Received 21/12/15

SOME INEQUALITIES FOR THE $\psi_{q,k}^{(m)}(x)$ AND $\psi_{p,q}^{(m)}(x)$ FUNCTIONS

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ABSTRACT. In this work, the authors establish some inequalities for the (q,k) and (p,q) deformations of the Polygamma functions denoted by $\psi_{q,k}^{(m)}(x)$ and $\psi_{p,q}^{(m)}(x)$ respectively. The approache makes use of some monotonicity properties of certain functions involving these two functions. The results extend and generalize some existing results.

1. Introduction

The classical Euler's Gamma function $\Gamma(x)$, which is an extension of the factorial notation to non-integer values, may be defined for $x > 0$ as

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{x}{n}\right)^{-1}\right].$$

Closely linked to the Gamma function is the Digamma or Psi function $\psi(x)$, which is defined for $x > 0$ as the logarithmic derivative of the Gamma function. That is,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is well known that this function satisfies the series representation:

$$\psi(x) = -\gamma + (x-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}$$

where γ is the Euler-Mascheroni's constant. Let $\psi^{(m)}(x)$ be the m-th derivative of $\psi(x)$ and $\psi^{(0)}(x) \equiv \psi(x)$. The functions $\psi^{(m)}(x)$ are called the Polygamma functions, and they also satisfy the series relation

$$\psi^{(m)}(x) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}}.$$

The (q,k)-deformation of the Gamma function is defined [1] for $q \in (0,1)$, $k > 0$ and $x > 0$ by

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}(1-q)^{\frac{x}{k}-1}}$$

2010 *Mathematics Subject Classification.* 33B15, 26A48.

Key words and phrases. Gamma function, Digamma function, Polygamma function, (q,k)-deformation, (p,q)-deformation, inequality.

where $(x+y)_{q,k}^n := \prod_{i=0}^{n-1}(x+q^{ik}y)$, $(1+x)_{q,k}^{\infty} := \prod_{i=0}^{\infty}(1+q^{ik}x)$, and $(1+x)_{q,k}^t := \frac{(1+x)_{q,k}^{\infty}}{(1+q^t x)_{q,k}^{\infty}}$ for $x, y, t \in \mathbb{R}$ and $n \in \mathbb{N}$.
The (q, k)-deformations of the Digamma and Polygamma functions are respectively defined as

$$\psi_{q,k}(x) = \frac{d}{dx} \ln \Gamma_{q,k}(x) = -\frac{1}{k} \ln(1-q) + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nkx}}{1-q^{nk}}$$

and

$$\psi_{q,k}^{(m)}(x) = (\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m k^m q^{nkx}}{1-q^{nk}} \tag{1}$$

where $\psi_{q,k}^{(0)}(x) \equiv \psi_{q,k}(x)$.

Another two parameter deformation of the Gamma function is the (p, q)-deformation which is defined [2] for $p \in \mathbb{N}$, $q \in (0, 1)$ and $x > 0$ as

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \dots [x+p]_q}$$

where $[p]_q = \frac{1-q^p}{1-q}$. Likewise, the (p, q)-deformations of the Digamma and Polygamma functions are respectively defined as

$$\psi_{p,q}(x) = \frac{d}{dx} \ln \Gamma_{p,q}(x) = \ln[p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nx}}{1-q^n}$$

and

$$\psi_{p,q}^{(m)}(x) = (\ln q)^{m+1} \sum_{n=1}^{p} \frac{n^m q^{nx}}{1-q^n} \tag{2}$$

where $\psi_{p,q}^{(0)}(x) \equiv \psi_{p,q}(x)$. It follows easily from (1) and (2) that

$$\psi_{q,k}^{(m)}(x) = \begin{cases} > 0 & \text{if } m \text{ is odd} \\ < 0 & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad \psi_{p,q}^{(m)}(x) = \begin{cases} > 0 & \text{if } m \text{ is odd} \\ < 0 & \text{if } m \text{ is even} \end{cases}.$$

In [5, Theorem 2.3], the authors established the following inequalities for the Digamma function.

$$\frac{[\psi(a)]^{\alpha}}{[\psi(c)]^{\beta}} \le \frac{[\psi(a+bx)]^{\alpha}}{[\psi(c+dx)]^{\beta}} \le \frac{[\psi(a+b)]^{\alpha}}{[\psi(c+d)]^{\beta}} \tag{3}$$

for $x \in [0, 1]$, where a, b, c, d, α, β are positive real numbers such that $\beta d \le \alpha b$, $a + bx \le c + dx$, $\psi(a + bx) > 0$ and $\psi(c + dx) > 0$.

Also, in [7, Theorem 3.7], the following results for the k-deformation of the Digamma function were established.

$$\frac{[\psi_k(a)]^{\alpha}}{[\psi_k(c)]^{\beta}} \le \frac{[\psi_k(a+bx)]^{\alpha}}{[\psi_k(c+dx)]^{\beta}} \le \frac{[\psi_k(a+b)]^{\alpha}}{[\psi_k(c+d)]^{\beta}} \tag{4}$$

for $x \in [0,1]$, where a, b, c, d, α, β are positive real numbers such that $a \leq c$, $b \leq d$, $\beta d \leq \alpha b$, $a+bx \leq c+dx$, $\psi_k(a+bx) > 0$ and $\psi_k(c+dx) > 0$.

Then, in [6], the authors extended the results to the (q,k) and (p,q)-deformations of the Digamma function.

In this paper, our main goal is to establish similar results for the (q,k) and (p,q)-deformations of the Polygamma functions. Our results will provide extensions and generalizations of the known results.

2. Some Lemmas

In this section, we present some lemmas that will be needed in order to establish our results.

Lemma 2.1. *Let $q \in (0,1)$, $k > 0$, $s > 1$, $\frac{1}{s}+\frac{1}{t} = 1$ and $m,n \in \mathbb{N}$ such that $\frac{m+n}{2} \in \mathbb{N}$. Then for $x > 0$, $y > 0$, we have*

$$\psi_{q,k}^{(\frac{m}{s}+\frac{n}{t})}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \left[\psi_{q,k}^{(m)}(x)\right]^{\frac{1}{s}} \left[\psi_{q,k}^{(n)}(y)\right]^{\frac{1}{t}}. \tag{5}$$

Proof. See Theorem 2.1 of [3].

Lemma 2.2. *Suppose that m is a positive integer, $x > 0$, $q \in (0,1)$ and $k > 0$. Then*

$$\psi_{q,k}^{(m)}(x)\psi_{q,k}^{(m+2)}(x) - \left[\psi_{q,k}^{(m+1)}(x)\right]^2 \geq 0.$$

Proof. This follows from Lemma 2.1 by letting $x = y$, $s = t = 2$ and $n = m+2$.

Lemma 2.3. *Let $q \in (0,1)$, $k > 0$ and $x \geq 0$. Suppose that m is a positive odd integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then,*

$$\alpha b\psi_{q,k}^{(m)}(c+dx)\psi_{q,k}^{(m+1)}(a+bx) - \beta d\psi_{q,k}^{(m)}(a+bx)\psi_{q,k}^{(m+1)}(c+dx) \leq 0.$$

Proof. Let $U(t) = \frac{\psi_{q,k}^{(m+1)}(t)}{\psi_{q,k}^{(m)}(t)}$ for $t > 0$, $q \in (0,1)$, $k > 0$ and m a positive odd integer. Then,

$$U'(t) = \frac{\psi_{q,k}^{(m)}(t)\psi_{q,k}^{(m+2)}(t) - [\psi_{q,k}^{(m+1)}(t)]^2}{[\psi_{q,k}^{(m)}(t)]^2}$$

and by Lemma 2.2, $U'(t) \geq 0$. This implies that U is increasing. Then for $0 < x \leq y$, we obtain

$$\frac{\psi_{q,k}^{(m+1)}(x)}{\psi_{q,k}^{(m)}(x)} \leq \frac{\psi_{q,k}^{(m+1)}(y)}{\psi_{q,k}^{(m)}(y)}$$

which implies

$$\psi_{q,k}^{(m)}(y)\psi_{q,k}^{(m+1)}(x) \leq \psi_{q,k}^{(m)}(x)\psi_{q,k}^{(m+1)}(y) < 0.$$

This together with the fact that $0 < \beta d \leq \alpha b$ yields

$$\alpha b\psi_{q,k}^{(m)}(y)\psi_{q,k}^{(m+1)}(x) \leq \beta d\psi_{q,k}^{(m)}(x)\psi_{q,k}^{(m+1)}(y) < 0.$$

Thus,

$$\alpha b\psi_{q,k}^{(m)}(y)\psi_{q,k}^{(m+1)}(x) - \beta d\psi_{q,k}^{(m)}(x)\psi_{q,k}^{(m+1)}(y) \leq 0.$$

Replacing x and y respectively by $a+bx$ and $c+dx$ completes the proof.

Lemma 2.4. *Let $p \in \mathbb{N}$, $q \in (0,1)$, $s > 1$, $\frac{1}{s}+\frac{1}{t}=1$ and $m,n \in \mathbb{N}$ such that $\frac{m+n}{2} \in \mathbb{N}$. Then for $x>0$, $y>0$, we have*

$$\psi_{p,q}^{(\frac{m}{s}+\frac{n}{t})}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \left[\psi_{p,q}^{(m)}(x)\right]^{\frac{1}{s}}\left[\psi_{p,q}^{(n)}(y)\right]^{\frac{1}{t}}. \tag{6}$$

Proof. See Theorem 2.1 of [4].

Lemma 2.5. *Suppose that m is a positive integer, $x>0$, $p \in \mathbb{N}$ and $q \in (0,1)$. Then*

$$\psi_{p,q}^{(m)}(x)\psi_{p,q}^{(m+2)}(x) - \left[\psi_{p,q}^{(m+1)}(x)\right]^2 \geq 0.$$

Proof. Follows from Lemma 2.4 by letting $x=y$, $s=t=2$ and $n=m+2$.

Lemma 2.6. *Let $p \in \mathbb{N}$, $q \in (0,1)$ and $x \geq 0$. Suppose that m is a positive odd integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then,*

$$\alpha b\psi_{p,q}^{(m)}(c+dx)\psi_{p,q}^{(m+1)}(a+bx) - \beta d\psi_{p,q}^{(m)}(a+bx)\psi_{p,q}^{(m+1)}(c+dx) \leq 0.$$

Proof. Let $W(t)=\frac{\psi_{p,q}^{(m+1)}(t)}{\psi_{p,q}^{(m)}(t)}$ for $t>0$, $p \in \mathbb{N}$, $q \in (0,1)$ and m a positive odd integer. Then, by using Lemma 2.5 it can similarly be shown that W is increasing. The rest of the proof follows a similar procedure as that of Lemma 2.3. As a result, we omit them.

Lemma 2.7. *Let $q \in (0,1)$, $k>0$ and $x \geq 0$. Suppose that m is a positive even integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then,*

$$\alpha b\frac{\psi_{q,k}^{(m+1)}(a+bx)}{\psi_{q,k}^{(m)}(a+bx)} - \beta d\frac{\psi_{q,k}^{(m+1)}(c+dx)}{\psi_{q,k}^{(m)}(c+dx)} \leq 0.$$

Proof. Similar to the proof of Lemma 2.3.

Lemma 2.8. *Let $p \in \mathbb{N}$, $q \in (0,1)$ and $x \geq 0$. Suppose that m is a positive even integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then,*

$$\alpha b\frac{\psi_{p,q}^{(m+1)}(a+bx)}{\psi_{p,q}^{(m)}(a+bx)} - \beta d\frac{\psi_{p,q}^{(m+1)}(c+dx)}{\psi_{p,q}^{(m)}(c+dx)} \leq 0.$$

Proof. Similar to the proof of Lemma 2.6.

3. Main Results

We now present the main findings of the paper in this section.

Theorem 3.1. *Let $q \in (0,1)$, $k > 0$ and $x \geq 0$. Suppose that m is a positive odd integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then the inequalities*

$$\frac{\left[\psi_{q,k}^{(m)}(a)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c)\right]^{\beta}} \geq \frac{\left[\psi_{q,k}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+dx)\right]^{\beta}} \geq \frac{\left[\psi_{q,k}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+d)\right]^{\beta}} \tag{7}$$

are valid for $x \in [0,1]$.

Proof. Defined a function S by $S = \frac{\left[\psi_{q,k}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+dx)\right]^{\beta}}$. Next, let $u(x) = \ln S(x)$. Then,

$$u(x) = \alpha \ln \psi_{q,k}^{(m)}(a+bx) - \beta \ln \psi_{q,k}^{(m)}(c+dx)$$

implying

$$\begin{aligned} u'(x) &= \alpha b \frac{\psi_{q,k}^{(m+1)}(a+bx)}{\psi_{q,k}^{(m)}(a+bx)} - \beta d \frac{\psi_{q,k}^{(m+1)}(c+dx)}{\psi_{q,k}^{(m)}(c+dx)} \\ &= \frac{\alpha b \psi_{q,k}^{(m+1)}(a+bx)\psi_{q,k}^{(m)}(c+dx) - \beta d \psi_{q,k}^{(m+1)}(c+dx)\psi_{q,k}^{(m)}(a+bx)}{\psi_{q,k}^{(m)}(a+bx)\psi_{q,k}^{(m)}(c+dx)}. \end{aligned}$$

Then by Lemma 2.3, we conclude that $u'(x) \leq 0$. Thus, u is decreasing on $x \in [0,\infty)$. As a result, S is also decreasing on $x \in [0,\infty)$. Then, for $x \in [0,1]$ we have

$$S(0) \geq S(x) \geq S(1)$$

concluding the proof.

Corollary 3.2. *If $x > 1$, then the following inequality is satisfied.*

$$\frac{\left[\psi_{q,k}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+dx)\right]^{\beta}} \leq \frac{\left[\psi_{q,k}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+d)\right]^{\beta}} \tag{8}$$

Proof. For $x > 1$, we have $S(x) \leq S(1)$ establishing the result.

Theorem 3.3. *Let $p \in \mathbb{N}$, $q \in (0,1)$ and $x \geq 0$. Suppose that m is a positive odd integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. Then the inequalities*

$$\frac{\left[\psi_{p,q}^{(m)}(a)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c)\right]^{\beta}} \geq \frac{\left[\psi_{p,q}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+dx)\right]^{\beta}} \geq \frac{\left[\psi_{p,q}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+d)\right]^{\beta}} \tag{9}$$

holds true for $x \in [0,1]$.

Proof. Defined a function T by $T = \frac{\left[\psi_{p,q}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+dx)\right]^{\beta}}$. Next, let $h(x) = \ln T(x)$. Then,

$$h(x) = \alpha \ln \psi_{p,q}^{(m)}(a+bx) - \beta \ln \psi_{p,q}^{(m)}(c+dx).$$

That implies

$$\begin{aligned} h'(x) &= \alpha b \frac{\psi_{p,q}^{(m+1)}(a+bx)}{\psi_{p,q}^{(m)}(a+bx)} - \beta d \frac{\psi_{p,q}^{(m+1)}(c+dx)}{\psi_{p,q}^{(m)}(c+dx)} \\ &= \frac{\alpha b \psi_{p,q}^{(m+1)}(a+bx)\psi_{p,q}^{(m)}(c+dx) - \beta d \psi_{p,q}^{(m+1)}(c+dx)\psi_{p,q}^{(m)}(a+bx)}{\psi_{p,q}^{(m)}(a+bx)\psi_{p,q}^{(m)}(c+dx)} \leq 0 \end{aligned}$$

by Lemma 2.6. Thus, h is decreasing on $x \in [0, \infty)$ and consequently, T is also decreasing. Then, for $x \in [0, 1]$ we obtain

$$T(0) \geq T(x) \geq T(1)$$

proving the result.

Corollary 3.4. *If $x > 1$, then following inequality holds.*

$$\frac{\left[\psi_{p,q}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+dx)\right]^{\beta}} \leq \frac{\left[\psi_{p,q}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+d)\right]^{\beta}} \tag{10}$$

Proof. We have $T(x) \leq T(1)$ yielding the result.

Theorem 3.5. *Let $q \in (0, 1)$, $k > 0$ and $x \geq 0$. Suppose that m is a positive even integer, and a, b, c, d, α and β are positive real numbers such that $a+bx \leq c+dx$ and $\beta d \leq \alpha b$. If either:*
(i) $[\psi_{q,k}^{(m)}(a+bx)]^{\alpha} > 0$ and $[\psi_{q,k}^{(m)}(c+dx)]^{\beta} < 0$, or
(ii) $[\psi_{q,k}^{(m)}(a+bx)]^{\alpha} < 0$ and $[\psi_{q,k}^{(m)}(c+dx)]^{\beta} > 0$, then the inequalities

$$\frac{\left[\psi_{q,k}^{(m)}(a)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c)\right]^{\beta}} \leq \frac{\left[\psi_{q,k}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+dx)\right]^{\beta}} \leq \frac{\left[\psi_{q,k}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+d)\right]^{\beta}} \tag{11}$$

hold true for $x \in [0, 1]$. If either:
(i) $[\psi_{q,k}^{(m)}(a+bx)]^{\alpha} > 0$ and $[\psi_{q,k}^{(m)}(c+dx)]^{\beta} > 0$, or
(ii) $[\psi_{q,k}^{(m)}(a+bx)]^{\alpha} < 0$ and $[\psi_{q,k}^{(m)}(c+dx)]^{\beta} < 0$, then the inequalities (7) are satisfied.

Proof. Defined a function F by $F(x)=\frac{\left[\psi_{q,k}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{q,k}^{(m)}(c+dx)\right]^{\beta}}$, where m is a positive even integer. Then, $\psi_{q,k}^{(m)}(a+bx)<0$ and $\psi_{q,k}^{(m)}(c+d)<0$. Next,

$$\begin{aligned}F'(x)&=\frac{1}{[\psi_{q,k}^{(m)}(c+dx)]^{2\beta}}\Big[\alpha b\psi_{q,k}^{(m+1)}(a+bx)[\psi_{q,k}^{(m)}(a+bx)]^{\alpha-1}[\psi_{q,k}^{(m)}(c+dx)]^{\beta}\\&\quad-\beta d\psi_{q,k}^{(m+1)}(c+dx)[\psi_{q,k}^{(m)}(a+bx)]^{\alpha}[\psi_{q,k}^{(m)}(c+dx)]^{\beta-1}\Big]\\&=\frac{[\psi_{q,k}^{(m)}(a+bx)]^{\alpha}}{[\psi_{q,k}^{(m)}(c+dx)]^{\beta}}\left[\alpha b\frac{\psi_{q,k}^{(m+1)}(a+bx)}{\psi_{q,k}^{(m)}(a+bx)}-\beta d\frac{\psi_{q,k}^{(m+1)}(c+dx)}{\psi_{q,k}^{(m)}(c+dx)}\right]\geq 0\end{aligned}$$

resulting from Lemma 2.7. Thus, F is increasing on $x\in[0,\infty)$ and for $x\in[0,1]$ we have

$$F(0)\leq F(x)\leq F(1)$$

establishing the results.

Theorem 3.6. *Let $p\in\mathbb{N}$, $q\in(0,1)$ and $x\geq 0$. Suppose that m is a positive even integer, and a, b, c, d, α and β are positive real numbers such that $a+bx\leq c+dx$ and $\beta d\leq\alpha b$. If either:*
(i) $[\psi_{p,q}^{(m)}(a+bx)]^{\alpha}>0$ and $[\psi_{p,q}^{(m)}(c+dx)]^{\beta}<0$, or
(ii) $[\psi_{p,q}^{(m)}(a+bx)]^{\alpha}<0$ and $[\psi_{p,q}^{(m)}(c+dx)]^{\beta}>0$, then the inequalities

$$\frac{\left[\psi_{p,q}^{(m)}(a)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c)\right]^{\beta}}\leq\frac{\left[\psi_{p,q}^{(m)}(a+bx)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+dx)\right]^{\beta}}\leq\frac{\left[\psi_{p,q}^{(m)}(a+b)\right]^{\alpha}}{\left[\psi_{p,q}^{(m)}(c+d)\right]^{\beta}}\tag{12}$$

hold true for $x\in[0,1]$. If either:
(i) $[\psi_{p,q}^{(m)}(a+bx)]^{\alpha}>0$ and $[\psi_{p,q}^{(m)}(c+dx)]^{\beta}>0$, or
(ii) $[\psi_{p,q}^{(m)}(a+bx)]^{\alpha}<0$ and $[\psi_{p,q}^{(m)}(c+dx)]^{\beta}<0$, then the inequalities (9) are satisfied.

Proof. Omitted due to its resemblance with the proof of Theorem 3.5.

4. Concluding Remarks

In this section, we present some remarks concerning our results.

Remark 4.1. If we allow $q\to 1^{-}$ in Theorem 3.1, then we obtain identical results for the k-deformation of the Polygamma function.

Remark 4.2. If we set $k=1$ in Theorem 3.1, or if we allow $p\to\infty$ in Theorem 3.3, then we obtain identical results for the q-deformation of the Polygamma function.

Remark 4.3. If we allow $q\to 1^{-}$ in Theorem 3.3, then we obtain identical results for the k-deformation of the Polygamma function.

Remark 4.4. The p, q and k-deformations of the results in Theorem 3.5 and Theorem 3.6 are obtained by similar parameter adjustments.

By these remarks, the previous results have been extended and generalized.

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