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SOME MULTIPLICATIVE REVERSES OF CALLEBAUT AND HÖLDER INEQUALITIES FOR ISOTONIC FUNCTIONALS

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Abstract. In this paper we obtain some multiplicative reverses of Callebaut and Hölder inequalities for isotonic functionals via a reverse of Young's inequality we have established recently. Applications for integrals and n-tuples of real numbers are provided as well.

1. Introduction

In the recent paper [9] we established the following multiplicative refinement and reverse of the celebrated Young's inequality for positive numbers a, b with $\frac{b}{a}\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$

$$\begin{cases} g_\nu(M)\,a^{1-\nu}b^\nu \text{ if } M<1,\\ a^{1-\nu}b^\nu \text{ if } m\le 1\le M,\\ g_\nu(m)\,a^{1-\nu}b^\nu \text{ if } 1<m.\end{cases} \tag{1.1}$$

$$\le(1-\nu)\,a+\nu b$$

$$\le\begin{cases} g_\nu(m)\,a^{1-\nu}b^\nu \text{ if } M<1,\\ \max\{g_\nu(m),g_\nu(M)\}\,a^{1-\nu}b^\nu \text{ if } m\le 1\le M,\\ g_\nu(M)\,a^{1-\nu}b^\nu \text{ if } 1<m,\end{cases},$$

where

$$g_\nu(x)=\frac{1-\nu+\nu x}{x^\nu}=(1-\nu)\,x^{-\nu}+\nu x^{1-\nu},\ x>0. \tag{1.2}$$

We recall that *Specht's ratio* is defined by [22]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ 1 \text{ if } h=1.\end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

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The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.4}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (1.4) is due to Tominaga [24] while the first one is due to Furuichi [13].

On making use of (1.1) and (1.4) we have the following lower and upper bounds in terms of Specht's ratio:

$$\begin{cases} S(M^{r})a^{1-\nu}b^{\nu} \text{ if } M<1,\\ \\ a^{1-\nu}b^{\nu} \text{ if } m\leq 1\leq M,\\ \\ S(m^{r})a^{1-\nu}b^{\nu} \text{ if } 1<m.\end{cases} \tag{1.5}$$

$$\leq(1-\nu)a+\nu b$$

$$\leq\begin{cases} S(m)a^{1-\nu}b^{\nu} \text{ if } M<1,\\ \\ \max\{S(m),S(M)\}a^{1-\nu}b^{\nu} \text{ if } m\leq 1\leq M,\\ \\ S(M)a^{1-\nu}b^{\nu} \text{ if } 1<m,\end{cases},$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

We consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ \ h>0. \tag{1.6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds.

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.7}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.7) was obtained by Zou et al. in [25] while the second by Liao et al. [16].

By making use of (1.1) and (1.4) we have the following lower and upper bounds in terms of Kantorovich's constant:

$$\begin{cases} K^r(M)\,a^{1-\nu}b^\nu \text{ if } M<1, \\ \\ a^{1-\nu}b^\nu \text{ if } m\le 1\le M, \\ \\ K^r(m)\,a^{1-\nu}b^\nu \text{ if } 1<m. \end{cases} \tag{1.8}$$

$$\le (1-\nu)\,a+\nu b$$

$$\le \begin{cases} K^R(m)\,a^{1-\nu}b^\nu \text{ if } M<1, \\ \\ \max\left\{K^R(m),K^R(M)\right\}a^{1-\nu}b^\nu \text{ if } m\le 1\le M, \\ \\ K^R(M)\,a^{1-\nu}b^\nu \text{ if } 1<m, \end{cases},$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

Let L be a *linear class* of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) f, $g\in L$ imply $(\alpha f+\beta g)\in L$ for all α, $\beta\in\mathbb{R}$;
(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1$, $t\in E$ then $f_0\in L$.

An *isotonic linear functional* $A:L\to\mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all f, $g\in L$ and α, $\beta\in\mathbb{R}$.
(A2) If $f\in L$ and $f\ge 0$, then $A(f)\ge 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1})=1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [18] and [19]). For other inequalities for isotonic functionals see [1], [4]-[17] and [20]-[23].

We note that common examples of such isotonic linear functionals A are given by

$$A(g)=\int_E g\,d\mu \text{ or } A(g)=\sum_{k\in E}p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k\ge 0$, $k\in E$).

In this paper we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via the reverse of Young's inequality obtained in (1.1). Applications for integrals and n-tuples of real numbers are provided as well.

2. Reverses of Callebaut's Inequality

The functional version of *Callebaut's inequality* states that

$$A^2(fg)\le A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right)\le A\left(f^2\right)A\left(g^2\right) \tag{2.1}$$

provided that f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}$, $fg\in L$ for some $\nu\in[0,1]$. For the discrete and integral of one real variable versions see [3].

We start with the following result:

Theorem 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.2}$$

for some constants m, M, then

$$\begin{aligned}(1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) \\ \leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right),\end{aligned} \tag{2.3}$$

where g_ν is defined by (1.2).

In particular,

$$\begin{aligned}A\left(f^2\right) A\left(g^2\right) \leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} \\ \times A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right).\end{aligned} \tag{2.4}$$

Proof. For any x, $y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

Consider

$$a = \frac{f^2(x)}{g^2(x)}, \; b = \frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\left(\frac{m}{M}\right)^2, \left(\frac{M}{m}\right)^2\right]$ and by the inequality (1.1) we have

$$\begin{aligned}(1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} \\ \leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\end{aligned} \tag{2.5}$$

for any x, $y \in E$.

Now, if we multiply (2.5) by $g^2(x)g^2(y) > 0$ then we get

$$\begin{aligned}(1-\nu) g^2(y) f^2(x) + \nu f^2(y) g^2(x) \\ \leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^{2(1-\nu)}(x) g^{2\nu}(x) f^{2\nu}(y) g^{2(1-\nu)}(y)\end{aligned}$$

for any x, $y \in E$.

Fix $y \in E$. Then by (2.6) we have in the order of L that

$$\begin{aligned}(1-\nu) g^2(y) f^2 + \nu f^2(y) g^2 \\ \leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^{2\nu}(y) g^{2(1-\nu)}(y) f^{2(1-\nu)} g^{2\nu}.\end{aligned} \tag{2.6}$$

If we take the functional A in (2.6) then we get

$$\begin{aligned}&(1-\nu) g^2(y) A(f^2) + \nu f^2(y) A(g^2)\\ &\leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^{2\nu}(y) g^{2(1-\nu)}(y) A\left(f^{2(1-\nu)} g^{2\nu}\right)\end{aligned}$$

for any $y \in E$.

This inequality can be written in the order of L as

$$\begin{aligned}&(1-\nu) A(f^2) g^2 + \nu A(g^2) f^2\\ &\leq \max\left\{g_\nu\left(\left(\frac{m}{M}\right)^2\right), g_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^{2(1-\nu)} g^{2\nu}\right) f^{2\nu} g^{2(1-\nu)}.\end{aligned} \tag{2.7}$$

Now, if we take the functional B in (2.7), then we get the desired result (2.3). □

On using the inequality (2.3) and (1.5) we get the following reverse of Callebaut inequality in terms of Specht's ratio for $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ satisfying the condition (2.2):

$$\begin{aligned}&(1-\nu) A(f^2) B(g^2) + \nu A(g^2) B(f^2)\\ &\leq S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right),\end{aligned} \tag{2.8}$$

where $\nu \in [0,1]$ and since

$$S\left(\left(\frac{m}{M}\right)^2\right) = S\left(\left(\frac{M}{m}\right)^2\right)$$

In particular,

$$A(f^2) A(g^2) \leq S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right). \tag{2.9}$$

On using the inequality (2.3) and (1.8) we get the following reverse of Callebaut inequality in terms of Kantorovich's constant for $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ satisfying the condition (2.2):

$$\begin{aligned}&(1-\nu) A(f^2) B(g^2) + \nu A(g^2) B(f^2)\\ &\leq K^R\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right),\end{aligned} \tag{2.10}$$

where $\nu \in [0,1]$, $R = \max\{1-\nu, \nu\}$ and since

$$K^R\left(\left(\frac{m}{M}\right)^2\right) = K^R\left(\left(\frac{M}{m}\right)^2\right).$$

In particular,

$$A(f^2) A(g^2) \leq K^R\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right). \tag{2.11}$$

3. Reverses of Hölder's Inequality

We have the following additive reverse of Hölder's inequality:

Theorem 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g : E \to \mathbb{R}$ are such that $fg, f^p, g^q \in L$ and*

$$0 < m_1 \le f \le M_1 < \infty, \; 0 < m_2 \le g \le M_2 < \infty, \tag{3.1}$$

then

$$\begin{aligned}(1 \le) \frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)} \\ \le \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1} \right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right) \right\},\end{aligned} \tag{3.2}$$

where $g_{\frac{1}{p}}$ is defined by

$$g_{\frac{1}{p}}(x) = \frac{1}{q}x^{-\frac{1}{p}} + \frac{1}{p}x^{\frac{1}{q}}, \; x > 0. \tag{3.3}$$

Proof. Observe that, by (3.1) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p \text{ and } \left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (1.1) for $b = \frac{f^p}{A(f^p)}$, $a = \frac{g^q}{A(g^q)}$, $\nu = \frac{1}{p}$, $M = \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q$ and $m = \left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1}$ we have

$$\begin{aligned}\frac{1}{q}\frac{g^q}{A(g^q)} + \frac{1}{p}\frac{f^p}{A(f^p)} \\ \le \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1} \right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right) \right\} \\ \times \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}.\end{aligned} \tag{3.4}$$

If we take the functional A in (3.4), then we get

$$\begin{aligned}\frac{1}{q}\frac{A(g^q)}{A(g^q)} + \frac{1}{p}\frac{A(f^p)}{A(f^p)} \\ \le \max\left\{ g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right]^{-1} \right), g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q \right) \right\} \\ \times \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}},\end{aligned}$$

which is equivalent to the desired result (3.2). □

On using the inequality (3.2) and (1.5) we get the following reverse of Hölder's inequality in terms of Specht's ratio for $f \geq 0$, $g > 0$, fg, f^p, $g^q \in L$ satisfying the condition (3.1):

$$(1 \leq) \frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)} \leq S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right),$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and since

$$S\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right) = S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right).$$

On using the inequality (3.2) and (1.5) we get the following reverse of Hölder's inequality in terms of Kantorovich's constant for $f \geq 0$, $g > 0$, fg, f^p, $g^q \in L$ satisfying the condition (3.1):

$$(1 \leq) \frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)} \leq K^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right),$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and since

$$K^T\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right) = K^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right).$$

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\, w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.1}$$

If f^2, $g^2 \in L_w(\Omega, \mu)$, then by (2.4) we have

$$\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu \leq \max\left\{g_s\left(\left(\frac{m}{M}\right)^2\right), g_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \times \int_\Omega wf^{2(1-s)}g^{2s} d\mu \int_\Omega wf^{2s}g^{2(1-s)} d\mu, \tag{4.2}$$

for any $s \in [0, 1]$, where g_s is defined by (1.2).

From (4.2) we have

$$\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu \leq S\left(\left(\frac{M}{m}\right)^2\right) \int_\Omega wf^{2(1-s)}g^{2s} d\mu \int_\Omega wf^{2s}g^{2(1-s)} d\mu, \tag{4.3}$$

and

$$\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu\le K^R\left(\left(\frac{M}{m}\right)^2\right)\int_\Omega wf^{2(1-s)}g^{2s}d\mu\int_\Omega wf^{2s}g^{2(1-s)}d\mu \tag{4.4}$$

for any $s\in[0,1]$.

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty\ \mu\text{-a.e. on }\Omega. \tag{4.5}$$

Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (3.2) we have the following reverse of Hölder's inequality

$$\begin{aligned}(1\le)\,&\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\\ &\le\max\left\{g_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right),g_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}\end{aligned} \tag{4.6}$$

where $g_{\frac{1}{p}}$ is defined by (3.3).

From (4.6) we have

$$(1\le)\,\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right) \tag{4.7}$$

and

$$(1\le)\,\frac{\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le K^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right) \tag{4.8}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ and the probability distribution $p=(p_1,...,p_n)$, i.e. $p_i\ge0$ for any $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

If there exist the constants m, $M>0$ such that

$$0<m\le\frac{a_i}{b_i}\le M<\infty\text{ for any }i\in\{1,...,n\},$$

then by (4.2), for the counting discrete measure, we have

$$\begin{aligned}\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2&\le\max\left\{g_s\left(\left(\frac{m}{M}\right)^2\right),g_s\left(\left(\frac{M}{m}\right)^2\right)\right\}\\ &\quad\times\sum_{i=1}^n p_ia_i^{2(1-s)}b_i^{2s}\sum_{i=1}^n p_ia_i^{2s}b_i^{2(1-s)},\end{aligned} \tag{5.1}$$

for any $s\in[0,1]$, where g_s is defined by (1.2).

From (5.1) we have

$$\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2\le S\left(\left(\frac{M}{m}\right)^2\right)\sum_{i=1}^n p_ia_i^{2(1-s)}b_i^{2s}\sum_{i=1}^n p_ia_i^{2s}b_i^{2(1-s)} \tag{5.2}$$

and

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 \leq K^R \left(\left(\frac{M}{m} \right)^2 \right) \sum_{i=1}^{n} p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^{n} p_i a_i^{2s} b_i^{2(1-s)} \tag{5.3}$$

for any $s \in [0,1]$.

If there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq a_i \leq M_1 < \infty,\ 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1, ..., n\} \tag{5.4}$$

and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.6) we have the following reverse of Hölder's inequality

$$\begin{aligned} (1 \leq) & \frac{\left(\sum_{i=1}^{n} p_i a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q \right)^{1/q}}{\sum_{i=1}^{n} p_i a_i b_i} \\ & \leq \max \left\{ g_{\frac{1}{p}} \left(\left[\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right]^{-1} \right), g_{\frac{1}{p}} \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right) \right\} \end{aligned} \tag{5.5}$$

where $g_{\frac{1}{p}}$ is defined by (3.3).

From (5.5) we have

$$(1 \leq) \frac{\left(\sum_{i=1}^{n} p_i a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q \right)^{1/q}}{\sum_{i=1}^{n} p_i a_i b_i} \leq S \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right) \tag{5.6}$$

and

$$(1 \leq) \frac{\left(\sum_{i=1}^{n} p_i a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q \right)^{1/q}}{\sum_{i=1}^{n} p_i a_i b_i} \leq K^T \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^q \right) \tag{5.7}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

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