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A REFINEMENT OF AN INEQUALITY FOR POSITIVE OPERATORS ON PSEUDO-HILBERT SPACES

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Abstract. In this paper, several inequalities for positive definite operators defined on pseudo-Hilbert spaces and Hilbert spaces respectively will be presented under suitable assumptions, starting from some refinements of the Kittaneh-Manasrah inequality which improves the well-known inequality of Young. The results will improve the inequalities presented before for positive operators in Hilbert spaces.

1. **Introduction**

In this paper, it is necessary to recall the following results which are given in the papers [4] and [5] and will be used below in the demonstration of our results. In demonstrations of the main section the same method as in the paper [3] will be utilized.

Lemma 1. *([4]) Let a and b be such that $a, b \geq 0$ and $0 \leq \nu \leq 1$. Then the following inequality holds:*

$$\nu a^2 + (1-\nu)b^2 \leq (a^\nu b^{1-\nu})^2 + s_0(a-b)^2,$$

where $s_0 = \max\{\nu, 1-\nu\}$.

Lemma 2. *([5]) For all x, y positive real numbers and $\lambda \in (0, 1,)$ we have the inequality*

$$2rE\left(x, y, \frac{1}{2}\right) \leq E(x, y, \lambda) \leq 2(1-r)E\left(x, y, \frac{1}{2}\right),$$

where

$$E(x, y, \lambda) = \lambda \exp x + (1-\lambda) \exp y - \exp(\lambda x + (1-\lambda)y) - \frac{\lambda(1-\lambda)}{2}(x-y)^2$$

and $r = \min\{\lambda, 1-\lambda\}$.

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Theorem 1. *([5]) For $a, b \geq 1$, and $\lambda \in (0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2 + A_1(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B_1(\lambda)\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ *and* $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

First, it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(H; < .,. >)$ as in [3] and the references therein. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;

(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [3] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$(1) \qquad f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A)$$

in the operator order of $B(H)$.

We recall the definitions of pseudo-Hilbert spaces (so called Loynes Z- spaces) and of the admissible spaces in the Loynes sense and then as in [2] we use the functional calculus with functions of the class C_1 in results which will be proved below.

A locally convex space Z- is called admissible in the Loynes sense if the following conditions are satisfied:

(A.1) Z- is complete;

(A.2) there is a closed convex cone in Z, denoted Z_+, defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);

(A.3) there is an involution in Z, $z \to z^*$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$) such that $z \in Z_+$ implies $z^* = z$;

(A.4) the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);

(A.6) any monotonously decreasing sequence in Z_+ is convergent.

Let Z be an admissible space in the Loynes sense. A topological linear space $\mathcal{H}$ is called pre-Loynes Z-space if it satisfies the following properties:

(L1) $\mathcal{H}$ is endowed with an Z- valued inner product (gramian), i.e. there exists an application $(h,k) \in \mathcal{H} \times \mathcal{H} \to [h,k] \in Z$ having the properties:

(G.1) $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$;

(G.2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G.3) $[\lambda h, k] = \lambda[h,k]$;

(G.4) $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbf{C}$.

(L.2) The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $h \in \mathcal{H} \to [h,h] \in Z$ is continous.

Moreover, if $\mathcal{H}$ is a complete spaces with this topology, then $\mathcal{H}$ is called Loynes Z- space (pseudo-Hilbert space).

Let $A \in \mathcal{B}_h^*(\mathcal{H})$, where $\mathcal{H}$ is now a pseudo-Hilbert space (so called Loynes Z-spaces) and

$$m_A := \sup\{\mu : \mu[h,h] \leq [Ah,h], h \in \mathcal{H}\},$$

$$M_A := \inf\{\nu : [Ah,h] \leq \nu[h,h], h \in \mathcal{H}\}.$$

We say that the function f is in the class C_1 and we denote $f \in C_1[m_A, M_A]$ if f is positive and superior semicontinuous on $[m_A, M_A]$.

We will denote by $\overline{\mathcal{A}(A)^t}$ the strong closing of $\mathcal{A}(A)$ in $\mathcal{B}^*(\mathcal{H})$.

The mapping

$$f : C_1[m_A, M_A] \to \overline{\mathcal{A}(A)^t}, \quad f \to f(A)$$

by which to a function $f \in C_1[m_A, M_A]$ we associate the gramian self-adjoint operator denoted by $f(A)$ and defined by $f(A) = \lim_{n\to\infty} p_n(A)$ where p_n is a decreasing sequence of polynomials p_n with $f(\lambda) = \lim_{n\to\infty} p_n(\lambda)$ for any $\lambda \in [m_A, M_A]$, is called functionals calculus with functions in class C_1.

Theorem 2. *(Lemma 2.1.1[2]) The functional calculus with functions of the class C_1 has the following immediate properties:*

(i) the mapping $f \to f(A)$ is monotone;

(ii) $f \to f(A)$ is function of positive type and positively homogeneous;

(iii) $f \to f(A)$ is additive and multiplicative(all these three properties being inherited by passing to the limit from the functional calculus with polynomials defined at the beginning);

(iv) In addition, the functional calculus with functions of the class C_1 extends the functional calculus with continuous and positive functions on $\sigma(A)$ defined in Corollary 1.5.6,

$$f : C_+(\sigma(A)) \to \mathcal{A}(A), \quad f \to f(A)$$

if $A \in \mathcal{B}_h^*(\mathcal{H})$.

Using the definition from [3], we say that the functions $f, g : [a,b] \to \mathbf{R}$ are *syncronous* (*asyncronous*) on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for each $t, s \in [a,b]$.

2. **Main results**

The following results present several inequalities for functions of positive operators when we have a classic inner product which takes its values in **C** and the second inner product (gramian) takes its values in an admissible space Z.

Proposition 1. *Let A be a positive definite operator on the Hilbert space H, $A \in B(H)$ and B a positive operator on the pseudo-Hilbert space K, $B \in \mathcal{B}_h^*(\mathcal{K})$. Then we have*

$$\nu < A^2x, x > [y, y] + (1-\nu) < x, x > [B^2y, y] \leq$$
$$\leq < A^{2\nu}x, x > [B^{2(1-\nu)}y, y] + s_0(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y]),$$

for each $x \in H$ and $y \in \mathcal{K}$, where $0 \leq \nu \leq 1$ and $s_0 = \max\{\nu, 1-\nu\}$.

Proof. We consider the continue function $f(a) = (a^\nu b^{1-\nu})^2 + s_0(a-b)^2 - (\nu a^2 + (1-\nu)b^2)$, which is positive for $a \geq 0$ and we fix $b \geq 0$ and then by the property (1) for each $x \in H$ we have that

$$< (\nu A^2 + (1-\nu)b^2 I)x, x > \leq < [A^{2\nu}b^{2(1-\nu)} + s_0(A^2 - 2Ab + b^2 I)]x, x >$$

which is equivalent with

$$\nu < A^2x, x > +(1-\nu)b^2 < x, x > \leq$$
$$\leq b^{2(1-\nu)} < A^{2\nu}x, x > +s_0[< A^2x, x > -2b < Ax, x > +b^2 < x, x >]$$

for each $b > 0$.

If we apply now Theorem 2 for last inequality taking into account that

$$\nu < A^2x, x > +(1-\nu)b^2 < x, x > \geq 0,$$

then for any $y \in \mathcal{K}$ we get

$$[(\nu < A^2x, x > +(1-\nu) < x, x > B^2)y, y] \leq$$
$$\leq [(B^{2(1-\nu)} < A^{2\nu}x, x > +s_0(< A^2x, x > I_K - 2B < Ax, x > +B^2 < x, x >))y, y]$$

and this inequality is equivalent with

$$\nu < A^2x, x > [y, y] + (1-\nu) < x, x > [B^2y, y] \leq$$
$$\leq < A^{2\nu}x, x > [B^{2(1-\nu)}y, y] + s_0(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])$$

for each $x \in H$ and $y \in \mathcal{K}$.

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Theorem 3. *Let A be a positive definite operator on the Hilbert space H, $A \in B(H)$ and B a positive operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in \mathcal{B}_h^*(\mathcal{K})$. Then the following inequality holds:*

$$r\{< \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] - 2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] -$$
$$-\frac{1}{4}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])\} \leq$$
$$\leq \lambda < \exp(A)x, x > [y, y] + (1-\lambda) < x, x > [\exp(B)y, y] - < \exp(\lambda A)x, x > \cdot$$
$$\cdot [\exp((1-\lambda)B)y, y] - \frac{\lambda(1-\lambda)}{2}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y]) \leq$$
$$\leq (1-r)\{< \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] - 2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] -$$

$$-\frac{1}{4}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])\}.$$

for each $x \in H$ *and* $y \in \mathcal{K}$ *where* $r = \min\{\lambda, 1-\lambda\}$.

Proof. We write and then use the inequality from Lemma 2 with x replaced by a and y replaced by b obtaining:

$$r\left[\exp(a) + \exp(b) - 2\exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2\right] \leq$$

$$\leq \lambda\exp(a) + (1-\lambda)\exp(b) - \exp(\lambda a + (1-\lambda)b) - \frac{\lambda(1-\lambda)}{2}(a-b)^2 \leq$$

$$\leq (1-r)\left[\exp(a) + \exp(b) - 2\exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2\right].$$

We fix $b > 0$ and apply the property (1) for previous inequality obtaining :

$$< r[\exp(A) + \exp(b)1_H - 2\exp(\frac{b}{2})\exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)]x, x > \leq$$

$$\leq < [\lambda\exp(A) + (1-\lambda)\exp(b)1_H - \exp(\lambda A)\exp((1-\lambda)b) - \frac{\lambda(1-\lambda)}{2}(A^2 - 2bA + b^2 1_H)]x, x >$$

$$\leq < (1-r)[\exp(A) + \exp(b)1_H - 2\exp(\frac{b}{2})\exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)]x, x >$$

which is equivalent with the following

$$r[< \exp(A)x, x > + \exp(b) < x, x > -2\exp(\frac{b}{2}) < \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2b < Ax, x > + b^2 < x, x >)] \leq$$

$$\leq \lambda < \exp(A)x, x > + (1-\lambda)\exp(b) < x, x > - \exp((1-\lambda)b) < \exp(\lambda A)x, x > -$$

$$-\frac{\lambda(1-\lambda)}{2}(< A^2x, x > -2b < Ax, x > + b^2 < x, x >) \leq$$

$$\leq (1-r)[< \exp(A)x, x > + \exp(b) < x, x > -2\exp(\frac{b}{2}) < \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2b < Ax, x > + b^2 < x, x >)],$$

for any $x \in H$.

If we apply Theorem 2 for previous inequality for the variable b, then we have for any $y \in \mathcal{K}$ that

$$r\{< \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] - 2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] -$$

$$-\frac{1}{4}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])\} \leq$$

$$\leq \lambda < \exp(A)x, x > [y, y] + (1-\lambda) < x, x > [\exp(B)y, y] - < \exp(\lambda A)x, x > [\exp((1-\lambda)B)y, y] -$$

$$-\frac{\lambda(1-\lambda)}{2}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y]) \leq$$

$$\leq (1-r)\{< \exp(A)x, x > [y, y] + < x, x > [\exp(B)y, y] - 2 < \exp(\frac{A}{2})x, x > [\exp(\frac{B}{2})y, y] -$$

$$-\frac{1}{4}(< A^2x, x > [y, y] - 2 < Ax, x > [By, y] + < x, x > [B^2y, y])\}.$$

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Proposition 2. *Let A be a positive definite operators on the Hilbert space H, $A \in B(H)$ and B a positive operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in \mathcal{B}_h^*(\mathcal{K})$. If $Sp(A)$, $Sp(B) \subseteq [1, \infty)$, and $\lambda \in (0,1)$ then we have*

$$r\Big(< Ax, x > [y, y]+ < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y]\Big) +$$
$$+A_1(\lambda)\left[< (\log^2 A)x, x > [y, y]+ < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y]\right] \leq$$
$$\leq \lambda < Ax, x > [y, y] + (1-\lambda) < x, x > [By, y]- < A^{\lambda}x, x > [B^{1-\lambda}y, y] \leq$$
$$\leq (1-r)\Big(< Ax, x > [y, y]+ < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y]\Big) +$$
$$+B_1(\lambda)\left[< (\log^2 A)x, x > y, y+ < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y]\right]$$

for each $x \in H$ and $y \in K$ where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

Proof. We consider the continue and positive functions $f(a) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - r(a + b - 2a^{\frac{1}{2}}b^{\frac{1}{2}}) - A_1(\lambda)[\log^2 a + \log^2 b - 2\log a \log b]$, and $g(a) = (1-r)(a + b - 2a^{\frac{1}{2}}b^{\frac{1}{2}}) + B_1(\lambda)[\log^2 a + \log^2 b - 2\log a \log b] - \lambda a - (1-\lambda)b + a^{\lambda}b^{1-\lambda}$ which are positive for $a \geq 1$ and we fix $b \geq 1$ and then by the property (1) for each $x \in H$ we have that

$$r\Big(< Ax, x > +b < x, x > -2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >\Big) +$$
$$+A_1(\lambda)\left[< (\log^2 A)x, x > + \log^2 b < x, x > -2\log b < (\log A)x, x >\right] \leq$$
$$\leq \lambda < Ax, x > +(1-\lambda)b < x, x > -b^{1-\lambda} < A^{\lambda}x, x > \leq$$
$$\leq (1-r)\Big(< Ax, x > +b < x, x > -2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >\Big) +$$
$$+B_1(\lambda)\left[< (\log^2 A)x, x > + \log^2 b < x, x > -2\log b < (\log A)x, x >\right]$$

for each $b > 1$.

If we apply now Theorem 2 for last inequality, then for any $y \in \mathcal{K}$ we get

$$r\Big(< Ax, x > [y, y]+ < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y]\Big) +$$
$$+A_1(\lambda)\left[< (\log^2 A)x, x > [y, y]+ < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y]\right] \leq$$
$$\leq \lambda < Ax, x > [y, y] + (1-\lambda) < x, x > [By, y]- < A^{\lambda}x, x > [B^{1-\lambda}y, y] \leq$$
$$\leq (1-r)\Big(< Ax, x > [y, y]+ < x, x > [By, y] - 2 < A^{\frac{1}{2}}x, x > [B^{\frac{1}{2}}y, y]\Big) +$$
$$+B_1(\lambda)\left[< (\log^2 A)x, x > [y, y]+ < x, x > [(\log^2 B)y, y] - 2 < (\log A)x, x > [(\log B)y, y]\right]$$

for each $x \in H$ and $y \in \mathcal{K}$.

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Next particular case of Proposition 2 may be of interest as well:

Remark 1. *Under previous conditions, if we consider $\mathcal{K}$ as being the Hilbert space H, $y = x$ and $A = B$ then the above inequality becomes:*

$$2r\left[< Ax, x > - \Big(< A^{\frac{1}{2}}x, x >\Big)^2\right] + 2A_1(\lambda)\left[< (\log^2 A)x, x > - (< (\log A)x, x >)^2\right] \leq$$
$$\leq 1- < A^{1-\lambda}x, x >< A^{\lambda}x, x > \leq$$
$$2(1-r)\left[< Ax, x > - \Big(< A^{\frac{1}{2}}x, x >\Big)^2\right] + 2B_1(\lambda)\left[< (\log^2 A)x, x > - (< (\log A)x, x >)^2\right].$$

In the following, we can think to rewrite some results as Theorem 1, see [3], in the case when we have an inner product which take its values in **C** and the second takes values in an admissible space Z.

Theorem 4. *Let A be a selfadjoint operator on the Hilbert space H, $A \in B(H)$ and B a selfadjoint operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in \mathcal{B}^*(\mathcal{K})$ with $Sp(A)$, $Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbf{R}$ are continuous, positive and syncronous (asyncronous) on $[m, M]$, then*

$$< f(A)g(A)x, x > [y, y] + < x, x > [f(B)g(B)y, y] \geq (\leq)$$

$$< f(A)x, x > [g(B)y, y] + < g(A)x, x > [f(B)y, y],$$

for any $x \in H$ and $y \in \mathcal{K}$.

Proof. We will use the same method as in [3] and we take into account only the case of syncronous functions. Therefore, if we fix $s \in [m, M]$ and apply (1) for inequality

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t)$$

where $t, s \in [m, M]$ then we have for each $x \in H$ that

$$< f(A)g(A)x, x > + f(s)g(s) < x, x > \geq g(s) < f(A)x, x > + f(s) < g(A)x, x >$$

for each $s \in [m, M]$.

Now if we apply Theorem 2 for previous inequality

$$< f(A)g(A)x, x > I_{\mathcal{K}} + f(B)g(B) < x, x > \geq g(B) < f(A)x, x > + f(B) < g(A)x, x >,$$

and then we have for any $y \in \mathcal{K}$ that

$$[(< f(A)g(A)x, x > I_{\mathcal{K}} + f(B)g(B) < x, x >)y, y] \geq$$

$$\geq [(g(B) < f(A)x, x > + f(B) < g(A)x, x >)y, y]$$

or by calculus

$$< f(A)g(A)x, x > [y, y] + < x, x > [f(B)g(B)y, y] \geq$$

$$\geq < f(A)x, x > [g(B)y, y] + < g(A)x, x > [f(B)y, y].$$

∎

Remark 2. *Under previous conditions, if we consider $\mathcal{K}$ as being the Hilbert space H, $y = x$ and $A = B$ then the above inequality becomes:*

$$< x, x >< f(A)g(A)x, x > \geq (\leq) < f(A)x, x >< g(A)x, x > .$$

Remark 3. *Under previous conditions, if we consider $\mathcal{K}$ as being the Hilbert space H, $y = x$ and $A = B$ then the above inequality becomes:*

$$< f(A)g(A)x, x > \geq (\leq) < f(A)x, x >< g(A)x, x >,$$

for any $x \in H$ with $||x|| = 1$.

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space H, $A \in B(H)$ and B a selfadjoint operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in \mathcal{B}^*(\mathcal{K})$ with $Sp(A)$, $Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbf{R}_+$ are continuous, positive and syncronous on $[m, M]$, then*

$$[f(B)g(B)y, y] + f(< Ax, x >)g(< Ax, x >)[y, y] \geq$$
$$\geq f(< Ax, x >)[g(B)y, y] + g(< Ax, x >)[f(B)y, y]$$

for any $x \in H$ and $y \in \mathcal{K}$.

Proof. As in [3], using the hypothesis that f, g are syncronous and $m \leq< Ax, x >\leq M$ for any $x \in H$ with $||x|| = 1$ we have

$$(f(t) - f(< Ax, x >)) (g(t) - g(< Ax, x >)) \geq 0$$

for any $t \in [a, b]$ or by calculus

$$f(t)g(t) + f(< Ax, x >)g(< Ax, x >) \geq g(t)f(< Ax, x >) + f(t)g(< Ax, x >).$$

By functional calculus, Theorem 2 we find:

$$f(B)g(B)+f(< Ax, x >)g(< Ax, x >)I_{\mathcal{K}} \geq g(B)f(< Ax, x >)+f(B)g(< Ax, x >)$$

and from here,

$$[f(B)g(B)y, y] + f(< Ax, x >)g(< Ax, x >)[y, y] \geq$$
$$\geq f(< Ax, x >)[g(B)y, y] + g(< Ax, x >)[f(B)y, y],$$

for any $y \in \mathcal{K}$.

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