SOME ADDITIVE REVERSES OF CALLEBAUT AND HÖLDER INEQUALITIES FOR ISOTONIC FUNCTIONALS

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Abstract. In this paper we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via a reverse of Young's inequality we have established recently. Applications for integrals and n-tuples of real numbers are provided as well.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f,\, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha,\, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1,\, t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f,\, g \in L$ and $\alpha,\, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [18] and [19]). For other inequalities for isotonic functionals see [1], [4]-[17] and [20]-[23].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0,\, k \in E$).

As is known to all, the famous *Young inequality* for scalars says that if $a,\, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We consider the function $f_\nu : [0, \infty) \to [0, \infty)$ defined for $\nu \in (0, 1)$ by

$$f_\nu(x) = 1 - \nu + \nu x - x^{\nu}. \tag{1.2}$$

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For $[m,M]\subset[0,\infty)$ define

$$\Delta_{\nu}(m,M):=\begin{cases} f_{\nu}(m) \text{ if } M<1, \\ \max\{f_{\nu}(m), f_{\nu}(M)\} \text{ if } m\leq 1\leq M, \\ f_{\nu}(M) \text{ if } 1<m \end{cases} \tag{1.3}$$

and

$$\delta_{\nu}(m,M):=\begin{cases} f_{\nu}(M) \text{ if } M<1, \\ 0 \text{ if } m\leq 1\leq M, \\ f_{\nu}(m) \text{ if } 1<m. \end{cases} \tag{1.4}$$

In the recent paper [9] we obtained the following refinement and reverse for the *additive Young's inequality:*

$$\delta_{\nu}(m,M)\,a\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\Delta_{\nu}(m,M)\,a, \tag{1.5}$$

for positive numbers a, b with $\frac{b}{a}\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$ where $\Delta_{\nu}(m,M)$ and $\delta_{\nu}(m,M)$ are defined by (1.3) and (1.4) respectively.

Kittaneh and Manasrah [14], [15] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.6}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.6) to an identity.

Using (1.5) and (1.6) we have the simpler, however coarser bounds:

$$r\times\begin{cases} \left(1-\sqrt{M}\right)^{2}a \text{ if } M<1, \\ 0 \text{ if } m\leq 1\leq M, \\ \left(\sqrt{m}-1\right)^{2}a \text{ if } 1<m, \end{cases} \tag{1.7}$$

$$\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}$$

$$\leq R\times\begin{cases} \left(1-\sqrt{m}\right)^{2}a \text{ if } M<1, \\ \max\left\{\left(1-\sqrt{m}\right)^{2},\left(\sqrt{M}-1\right)^{2}\right\}a \text{ if } m\leq 1\leq M, \\ \left(\sqrt{M}-1\right)^{2}a \text{ if } 1<m. \end{cases},$$

We recall that *Specht's ratio* is defined by [22]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.8}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^\nu \le (1-\nu) a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu} b^\nu, \tag{1.9}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [24] while the first one is due to Furuichi [13].

On making use of (1.5) and (1.9) we have the following lower and upper bounds in terms of Specht's ratio:

$$\begin{cases} [S(M^r) - 1] M^\nu a \text{ if } M < 1, \\ \\ 0 \text{ if } m \le 1 \le M, \\ \\ [S(m^r) - 1] m^\nu a \text{ if } 1 < m, \end{cases} \tag{1.10}$$

$$\le (1-\nu) a + \nu b - a^{1-\nu} b^\nu$$

$$\le \begin{cases} [S(m) - 1] m^\nu a \text{ if } M < 1, \\ \\ \max\{[S(m) - 1] m^\nu, [S(M) - 1] M^\nu\} a \text{ if } m \le 1 \le M, \\ \\ [S(M) - 1] M^\nu a \text{ if } 1 < m. \end{cases}$$

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.11}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds.

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \le (1-\nu) a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \tag{1.12}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.12) was obtained by Zou et al. in [25] while the second by Liao et al. [16].

By making use of (1.5) and (1.9) we have the following lower and upper bounds in terms of Kantorovich's constant:

$$
(1.13)\qquad \begin{cases} \left[K^{r}\left(M\right)-1\right]M^{\nu}a \text{ if } M<1, \\ \\ 0 \text{ if } m\leq 1\leq M, \\ \\ \left[K^{r}\left(m\right)-1\right]m^{\nu}a \text{ if } 1<m, \end{cases}
$$

$$
\leq \left(1-\nu\right)a+\nu b-a^{1-\nu}b^{\nu}
$$

$$
\leq \begin{cases} \left[K^{R}\left(m\right)-1\right]m^{\nu}a \text{ if } M<1, \\ \\ \max\left\{\left[K^{R}\left(m\right)-1\right]m^{\nu},\left[K^{R}\left(M\right)-1\right]M^{\nu}\right\}a \text{ if } m\leq 1\leq M, \\ \\ \left[K^{R}\left(M\right)-1\right]M^{\nu}a \text{ if } 1<m. \end{cases}
$$

In this paper we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via the reverse of Young's inequality obtained in (1.5). Applications for integrals and n-tuples of real numbers are provided as well.

2. Reverses of Callebaut's Inequality

The functional version of *Callebaut's inequality* states that

$$
(2.1)\qquad A^{2}\left(fg\right)\leq A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right)\leq A\left(f^{2}\right)A\left(g^{2}\right)
$$

provided that f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}$, $fg\in L$ for some $\nu\in[0,1]$. For the discrete and integral of one real variable versions see [3].

We start with the following result:

Theorem 1. *Let A, $B: L\to\mathbb{R}$ be two normalised isotonic functionals. If f, $g: E\to\mathbb{R}$ are such that $f\geq 0$, $g>0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and*

$$
(2.2)\qquad 0<m\leq\frac{f}{g}\leq M<\infty
$$

for some constants m, M, then

$$
(2.3)\qquad \begin{aligned} &(0\leq)\left(1-\nu\right)A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right) \\ &-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq\max\left\{f_{\nu}\left(\left(\frac{m}{M}\right)^{2}\right),f_{\nu}\left(\left(\frac{M}{m}\right)^{2}\right)\right\}A\left(f^{2}\right)B\left(g^{2}\right), \end{aligned}
$$

where f_ν is defined by (1.2).

In particular,

$$
(2.4)\qquad \begin{aligned} &(0\leq)A\left(f^{2}\right)A\left(g^{2}\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq\max\left\{f_{\nu}\left(\left(\frac{m}{M}\right)^{2}\right),f_{\nu}\left(\left(\frac{M}{m}\right)^{2}\right)\right\}A\left(f^{2}\right)A\left(g^{2}\right). \end{aligned}
$$

Proof. For any x, $y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)},\ \frac{f^2(y)}{g^2(y)} \leq M^2.$$

Consider

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\left(\frac{m}{M}\right)^2, \left(\frac{M}{m}\right)^2\right]$ and by the inequality (1.5) we have

$$(0 \leq)\,(1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\}\frac{f^2(x)}{g^2(x)}, \tag{2.5}$$

for any x, $y \in E$.

Now, if we multiply (2.5) by $g^2(x)\,g^2(y) > 0$ then we get

$$(1-\nu) f^2(x)\,g^2(y) + \nu g^2(x)\,f^2(y) - f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y) \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^2(x)\,g^2(y) \tag{2.6}$$

for any x, $y \in E$.

Fix $y \in E$. Then by (2.5) we have in the order of L that

$$(1-\nu)\,g^2(y)\,f^2 + \nu f^2(y)\,g^2 - f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu} \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y)\,f^2. \tag{2.7}$$

If we take the functional A in (2.6) then we get

$$(1-\nu)\,g^2(y)\,A\left(f^2\right) + \nu f^2(y)\,A\left(g^2\right) - f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right) \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y)\,A\left(f^2\right)$$

for any $y \in E$.

This inequality can be written in the order of L as

$$(1-\nu)\,A\left(f^2\right)g^2 + \nu A\left(g^2\right)f^2 - A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right)g^2. \tag{2.8}$$

Now, if we take the functional B in (2.8), then we get the desired result (2.3). □

Corollary 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (2.2) holds* *true, then*

$$(0 \le) \frac{1}{2}\left[A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right)\right]-A\left(fg\right)B\left(fg\right) \le \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right)B\left(g^2\right). \tag{2.9}$$

In particular,

$$(0 \le) A\left(f^2\right)A\left(g^2\right)-A^2\left(fg\right) \le \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right)A\left(g^2\right), \tag{2.10}$$

or, equivalently

$$(0 \le) 1-\frac{A^2\left(fg\right)}{A\left(f^2\right)A\left(g^2\right)} \le \frac{1}{2}\left(\frac{M}{m}-1\right)^2. \tag{2.11}$$

Proof. Observe that

$$f_{\frac{1}{2}}\left(\left(\frac{m}{M}\right)^2\right)=\frac{m^2+M^2}{2M^2}-\frac{m}{M}=\frac{\left(M-m\right)^2}{2M^2}$$

and

$$f_{\nu}\left(\left(\frac{M}{m}\right)^2\right)=\frac{m^2+M^2}{2m^2}-\frac{M}{m}=\frac{\left(M-m\right)^2}{2m^2}.$$

Therefore

$$\max\left\{f_{\nu}\left(\left(\frac{m}{M}\right)^2\right),f_{\nu}\left(\left(\frac{M}{m}\right)^2\right)\right\}=\frac{\left(M-m\right)^2}{2m^2}=\frac{1}{2}\left(\frac{M}{m}-1\right)^2$$

and by (2.3) we get the desired result (2.9). □

Remark 1. *We observe that the inequality (2.10) can be written as*

$$A\left(f^2\right)A\left(g^2\right)\left[1-\frac{1}{2}\left(\frac{M}{m}-1\right)^2\right] \le A^2\left(fg\right). \tag{2.12}$$

We observe that the function $\varphi:[1,\infty)\to\mathbb{R}$, $\varphi(t)=1-\frac{1}{2}(t-1)^2$ is positive for $t\in\left(1,1+\sqrt{2}\right)$ and negative for $t\in[1,\infty)$. Therefore, the inequality (2.12) is of interest only in the case that $\frac{M}{m}\in\left(1,1+\sqrt{2}\right)$.

On using the inequality (2.3) and (1.7) we get

$$\begin{aligned}(0 \le) & (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right) \\ & -A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ & \le R\max\left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\}A\left(f^2\right)B\left(g^2\right)\end{aligned} \tag{2.13}$$

and since

$$\max\left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\}=\left(\frac{M}{m}-1\right)^2,$$

then we get from (2.13) that

$$(0\leq)\,(1-\nu)\,A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \leq R\left(\frac{M}{m}-1\right)^{2}A\left(f^{2}\right)B\left(g^{2}\right) \tag{2.14}$$

provided $f\geq 0$, $g>0$, f^{2}, g^{2}, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

On using the inequality (2.3) and (1.10) we get the following reverse of Callebaut inequality in terms of Specht's ratio

$$(0\leq)\,(1-\nu)\,A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \leq \max\left\{\left[S\left(\left(\frac{m}{M}\right)^{2}\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[S\left(\left(\frac{M}{m}\right)^{2}\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\} \times A\left(f^{2}\right)B\left(g^{2}\right), \tag{2.15}$$

provided $f\geq 0$, $g>0$, f^{2}, g^{2}, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

Finally, on using the inequality (2.3) and (1.13) we get the following reverse of Callebaut inequality in terms of Kantorovich's constant

$$(0\leq)\,(1-\nu)\,A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \leq \max\left\{\left[K^{R}\left(\left(\frac{m}{M}\right)^{2}\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[K^{R}\left(\left(\frac{M}{m}\right)^{2}\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\} \times A\left(f^{2}\right)B\left(g^{2}\right), \tag{2.16}$$

provided $f\geq 0$, $g>0$, f^{2}, g^{2}, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

3. Reverses of Hölder's Inequality

We have the following additive reverse of Hölder's inequality:

Theorem 2. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If f, $g:E\to\mathbb{R}$ are such that fg, f^{p}, $g^{q}\in L$ and*

$$0<m_{1}\leq f\leq M_{1}<\infty,\ 0<m_{2}\leq g\leq M_{2}<\infty, \tag{3.1}$$

then

$$(0\leq)\,1-\frac{A\left(fg\right)}{\left[A\left(f^{p}\right)\right]^{1/p}\left[A\left(g^{q}\right)\right]^{1/q}} \leq \max\left\{f_{\frac{1}{p}}\left(\left[\left(\frac{M_{1}}{m_{1}}\right)^{p}\left(\frac{M_{2}}{m_{2}}\right)^{q}\right]^{-1}\right),f_{\frac{1}{p}}\left(\left(\frac{M_{1}}{m_{1}}\right)^{p}\left(\frac{M_{2}}{m_{2}}\right)^{q}\right)\right\}, \tag{3.2}$$

where $f_{\frac{1}{p}}$ is defined by

$$f_{\frac{1}{p}}\left(x\right)=\frac{1}{q}+\frac{1}{p}x-x^{\frac{1}{p}}. \tag{3.3}$$

Proof. Observe that, by (3.1) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p \text{ and } \left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (1.5) for $b = \frac{f^p}{A(f^p)}$, $a = \frac{g^q}{A(g^q)}$, $\nu = \frac{1}{p}$, $M = \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q$ and $m = \left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}$ we have

$$(3.4)\quad 0 \le \frac{1}{q}\frac{g^q}{A(g^q)} + \frac{1}{p}\frac{f^p}{A(f^p)} - \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}$$
$$\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{g^q}{A(g^q)}.$$

If we take the functional A in (3.4), then we get

$$0 \le \frac{1}{q}\frac{A(g^q)}{A(g^q)} + \frac{1}{p}\frac{A(f^p)}{A(f^p)} - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}$$
$$\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{A(g^q)}{A(g^q)},$$

which is equivalent to the desired result (3.1). □

The following reverse of Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional, f, $g : E \to \mathbb{R}$ are such that fg, f^2, $g^2 \in L$ and the condition (3.1) is valid, then*

$$(3.5)\qquad (0 \le)\, 1 - \frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}} \le \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}.$$

Proof. For $p = 2$ we have $f_{\frac{1}{2}}(x) = \frac{1+x}{2} - \sqrt{x}$, $x \ge 0$. Then

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right) = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}$$

and

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right) = \frac{(M_1M_2 - m_1m_2)^2}{2M_1^2M_2^2}$$

and since

$$\max\left\{ f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right), f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right)\right\} = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2},$$

then by (3.2) we get the desired result (3.5). □

Using the inequality (3.5) and (1.7) we get

$$
\begin{aligned}
(3.6)\qquad & (0\leq)\,1-\frac{A(fg)}{[A(f^{p})]^{1/p}[A(g^{q})]^{1/q}}\\
&\leq T\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^{2},\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^{2}\right\},
\end{aligned}
$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Since

$$
\begin{aligned}
&\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^{2},\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^{2}\right\}\\
&=\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^{2},
\end{aligned}
$$

then by (3.6) we have the inequality

$$
(3.7)\qquad (0\leq)\,1-\frac{A(fg)}{[A(f^{p})]^{1/p}[A(g^{q})]^{1/q}}\leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^{2},
$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg,\ f^{p},\ g^{q}\in L$ and they satisfy the condition (3.1).

Using the inequality (3.5) and (1.10) we get

$$
\begin{aligned}
(3.8)\qquad & (0\leq)\,1-\frac{A(fg)}{[A(f^{p})]^{1/p}[A(g^{q})]^{1/q}}\\
&\leq\max\left\{\left[S\left(\left[\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right.\\
&\left.\left[S\left(\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\},
\end{aligned}
$$

provided $f,g:E\to\mathbb{R}$ are such that $fg,\ f^{p},\ g^{q}\in L$ and they satisfy the condition (3.1).

Using the inequality (3.5) and (1.13) we get

$$
\begin{aligned}
(3.9)\qquad & (0\leq)\,1-\frac{A(fg)}{[A(f^{p})]^{1/p}[A(g^{q})]^{1/q}}\\
&\leq\max\left\{\left[K^{T}\left(\left[\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right.\\
&\left.\left[K^{T}\left(\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\},
\end{aligned}
$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg,\ f^{p},\ g^{q}\in L$ and they satisfy the condition (3.1).

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)| w(x) d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.1}$$

If f^2, $g^2 \in L_w(\Omega, \mu)$, then by (2.4) we have

$$\begin{aligned}(0 \leq) \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu - \int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu \\ \leq \max\left\{f_s\left(\left(\frac{m}{M}\right)^2\right), f_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu,\end{aligned} \tag{4.2}$$

for any $s \in [0, 1]$, where f_s is defined by (1.2), and, in particular,

$$(0 \leq) 1 - \frac{\left(\int_\Omega w f g d\mu\right)^2}{\int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu} \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)^2. \tag{4.3}$$

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty \ \mu\text{-a.e. on } \Omega. \tag{4.4}$$

Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (3.2) we have the following reverse of Hölder's inequality

$$\begin{aligned}(0 \leq) 1 - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^p d\mu\right)^{1/p} \left(\int_\Omega w g^q d\mu\right)^{1/q}} \\ \leq \max\left\{f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\},\end{aligned} \tag{4.5}$$

where $f_{\frac{1}{p}}$ is defined by (3.3).

In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(0 \leq) 1 - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^2 d\mu\right)^{1/2} \left(\int_\Omega w g^2 d\mu\right)^{1/2}} \leq \frac{(M_1 M_2 - m_1 m_2)^2}{2 m_1^2 m_2^2}. \tag{4.6}$$

From (3.7) we have, for $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, that

$$(0 \leq) 1 - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^p d\mu\right)^{1/p} \left(\int_\Omega w g^q d\mu\right)^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}} - 1\right)^2. \tag{4.7}$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If there exist the constants m, $M > 0$ such that

$$0 < m \leq \frac{a_i}{b_i} \leq M < \infty \text{ for any } i \in \{1, ..., n\},$$

then by (4.2), for the counting discrete measure, we have

$$\begin{aligned} (5.1) \qquad & (0 \leq) \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^n p_i a_i^{2s} b_i^{2(1-s)} \\ & \leq \max \left\{ f_s \left(\left(\frac{m}{M} \right)^2 \right), f_s \left(\left(\frac{M}{m} \right)^2 \right) \right\} \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2, \end{aligned}$$

for any $s \in [0, 1]$, where f_s is defined by (1.2).

In particular,

$$(5.2) \qquad (0 \leq) 1 - \frac{\left(\sum_{i=1}^n p_i a_i b_i \right)^2}{\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2} \leq \frac{1}{2} \left(\frac{M}{m} - 1 \right)^2.$$

If there exists the constants m_1, M_1, m_2, M_2 such that

$$(5.3) \qquad 0 < m_1 \leq a_i \leq M_1 < \infty, \ 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1, ..., n\}$$

and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.5) we have the following reverse of Hölder's inequality

$$\begin{aligned} (5.4) \qquad & (0 \leq) 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p \right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q \right)^{1/q}} \\ & \leq \max \left\{ f_{\frac{1}{p}} \left(\left[\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right]^{-1} \right), f_{\frac{1}{p}} \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right) \right\}, \end{aligned}$$

where $f_{\frac{1}{p}}$ is defined by (3.3).

In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(5.5) \qquad (0 \leq) 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2 \right)^{1/2}} \leq \frac{\left(M_1 M_2 - m_1 m_2 \right)^2}{2 m_1^2 m_2^2}.$$

From (4.7) we have, for $T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$, that

$$(5.6) \qquad (0 \leq) 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p \right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q \right)^{1/q}} \leq T \left(\left(\frac{M_1}{m_1} \right)^{\frac{p}{2}} \left(\frac{M_2}{m_2} \right)^{\frac{q}{2}} - 1 \right)^2,$$

provided a and b satisfy the condition (5.3).

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