Received 18/02/15

SOME GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities in inner product spaces $(H,\langle\cdot,\cdot\rangle)$ that provide upper bounds for the quantities

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right| \text{ and } \left|\langle x,y\rangle-\frac{1}{2}\left[\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle\right]\right|,$$

where $e,f\in H$ with $\|e\|=\|f\|=1$ and x,y are vectors in H satisfying some appropriate assumptions are given. Applications for discrete and integral inequalities are provided as well.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\|\geq\left|\langle x,y\rangle\right| \text{ for any } x,y\in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

In 1985 the author [4] (see also [19]) established the following refinement of (1.1):

$$\|x\|\,\|y\|\geq\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|+\left|\langle x,e\rangle\langle e,y\rangle\right|\geq\left|\langle x,y\rangle\right| \tag{1.2}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Using the triangle inequality for modulus we have

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\geq\left|\langle x,e\rangle\langle e,y\rangle\right|-\left|\langle x,y\rangle\right|$$

and by (1.2) we get

$$\begin{aligned}\|x\|\,\|y\| &\geq \left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|+\left|\langle x,e\rangle\langle e,y\rangle\right|\\ &\geq 2\left|\langle x,e\rangle\langle e,y\rangle\right|-\left|\langle x,y\rangle\right|,\end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\|x\|\,\|y\|+\left|\langle x,y\rangle\right|\right]\geq\left|\langle x,e\rangle\langle e,y\rangle\right| \tag{1.3}$$

that holds for any $x,y,e\in H$ with $\|e\|=1$.

In [5], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

1991 *Mathematics Subject Classification.* 46C05; 26D15; 26D10.

Key words and phrases. Inner product spaces, Schwarz's inequality, Buzano's inequality, Grüss' inequality.

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{1.4}$$

hold, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| . \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

For other Schwarz, Buzano and Grüss related inequalities in inner product spaces, see [1]-[3], [4]-[13], [17]-[20], [22]-[29], and the monographs [14], [15] and [16].

Motivated by the above results, we establish in this paper other upper bounds for the quantities

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \text{ and } \left| \langle x, y \rangle - \frac{1}{2} [\langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle e, y \rangle] \right|$$

provided $e, f \in H$ with $\|e\| = \|f\| = 1$ and x, y are vectors in H satisfying some appropriate assumptions.

Natural applications for discrete inequalities, power series and integral inequalities are also given.

2. Main Results

The following results hold:

Theorem 2. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e \in H$ with $\|e\| = 1$, then*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \min \left\{ \|x\| \left(\|y\|^2 - |\langle y, e \rangle|^2 \right)^{1/2}, \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| \right\} \\ & \leq \frac{1}{2} \left[\|x\| \left(\|y\|^2 - |\langle y, e \rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| \right] \\ & \leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e \rangle|^2 - |\langle y, e \rangle|^2 \right)^{1/2} . \end{aligned} \tag{2.1}$$

The inequalities are sharp.

Proof. Using Schwarz inequality we have

$$\|x\| \|y - \langle y, e \rangle e\| \geq |\langle x, y - \langle y, e \rangle e \rangle| = |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \tag{2.2}$$

and

$$\|x - \langle x, e \rangle e\| \|y\| \geq |\langle x - \langle x, e \rangle e, y \rangle| = |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \tag{2.3}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since

$$\|x - \langle x, e \rangle e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2 , \quad \|y - \langle y, e \rangle e\|^2 = \|y\|^2 - |\langle y, e \rangle|^2$$

then by (2.2) and (2.3) we get

$$\min\left\{\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2},\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right\} \geq |\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \tag{2.4}$$

for any $x,y,e\in H$ with $\|e\|=1$.

This proves the first inequality in (2.1).

Using the elementary inequality

$$\frac{1}{2}(a+b)\geq\min\{a,b\}$$

that holds for any real numbers $a,b\in\mathbb{R}$, we have the second inequality in (2.1).

By the Cauchy-Bunyakovsky-Schwarz inequality

$$ac+bd\leq\left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2}\text{ for }a,b,c,d\geq 0 \tag{2.5}$$

we have

$$\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\| \leq\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-|\langle x,e\rangle|^2-|\langle y,e\rangle|^2\right)^{1/2} \tag{2.6}$$

for any $x,y,e\in H$ with $\|e\|=1$.

This proves the last part of (2.1).

Observe that if we take in (2.1) $y=x$, then we get from all inequalities that

$$\|x\|^2-|\langle x,e\rangle|^2\leq\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|x\|,$$

which is sharp since for $x\perp y$, $\langle x,e\rangle=0$ it reduces to an equality. □

Remark 1. *If we use the triangle inequality*

$$|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|\leq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|$$

then we get from (2.1)

$$|\langle x,e\rangle\langle e,y\rangle|\leq|\langle x,y\rangle| +\min\left\{\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2},\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right\} \tag{2.7}$$

for any $x,y,e\in H$ with $\|e\|=1$.

The following lemma holds, see [6]:

Lemma 1. *Let a,x,A be vectors in the inner product space $(H,\langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ with $a\neq A$. Then*

$$\operatorname{Re}\langle A-x,x-a\rangle\geq 0 \tag{2.8}$$

if and only if

$$\left\|x-\frac{a+A}{2}\right\|\leq\frac{1}{2}\|A-a\|. \tag{2.9}$$

Proof. Define

$$I_1 := \operatorname{Re}\langle A - x, x - a\rangle \text{ and } I_2 := \frac{1}{4}\|A - a\|^2 - \left\| x - \frac{a + A}{2}\right\|^2 .$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}[\langle x, a\rangle + \langle A, x\rangle] - \operatorname{Re}\langle A, a\rangle - \|x\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$ showing the required equivalence. □

The following corollary is obvious:

Corollary 1. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta, \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0 \tag{2.10}$$

iff

$$\left\| x - \frac{\delta + \Delta}{2} \cdot e \right\| \leq \frac{1}{2}|\Delta - \delta| . \tag{2.11}$$

Remark 2. *If $H = \mathbb{C}$, then $\operatorname{Re}[(A - x)(\bar{x} - \bar{a})] \geq 0$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}|A - a|$, where $a, x, A \in \mathbb{C}$. If $H = \mathbb{R}$, and $A > a$ then $a \leq x \leq A$ if and only if $\left|x - \frac{a+A}{2}\right| \leq \frac{1}{2}(A - a)$.*

The following lemma is of interest [6].

Lemma 2. *Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation*

$$\|x\|^2 - |\langle x, e\rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2 \geq 0. \tag{2.12}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned}\langle x - \lambda e, x - \langle x, e\rangle e\rangle &= \|x\|^2 - |\langle x, e\rangle|^2 - \lambda\left[\langle e, x\rangle - \langle e, x\rangle \|e\|^2\right] \\ &= \|x\|^2 - |\langle x, e\rangle|^2 .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2 - |\langle x, e\rangle|^2\right]^2 &= |\langle x - \lambda e, x - \langle x, e\rangle e\rangle|^2 \leq \|x - \lambda e\|^2 \|x - \langle x, e\rangle e\|^2 \\ &= \|x - \lambda e\|^2 \left[\|x\|^2 - |\langle x, e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \|x - \lambda e\|^2 , \qquad \lambda \in \mathbb{K}. \tag{2.13}$$

Taking the infimum in (2.13) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2 .$$

Since, for $\lambda_0 = \langle x, e\rangle$, we get $\|x - \lambda_0 e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2$, then the representation (2.12) is proved. □

The following result also holds:

Corollary 2. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e \rangle \geq 0, \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{2.14}$$

hold, or, equivalently, the following assumptions

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{2.15}$$

are valid, then one has the inequality

$$\begin{aligned} |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| &\leq \frac{1}{2} \min \{ \|x\| |\Gamma - \gamma|, |\Phi - \varphi| \|y\| \} \\ &\leq \frac{1}{4} [\|x\| |\Gamma - \gamma| + |\Phi - \varphi| \|y\|] \\ &\leq \frac{1}{4} \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(|\Phi - \varphi|^2 + |\Gamma - \gamma|^2 \right)^{1/2}. \end{aligned} \tag{2.16}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Proof. Using the inequality (2.1) and Lemma 2 we have

$$\begin{aligned} &|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ &\leq \min \left\{ \|x\| \left(\|y\|^2 - |\langle y, e \rangle|^2 \right)^{1/2}, \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| \right\} \\ &= \min \left\{ \|x\| \inf_{\eta \in \mathbb{K}} \|y - \eta e\|, \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\| \|y\| \right\} \\ &\leq \frac{1}{2} \min \{ \|x\| |\Gamma - \gamma|, |\Phi - \varphi| \|y\| \}, \end{aligned}$$

which proves the first inequality in (2.16).

The rest follows as in the proof of Theorem 2.

For the sharpness of the constants, we take $y = x$ in (2.16) to get

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{2} \|x\| |\Phi - \varphi|$$

provided

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|.$$

Moreover, if we take $\varphi = -\Phi$, then we have the inequality

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \|x\| |\Phi| \tag{2.17}$$

provided $\|x\| \leq |\Phi|$.

Let $x = \Phi m$ with $m \in H$, $\|m\| = 1$ and $m \perp e$. Then $\|x\| = |\Phi|$, $\|x\|^2 - |\langle x, e \rangle|^2 = |\Phi|$ and the equality case is realized in (2.17). □

The following result also holds:

Theorem 3. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned} \left| \langle x, y \rangle - \frac{1}{2} \left[\langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle e, y \rangle \right] \right| & \\ \le \frac{1}{2} \left[\|x\| \left(\|y\|^2 - |\langle y, f \rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| \right] & \\ \le \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e \rangle|^2 - |\langle y, f \rangle|^2 \right)^{1/2} & \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} |\langle x, f \rangle \langle f, y \rangle - \langle x, e \rangle \langle e, y \rangle| & \\ \le \|x\| \left(\|y\|^2 - |\langle y, f \rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| & \\ \le \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e \rangle|^2 - |\langle y, f \rangle|^2 \right)^{1/2}. & \end{aligned} \tag{2.19}$$

The inequalities (2.18) are sharp.

Proof. Using Schwarz inequality we have

$$\|x\| \|y - \langle y, f \rangle f\| \ge |\langle x, y - \langle y, f \rangle f \rangle| = |\langle x, y \rangle - \langle x, f \rangle \langle f, y \rangle| \tag{2.20}$$

and

$$\|x - \langle x, e \rangle e\| \|y\| \ge |\langle x - \langle x, e \rangle e, y \rangle| = |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \tag{2.21}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

If we add the inequalities (2.20) and (2.21) and use the triangle inequality, then we get

$$\begin{aligned} \|x\| \|y - \langle y, f \rangle f\| + \|x - \langle x, e \rangle e\| \|y\| & \\ \ge |\langle x, y \rangle - \langle x, f \rangle \langle f, y \rangle| + |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| & \\ \ge \begin{cases} |2 \langle x, y \rangle - \langle x, f \rangle \langle f, y \rangle - \langle x, e \rangle \langle e, y \rangle|, \\ \\ |\langle x, f \rangle \langle f, y \rangle - \langle x, e \rangle \langle e, y \rangle| \end{cases} & \end{aligned} \tag{2.22}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Since

$$\|x - \langle x, e \rangle e\| = \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2}, \quad \|y - \langle y, f \rangle f\| = \left(\|y\|^2 - |\langle y, f \rangle|^2 \right)^{1/2}$$

then by (2.22) we have

$$\begin{aligned} \|x\| \left(\|y\|^2 - |\langle y, f \rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \|y\| & \\ \ge \begin{cases} |2 \langle x, y \rangle - \langle x, f \rangle \langle f, y \rangle - \langle x, e \rangle \langle e, y \rangle|, \\ \\ |\langle x, f \rangle \langle f, y \rangle - \langle x, e \rangle \langle e, y \rangle| \end{cases} & \end{aligned} \tag{2.23}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

By employing the inequality (2.5) we get

$$\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|y\|^2-|\langle y,f\rangle|^2+\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2} \geq \|x\|\left(\|y\|^2-|\langle y,f\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\| \tag{2.24}$$

for any $x,y,e,f\in H$ with $\|e\|=\|f\|=1$.

Making use of (2.23) and (2.24) we get the desired inequalities (2.18) and (2.19).

If we take $f=e$ in (2.18) then we get

$$\begin{aligned}
&|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \\
&\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right] \\
&\leq \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-|\langle x,e\rangle|^2-|\langle y,e\rangle|^2\right)^{1/2},
\end{aligned}$$

which by Theorem 2 are sharp. □

Remark 3. *If we use the triangle inequality*

$$\begin{aligned}
&\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle| \\
&\leq \left|\langle x,y\rangle-\frac{1}{2}[\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle]\right|,
\end{aligned}$$

then we get from (2.1)

$$\begin{aligned}
&\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle| \\
&\leq |\langle x,y\rangle|+\frac{1}{2}\left[\|x\|\left(\|y\|^2-|\langle y,f\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right]
\end{aligned} \tag{2.25}$$

for any $x,y,e,f\in H$ with $\|e\|=\|f\|=1$.

Corollary 3. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e,\ f\in H, \|e\|=\|f\|=1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle\Phi e-x,x-\varphi e\rangle\geq 0,\ \ \operatorname{Re}\langle\Gamma f-y,y-\gamma f\rangle\geq 0 \tag{2.26}$$

hold, or, equivalently, the following assumptions

$$\left\|x-\frac{\varphi+\Phi}{2}e\right\|\leq\frac{1}{2}|\Phi-\varphi|,\quad \left\|y-\frac{\gamma+\Gamma}{2}f\right\|\leq\frac{1}{2}|\Gamma-\gamma| \tag{2.27}$$

are valid, then one has the inequalities

$$\begin{aligned}
&\left|\langle x,y\rangle-\frac{1}{2}[\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle]\right| \\
&\leq \frac{1}{4}[\|x\|\,|\Gamma-\gamma|+|\Phi-\varphi|\,\|y\|] \\
&\leq \frac{1}{4}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(|\Gamma-\gamma|^2+|\Phi-\varphi|^2\right)^{1/2}
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned} (2.29) \qquad & |\langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ & \leq \frac{1}{2} [\|x\| |\Gamma - \gamma| + |\Phi - \varphi| \|y\|] \\ & \leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2\right)^{1/2} \left(|\Gamma - \gamma|^2 + |\Phi - \varphi|^2\right)^{1/2}. \end{aligned}$$

The constant $\frac{1}{4}$ in the right hand side of (2.28) is sharp.

3. Applications for Sequences and Power Series

Consider the Hilbert space $\mathbb{C}^n$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathbf{p}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{p}} := \sum_{j=1}^{n} p_j x_j \overline{y}_j,$$

where $\mathbf{p} = (p_1, ..., p_n)$ is a probability distribution, i.e. $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ and

$$\mathbf{x} = (x_1, ..., x_n), \ \mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n.$$

Assume that $\mathbf{e} = (e_1, ..., e_n), \mathbf{f} = (f_1, ..., f_n) \in \mathbb{C}^n$ with

$$\sum_{j=1}^{n} p_j |e_j|^2 = \sum_{j=1}^{n} p_j |f_j|^2 = 1. \tag{3.1}$$

Then for any $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n$ we have from (2.1) the inequality

$$
\begin{aligned}
(3.2)\qquad & \left|\sum_{j=1}^{n} p_j x_j \overline{y}_j - \sum_{j=1}^{n} p_j x_j \overline{e}_j \sum_{j=1}^{n} p_j e_j \overline{y}_j\right| \\
& \leq \min\left\{\left(\sum_{j=1}^{n} p_j |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |y_j|^2 - \left|\sum_{j=1}^{n} p_j y_j \overline{e}_j\right|^2\right)^{1/2},\right. \\
& \left.\left(\sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |x_j|^2 - \left|\sum_{j=1}^{n} p_j x_j \overline{e}_j\right|^2\right)^{1/2}\right\} \\
& \leq \frac{1}{2}\left[\left(\sum_{j=1}^{n} p_j |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |y_j|^2 - \left|\sum_{j=1}^{n} p_j y_j \overline{e}_j\right|^2\right)^{1/2}\right. \\
& \left. + \left(\sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |x_j|^2 - \left|\sum_{j=1}^{n} p_j x_j \overline{e}_j\right|^2\right)^{1/2}\right] \\
& \leq \frac{1}{2}\left(\sum_{j=1}^{n} p_j |x_j|^2 + \sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} \\
& \times \left(\sum_{j=1}^{n} p_j |x_j|^2 + \sum_{j=1}^{n} p_j |y_j|^2 - \left|\sum_{j=1}^{n} p_j x_j \overline{e}_j\right|^2 - \left|\sum_{j=1}^{n} p_j y_j \overline{e}_j\right|^2\right)^{1/2}
\end{aligned}
$$

while from (2.18) we get

$$\begin{aligned}
&\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\frac{1}{2}\left[\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j+\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right]\right| \quad (3.3)\\
&\leq\frac{1}{2}\left[\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{f}_j\right|^2\right)^{1/2}\right.\\
&+\left.\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2\right)^{1/2}\right]\\
&\leq\frac{1}{2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}\\
&\times\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{f}_j\right|^2\right)^{1/2}.
\end{aligned}$$

If we denote by $\mathcal{C}(0,1)$ the unit circle of radius 1 in $\mathbb{C}$, namely $\mathcal{C}(0,1)=\{z\in\mathbb{C}|\ |z|=1\}$, then for $\mathbf{e}=(e_1,...,e_n)$, $\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with $e_j,f_j\in\mathcal{C}(0,1)$ for any $j\in\{1,...,n\}$ we have that the condition (3.1) holds true and therefore the inequalities (3.2) and (3.3) are valid.

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$.

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\begin{aligned}
&\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1), \quad (3.4)\\
&\ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C},\\
&\sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.
\end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned}
&\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \quad (3.5)\\
&\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ z\in D(0,1),
\end{aligned}$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \; z \in D(0,1),$$

$$_2F_1(\alpha,\beta,\gamma,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0$$
$$z \in D(0,1),$$

where Γ is *Gamma function.*

Proposition 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $0 < p < R$, $u, v \in \mathcal{C}(0,1)$ and $x, y \in \mathbb{C}$ with $p|x|^2, p|y|^2 < R$ then we have the inequalities*

$$\begin{aligned}
(3.6)\qquad & \left| \frac{f(px\overline{y})}{f(p)} - \frac{f(px\overline{u})}{f(p)} \frac{f(pu\overline{y})}{f(p)} \right| \\
& \leq \min\left\{ \left(\frac{f\left(p|x|^2\right)}{f(p)} \right)^{1/2} \left(\frac{f\left(p|y|^2\right)}{f(p)} - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2}, \right. \\
& \left. \left(\frac{f\left(p|y|^2\right)}{f(p)} \right)^{1/2} \left(\frac{f\left(p|x|^2\right)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right\} \\
& \leq \frac{1}{2} \left[\left(\frac{f\left(p|x|^2\right)}{f(p)} \right)^{1/2} \left(\frac{f\left(p|y|^2\right)}{f(p)} - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right. \\
& \left. + \left(\frac{f\left(p|y|^2\right)}{f(p)} \right)^{1/2} \left(\frac{f\left(p|x|^2\right)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right] \\
& \leq \frac{1}{2} \left(\frac{f\left(p|x|^2\right)}{f(p)} + \frac{f\left(p|y|^2\right)}{f(p)} \right)^{1/2} \\
& \times \left(\frac{f\left(p|x|^2\right)}{f(p)} + \frac{f\left(p|y|^2\right)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2}
\end{aligned}$$

and

$$\left|\frac{f(px\overline{y})}{f(p)}-\frac{1}{2}\left[\frac{f(px\overline{u})}{f(p)}\frac{f(pu\overline{y})}{f(p)}+\frac{f(px\overline{v})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\right]\right| \tag{3.7}$$
$$\leq\frac{1}{2}\left[\left(\frac{f\left(p|x|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|y|^2\right)}{f(p)}-\left|\frac{f(py\overline{v})}{f(p)}\right|^2\right)^{1/2}\right.$$
$$\left.+\left(\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|x|^2\right)}{f(p)}-\left|\frac{f(px\overline{u})}{f(p)}\right|^2\right)^{1/2}\right]$$
$$\leq\frac{1}{2}\left(\frac{f\left(p|x|^2\right)}{f(p)}+\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}$$
$$\times\left(\frac{f\left(p|x|^2\right)}{f(p)}+\frac{f\left(p|y|^2\right)}{f(p)}-\left|\frac{f(px\overline{u})}{f(p)}\right|^2-\left|\frac{f(py\overline{v})}{f(p)}\right|^2\right)^{1/2}.$$

Proof. If $u,v\in\mathcal{C}(0,1)$ then for any $n\geq 0$ we have $u^n,v^n\in\mathcal{C}(0,1)$. Observe that for any $m\geq 1$ we have that

$$\frac{\sum_{n=0}^m a_np^n|u^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n|v^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n}{\sum_{n=0}^m a_np^n}=1.$$

Using the inequality (3.2) we have

$$\left|\frac{\sum_{n=0}^m a_np^n(x\overline{y})^n}{\sum_{n=0}^m a_np^n}-\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\frac{\sum_{n=0}^m a_np^n(u\overline{y})^n}{\sum_{n=0}^m a_np^n}\right| \tag{3.8}$$
$$\leq\min\left\{\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2},\right.$$
$$\left.\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right\}$$
$$\leq\frac{1}{2}\left[\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right.$$
$$\left.+\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right]$$
$$\leq\frac{1}{2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}+\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}$$
$$\times\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}+\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}$$

Since all the series whose partial sums are involved in inequality (3.8) are convergent, then by letting $m \to \infty$ in (3.8) we get the desired result (3.6).

The proof of the inequality (3.7) can be proved in the same way by utilizing (3.3) and we omit the details. $\square$

Remark 4. *The inequality (3.6) can provide some particular inequalities of interest. For instance, if we take* $f(z) = \exp(z)$, $z \in \mathbb{C}$, *then we get*

$$
\begin{aligned}
(3.9)\quad & \left|\exp\left[p\left(x\overline{y}-1\right)\right]-\exp\left[p\left(x\overline{u}+u\overline{y}-2\right)\right]\right| \\
&\leq \min\left\{\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2},\right. \\
&\left.\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right\} \\
&\leq \frac{1}{2}\left[\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right. \\
&\left.+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right] \\
&\leq \frac{1}{2}\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right)^{1/2} \\
&\times\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right. \\
&\left.-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2}
\end{aligned}
$$

for any $p > 0, u \in \mathcal{C}(0,1)$ *and* $x, y \in \mathbb{C}$.

If we take $u = v = 1$, *then from (3.9) we get*

$$
\begin{aligned}
(3.10)\quad & \left|\exp\left[p\left(x\overline{y}-1\right)\right]-\exp\left[p\left(x+\overline{y}-2\right)\right]\right| \\
&\leq \min\left\{\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2},\right. \\
&\left.\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x-1\right)\right]\right|^2\right)^{1/2}\right\} \\
&\leq \frac{1}{2}\left[\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2}\right. \\
&\left.+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x-1\right)\right]\right|^2\right)^{1/2}\right] \\
&\leq \frac{1}{2}\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right)^{1/2} \\
&\times\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right. \\
&\left.-\left|\exp\left[p\left(x-1\right)\right]\right|^2-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2}
\end{aligned}
$$

for any $p > 0$ *and* $x, y \in \mathbb{C}$.

Moreover, if we take in (3.10) $x = \overline{y} = z \in \mathbb{C}$, then we get

$$\left|\exp\left[p\left(z^2-1\right)\right] - \exp\left[2p\left(z-1\right)\right]\right| \leq \exp\left[\frac{1}{2}p\left(|z|^2-1\right)\right]\left(\exp\left[p\left(|z|^2-1\right)\right] - \left|\exp\left[p\left(z-1\right)\right]\right|^2\right)^{1/2} \tag{3.11}$$

for any $p > 0$ and $z \in \mathbb{C}$.

4. Applications for Integrals

Consider $L^2[a,b]$ the Hilbert space of all complex valued functions f with $\int_a^b |f(t)|^2\, dt < \infty$. The inner product is given by

$$\langle f, g\rangle_2 := \int_a^b f(t)\,\overline{g(t)}dt.$$

Assume that $h, k \in L^2[a,b]$ with

$$\int_a^b |h(t)|^2\, dt = \int_a^b |k(t)|^2\, dt = 1. \tag{4.1}$$

For instance, if $h(t) = \frac{1}{\sqrt{b-a}}\rho(t), k(t) = \frac{1}{\sqrt{b-a}}\varphi(t)$ with $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then $h, k \in L^2[a,b]$ and the condition (4.1) is satisfied.

Proposition 2. *Assume that $h, k \in L^2[a,b]$ with the property (4.1). Then for any $f, g \in L^2[a,b]$ we have the inequality*

$$\begin{aligned} &\Bigg|\int_a^b f(t)\,\overline{g(t)}dt \\ &-\frac{1}{2}\left[\int_a^b f(t)\,\overline{h(t)}dt\int_a^b h(t)\,\overline{g(t)}dt + \int_a^b f(t)\,\overline{k(t)}dt\int_a^b k(t)\,\overline{g(t)}dt\right]\Bigg| \\ &\leq \frac{1}{2}\left(\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt\right)^{1/2} \\ &\times\left(\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt - \left|\int_a^b f(t)\,\overline{h(t)}dt\right|^2 - \left|\int_a^b g(t)\,\overline{k(t)}dt\right|^2\right)^{1/2}. \end{aligned} \tag{4.2}$$

The proof follows by Theorem 3 for the inner product $\langle\cdot,\cdot\rangle_2$.

Remark 5. *If $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then we have the following inequalities for integral means*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt\right. \\
&-\frac{1}{2(b-a)^2}\left[\int_a^b f(t)\overline{\rho(t)}dt\int_a^b \rho(t)\overline{g(t)}dt\right. \\
&\left.\left.+\int_a^b f(t)\overline{\varphi(t)}dt\int_a^b \varphi(t)\overline{g(t)}dt\right]\right| \\
&\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right)^{1/2} \\
&\times\left[\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right. \\
&\left.-\left|\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\right|^2-\left|\frac{1}{b-a}\int_a^b g(t)\overline{\varphi(t)}dt\right|^2\right]^{1/2}
\end{aligned} \tag{4.3}$$

for any $f, g \in L^2[a,b]$.

If we take $\rho(t) = 1$, $\varphi(t) = sgn\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$, then $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$ and then we get from (4.3)

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt\right. \\
&-\frac{1}{2(b-a)^2}\left[\int_a^b f(t)\,dt\int_a^b \overline{g(t)}dt\right. \\
&\left.\left.+\int_a^b f(t)\,sgn\left(t-\frac{a+b}{2}\right)dt\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right]\right| \\
&\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right)^{1/2} \\
&\times\left[\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right. \\
&\left.-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2-\left|\frac{1}{b-a}\int_a^b g(t)\,sgn\left(t-\frac{a+b}{2}\right)dt\right|^2\right]^{1/2}
\end{aligned} \tag{4.4}$$

for any $f, g \in L^2[a,b]$.

On making use of Corollaries 2 and 3 one can state similar discrete and integral inequalities. However the details are not presented here.

References

[1] J. M. Aldaz, Strengthened Cauchy-Schwarz and Hölder inequalities. *J. Inequal. Pure Appl. Math.* **10** (2009), no. 4, Article 116, 6 pp.

[2] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405–409 (1974).

[3] A. De Rossi, A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets. *Math. Inequal. Appl.* **2** (1999), no. 2, 263–282.

[4] S. S. Dragomir, Some refinements of Schwartz inequality, Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania, 1-2 Noiembrie 1985, 13–16.

[5] S. S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237**(1999), 74-82.

[6] S. S. Dragomir, Some Grüss type inequalities in inner product spaces, *J. Inequal. Pure and Appl. Math.*, **4**(2) Art. 42, 2003.

[7] S. S. Dragomir, Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality. *Integral Transforms Spec. Funct.* **20** (2009), no. 9-10, 757–767.

[8] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. I. *J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 59, 15 pp.

[9] S. S. Dragomir, A potpourri of Schwarz related inequalities in inner product spaces. II. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 1, Article 14, 11 pp.

[10] S. S. Dragomir, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *J. Inequal. Pure Appl. Math.* **5** (2004), no. 3, Article 76, 18 pp.

[11] S. S. Dragomir, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 1, Art. 1, 18 pp.

[12] S. S. Dragomir, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Demonstratio Math.* **40** (2007), no. 2, 411–417.

[13] S. S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Tamkang J. Math.* **39** (2008), no. 1, 1–7.

[14] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces.* Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.

[15] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces.* Nova Science Publishers, Inc.,

[16] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces.* Springer Briefs in Mathematics. Springer, 2013. x+120 pp. ISBN: 978-3-319-01447-0; 978-3-319-01448-7.

[17] S. S. Dragomir and Anca C. Goşa, Quasilinearity of some composite functionals associated to Schwarz's inequality for inner products. *Period. Math. Hungar.* **64** (2012), no. 1, 11–24.

[18] S. S. Dragomir and B. Mond, Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. *Soochow J. Math.* **21** (1995), no. 4, 413–426.

[19] S. S. Dragomir and I. Sándor, Some inequalities in pre-Hilbertian spaces. *Studia Univ. Babeş-Bolyai Math.* **32** (1987), no. 1, 71–78.

[20] H. Gunawan, On n-inner products, n-norms, and the Cauchy-Schwarz inequality. *Sci. Math. Jpn.* **55** (2002), no. 1, 53–60

[21] K. E. Gustafson and D. K. M. Rao, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[22] E. R. Lorch, The Cauchy-Schwarz inequality and self-adjoint spaces. *Ann. of Math.* (2) **46**, (1945). 468–473.

[23] C. Lupu and D. Schwarz, Another look at some new Cauchy-Schwarz type inner product inequalities. *Appl. Math. Comput.* **231** (2014), 463–477.

[24] M. Marcus, The Cauchy-Schwarz inequality in the exterior algebra. *Quart. J. Math. Oxford Ser.* (2) **17** 1966 61–63.

[25] P. R. Mercer, A refined Cauchy-Schwarz inequality. *Internat. J. Math. Ed. Sci. Tech.* **38** (2007), no. 6, 839–842.

[26] F. T. Metcalf, A Bessel-Schwarz inequality for Gramians and related bounds for determinants. *Ann. Mat. Pura Appl.* (4) **68** 1965 201–232.

[27] T. Precupanu, On a generalization of Cauchy-Buniakowski-Schwarz inequality. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat.* (N.S.) **22** (1976), no. 2, 173–175.

[28] K. Trenčevski and R. Malčeski, On a generalized n-inner product and the corresponding Cauchy-Schwarz inequality. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 53, 10 pp.

[29] G.-B. Wang and J.-P. Ma, Some results on reverses of Cauchy-Schwarz inequality in inner product spaces. *Northeast. Math. J.* **21** (2005), no. 2, 207–211.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa