left(Ax,x\right)+\varphi\left(C^*y,x\right)+\varphi\left(Cx,y\right)+\varphi\left(By,y\right) \\ &= \varphi\left(Ax,x\right)+\varphi\left(By,y\right)+2\operatorname{Re}\varphi\left(Cx,y\right) \\ &\geq 2(\varphi\left(Ax,x\right))^{\frac{1}{2}}(\varphi\left(By,y\right))^{\frac{1}{2}}+2\operatorname{Re}\varphi\left(Cx,y\right) \\ &\geq 2\left|\varphi\left(Cx,y\right)\right|+2\operatorname{Re}\varphi\left(Cx,y\right) \\ &\geq 2\left|\varphi\left(Cx,y\right)\right|-2\left|\varphi\left(Cx,y\right)\right| \\ &= 0.\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.2. *If we put $C = AB$ in* (2.4), *then we obtain*

$$\left|\varphi\left(ABx,x\right)\right|^2 \leq \varphi\left(A^2x,x\right)\varphi\left(B^2y,y\right).$$

We will need the following definition to obtain our results. For more related details see [4, p. 1-5].

Definition 2.4. A functional $(\cdot,\cdot):\mathscr{V}\times\mathscr{V}\to\mathbb{C}$ is said to be a Hermitian form on linear space $\mathscr{V}$ if

(a) $(ax+by,z)=a\,(x,z)+b\,(y,z)$, for $a,b\in\mathbb{C}$ and $x,y,z\in\mathscr{V}$;
(b) $(x,y)=\overline{(y,x)}$, for all $x,y\in\mathscr{V}$.

Utilising the Cauchy Schwarz inequality we can state the following result that will be useful in the sequel (see [7, Theorem 2]).

Lemma 2.4. *Let $(\mathscr{V},\varphi(\cdot,\cdot))$ be a complex vector space. Then*

$$\begin{aligned}&\left(\|a\|_\varphi^2\|b\|_\varphi^2-\left|\varphi\left(a,b\right)\right|^2\right)\left(\|b\|_\varphi^2\|c\|_\varphi^2-\left|\varphi\left(b,c\right)\right|^2\right) \\ &\geq\left|\varphi\left(a,c\right)\|b\|_\varphi^2-\varphi\left(a,b\right)\varphi\left(b,c\right)\right|^2.\end{aligned} \tag{2.5}$$

Proof. Let us consider the mapping $p_b:\mathscr{V}\times\mathscr{V}\to\mathbb{C}$, with $p_b\left(a,c\right)=\varphi\left(a,c\right)\|b\|_\varphi^2-\varphi\left(a,b\right)\varphi\left(b,c\right)$, for each $b\in\mathscr{V}\setminus\{0\}$.

Obviously $p_b\left(\cdot,\cdot\right)$ is a nonnegative Hermitian form and then writing Schwarz's inequality

$$\left|p_b\left(a,c\right)\right|^2\leq p_b\left(a,a\right)p_b\left(c,c\right),\quad\left(a,c\in\mathscr{V}\right)$$

we obtain the desired inequality (2.5). □

The following refinement of the Schwarz inequality holds (see [8, Theorem 4]):

Theorem 2.2. *Let $a, b \in \mathscr{V}$ and $e \in \mathscr{V}$ with $\|e\|_\varphi = 1$, then*

$$\|a\|_\varphi \|b\|_\varphi \geq |\varphi(a,b)\,\varphi(a,e)\,\varphi(e,b)| + |\varphi(a,e)\,\varphi(e,b)| \geq |\varphi(a,b)|\,. \tag{2.6}$$

Proof. Applying the inequality (2.4), we can state that

$$\left(\|a\|_\varphi^2 - |\varphi(a,e)|^2\right)\left(\|b\|_\varphi^2 - |\varphi(b,e)|^2\right) \geq |\varphi(a,b) - \varphi(a,e)\,\varphi(e,b)|^2. \tag{2.7}$$

Utilising the elementary inequality for real numbers

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2,$$

we can easily see that

$$\left(\|a\|_\varphi \|b\|_\varphi - |\varphi(a,e)\,\varphi(e,b)|\right)^2 \geq \left(\|a\|_\varphi^2 - |\varphi(a,e)|^2\right)\left(\left|\|b\|_\varphi^2 - |\varphi(b,e)|^2\right|\right) \tag{2.8}$$

for any $a, b, e \in \mathscr{V}$ with $\|e\|_\varphi = 1$.

Since, by the Schwarz's inequality

$$|\varphi(a,e)\,\varphi(e,b)| \leq \|a\|_\varphi \|b\|_\varphi$$

hence, by (2.7) and (2.8) we deduce the first part of (2.6).

The second part of (2.6) is obvious. □

If $\varphi(x,y) = 0$, x is said to be φ-orthogonal to y, and notation $x \perp^\varphi y$ is used. If $\varphi(x,x) = 0$ implies $x = 0$, then the relation $\perp^\varphi$ is symmetric. The notation $\mathscr{U} \perp^\varphi \mathscr{W}$ means that $x \perp^\varphi y$ when $x \in \mathscr{U}$ and $y \in \mathscr{W}$. Also $\mathscr{U}^\perp$ is the set of all $y \in \mathscr{V}$ that are orthogonal to every $x \in \mathscr{U}$.

The following lemmas are known in the literature (see [21, p. 307-308]).

Lemma 2.5. *If $x, y \in \mathscr{V}$, and $\varphi(x,x) = 0$ implies $x = 0$, then*

$$\|y\|_\varphi \leq \|\lambda x + y\|_\varphi \ \ (\lambda \in \mathbb{C}),$$

if and and only if $x \perp^\varphi y$.

Lemma 2.6. *Every non empty closed convex set $\mathscr{U} \subset \mathscr{V}$ contains a unique x of minimal norm.*

The next assertion is interesting on its own right.

Theorem 2.3. *If $\mathscr{M}$ is a closed subspace of $\mathscr{V}$, then*

$$\mathscr{V} = \mathscr{M} \oplus \mathscr{M}^\perp.$$

3. φ-numerical range and φ-numerical radius

This section deals with the theory of sesquilinear forms, its generalizations and applications to numerical range and numerical radius of operators. The basic notions of numerical range and numerical radius can be found in [10]. Moreover, for a host of numerical radius inequalities, and for diverse applications of these inequalities, we refer to [6, 5, 20], and references therein. Before stating the results, we establish the notation some results from the literature.

Definition 3.1. The φ-numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$, given by

$$W_\varphi(A) = \left\{ \varphi(Ax, x) : \ x \in \mathscr{V}, \|x\|_\varphi = 1 \right\}.$$

Proposition 3.1. *The following properties of $W_\varphi(A)$ are immediate.*

(a) If φ is symmetric then, $W_\varphi(A^*) = \left\{ \overline{\lambda} : \ \lambda \in W_\varphi(A) \right\}$.

(b) $W_\varphi(\alpha I + \beta A) = \alpha + \beta W_\varphi(A)$.

(c) $W_\varphi(U^* A U) = W_\varphi(A)$, for any unitary operator U.

Further, we list some basic properties of $W_\varphi(A)$:

Proposition 3.2. *Let $A \in \mathcal{B}(\mathscr{V})$, φ be a sesqulinear form on vector space $\mathscr{V}$, then*

(a) $W_\varphi(A)$ is convex.

(b) $Sp(A) \subseteq \overline{W_\varphi(A)}$, where $Sp(A)$ denotes the spectrum of A.

(c) If φ is symmetric then, A is real if and only if $W_\varphi(A)$ is real.

Definition 3.2. The φ-numerical radius of an operator A on $\mathscr{V}$ given by

$$\omega_\varphi(A) = \sup \left\{ |\varphi(Ax, x)| : \ \|x\|_\varphi = 1 \right\}.$$

If $\varphi(x, x) = 0$ implies $x = 0$ then $\omega_\varphi(\cdot)$ is a norm on the $\mathcal{B}(\mathscr{V})$ of all bounded linear operators $A : \mathscr{V} \to \mathscr{V}$, that is

(a) $\omega_\varphi(A) \geq 0$ for any $A \in \mathcal{B}(\mathscr{V})$ and $\omega_\varphi(A) = 0$ if and only if $A = 0$;

(b) $\omega_\varphi(\lambda A) = |\lambda| \omega_\varphi(A)$ for any $\lambda \in \mathbb{C}$ and $A \in \mathcal{B}(\mathscr{V})$;

(c) $\omega_\varphi(A + B) \leq \omega_\varphi(A) + \omega_\varphi(B)$ for any $A, B \in \mathcal{B}(\mathscr{V})$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

Proposition 3.3. *For each $A \in \mathscr{B}(\mathscr{V})$*

$$\omega_\varphi(A) \leq \|A\|_\varphi \leq 2\omega_\varphi(A), \tag{3.1}$$

where

$$\|A\|_\varphi = \sup \left\{ |\varphi(Ax, y)| : \ \|x\|_\varphi = \|y\|_\varphi = 1 \right\}.$$

We are now ready to construct our main results of this section.

Theorem 3.1. *Let φ be a symmetric sesquilinear form. Then, A is self adjoint if and only if $W_\varphi(A)$ is real.*

Proof. If A is self adjoint, we have, for all $f \in \mathscr{V}$, $\varphi(Af, f) = \varphi(f, Af) = \overline{\varphi(Af, f)}$, and hence $W_\varphi(A)$ is real. Conversely, if $\varphi(Af, f)$ is real for all $f \in \mathscr{V}$, we have $\varphi(Af, f) = \varphi(f, Af) = 0 = \varphi((A - A^*) f, f)$. Thus the operator $A - A^*$ has only $\{0\}$ in its φ-numerical range. So $A - A^* = 0$ and $A = A^*$. □

Theorem 3.2. *Let $\mathscr{V}$ be a normed space and $A \in \mathcal{B}(\mathscr{V})$. If $R(A) \perp^\varphi R(A^*)$, then $\omega_\varphi(A) = \frac{1}{2}\|A\|_\varphi$.*

Proof. Let $x \in \mathscr{V}$, $\|x\|_\varphi = 1$. We can write $x = x_1 + x_2$, where $x_1 \in N(A)$, the null space of T, and $x_2 \in \overline{R(A^*)}$. Thus we have

$$\varphi(Ax, x) = \varphi(A(x_1 + x_2), x_1 + x_2) = \varphi(Ax_2, x_1).$$

Since $Ax_1 = 0$ and $\varphi(Ax_2, x_2) = \varphi(x_2, A^* x_2) = 0$. Thus

$$\begin{aligned} |\varphi(Ax, x)| &\leq \|A\|_\varphi \|x_1\| \|x_2\| \\ &\leq \frac{1}{2}\|A\|_\varphi (\|x_1\| + \|x_2\|) \quad \left(\text{by the inequality } \|a\| \|b\| \leq \frac{1}{2}\left(\|a\|^2 + \|b\|^2\right)\right) \\ &= \frac{1}{2}\|A\|_\varphi \quad (\text{since } \|x_1\| + \|x_2\| = 1). \end{aligned}$$

Since x is arbitrary, we have

$$\omega_\varphi(A) \leq \frac{1}{2}\|A\|_\varphi \leq \omega_\varphi(A).$$

This completes the proof. □

Our φ-numerical radius inequality for bounded operators can be stated as follows.

Theorem 3.3. *Let $A, X \in \mathcal{B}(\mathscr{V})$. Then*

$$\omega_\varphi(AXA^*) \leq \|A\|_\varphi^2 \, \omega_\varphi(X). \tag{3.2}$$

Proof. Let $x \in \mathscr{V}$ be a unit vector. Then

$$\begin{aligned} |\varphi(AXA^* x, x)| &= |\varphi(XA^* x, A^* x)| \\ &\leq \|A^* x\|_\varphi^2 \, \omega_\varphi(x) \\ &\leq \|A^*\|_\varphi^2 \, \omega_\varphi(x) \\ &= \|A\|_\varphi^2 \, \omega_\varphi(x). \end{aligned}$$

Now the result follows immediately by taking supremum over all unit vectors in $\mathscr{V}$. □

Remark 3.1. *Let $A, X \in \mathcal{B}(\mathscr{V})$. Then*

$$\omega_\varphi (AXA^*) \le \|A\|_\varphi^2 \|X\|_\varphi. \tag{3.3}$$

Note that, by (3.1) *we can easily see that inequality* (3.2) *is sharper than inequality* (3.3).

The following result holds (see [11, Theorem 1], for the case of inner product):

Theorem 3.4. *If $A, B \in \mathcal{B}(\mathscr{V})$ and φ is a bounded sesquilinear form, then*

$$\frac{1}{4}\|A^*A + AA^*\|_\varphi \le (\omega_\varphi(A))^2 \le \|A^*A + AA^*\|_\varphi.$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self adjoint, and $A^*A+AA^* = 2(B^2 + C^2)$. Let x be any vector in $\mathscr{V}$. Then by the convexity of the function $f(t) = t^2$, we have

$$\begin{aligned}
|\varphi(Ax, x)|^2 &= (\varphi(Bx, x))^2 + (\varphi(Cx, x))^2 \\
&\ge \frac{1}{2}(|\varphi(Bx, x)| + |\varphi(Cx, x)|)^2 \\
&\ge \frac{1}{2}|\varphi((B \pm C)x, x)|^2
\end{aligned}$$

takking supremum over $x \in \mathscr{V}$ with $\|x\|_\varphi = 1$, produces

$$\frac{1}{2}\|B \pm C\|_\varphi^2 \le (\omega_\varphi(A))^2.$$

Since

$$\begin{aligned}
2(\omega_\varphi(A))^2 &\ge \frac{1}{2}\left(\|B + C\|_\varphi^2 + \|B - C\|_\varphi^2\right) \\
&\ge \frac{1}{2}\left\|(B + C)^2 + (B - C)^2\right\|_\varphi \\
&= \left\|B^2 + C^2\right\|_\varphi \\
&= \frac{1}{2}\|A^*A + AA^*\|_\varphi
\end{aligned}$$

and hence

$$(\omega_\varphi(A))^2 \le \frac{1}{4}\|A^*A + AA^*\|_\varphi.$$

On the other hand

$$|\varphi(Ax, x)|^2 = (\varphi(Bx, x))^2 + (\varphi(Cx, x))^2 \le 2\left\|B^2 + C^2\right\|_\varphi.$$

Now by taking the supremum over $x \in \mathscr{V}$, with $\|x\|_\varphi = 1$ in the above inequality we infer that Theorem 3.4. □

Now we state, another related φ-numerical radius inequality that has been given in [6, Theorem 36], for Hilbert sapce case.

Theorem 3.5. *Let $A \in \mathcal{B}(\mathscr{V})$, then*

$$\omega_\varphi^2(A) \le \frac{1}{2}\left(\omega_\varphi\left(A^2\right) + \|A\|_\varphi^2\right). \tag{3.4}$$

Proof. By Theorem 2.2 observing that

$$|\varphi(a,b) - \varphi(a,e)\varphi(e,b)| \ge |\varphi(a,e)\varphi(e,b)| - |\varphi(a,b)|,$$

hence by first inequality in (2.6) we deduce

$$\frac{1}{2}\left(\|a\|_\varphi \|b\|_\varphi + |\varphi(a,b)|\right) \ge |\varphi(a,e)\varphi(e,b)|. \tag{3.5}$$

Choose in (3.5), $e = x, \|x\|_\varphi = 1, a = Ax$ and $b = A^*x$ to get

$$\frac{1}{2}\|Ax\|_\varphi \|A^*x\|_\varphi + \left|\varphi\left(A^2x, x\right)\right| \ge |\varphi(Ax,x)|^2, \tag{3.6}$$

for any $x \in V$ with $\|x\|_\varphi = 1$.

Taking the supremum in (3.6) over $x \in \mathscr{V}$ with $\|x\|_\varphi = 1$, we deduce the desired inequality (3.4). □

Remark 3.2. *All of the inequalities which are obtained by Dragomir in* [6, Section 3] *can be extended to vector space in a similar way. The details are left to the interested readers.*

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