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VECTOR INEQUALITIES FOR A PROJECTION IN HILBERT SPACES AND APPLICATIONS

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Abstract. In this paper we establish some vector inequalities related to Schwarz and Buzano results. We show amongst others that in an inner product space H we have the inequality

$$\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right]\geq|\langle Px,y\rangle|$$

for any vectors x,y and a projection $P:H\to H$.

Applications for norm and numerical radius inequalities of two bounded operators are given as well.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\|y\|\geq|\langle x,y\rangle| \text{ for any } x,y\in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

In 1985 the author [5] (see also [23]) established the following refinement of (1.1):

$$\|x\|\|y\|\geq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,y\rangle| \tag{1.2}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Using the triangle inequality for modulus we have

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|$$

and by (1.2) we get

$$\begin{aligned}\|x\|\|y\| &\geq |\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\\ &\geq 2|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|,\end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right]\geq|\langle x,e\rangle\langle e,y\rangle| \tag{1.3}$$

that holds for any $x,y,e\in H$ with $\|e\|=1$.

For other Schwarz and Buzano related inequalities in inner product spaces, see [1]-[4], [5]-[14], [21]-[25], [29]-[38], and the monographs [16], [17] and [18].

Now, let us recall some basic facts on *orthogonal projection* that will be used in the sequel.

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If K is a subset of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, the set of *vectors orthogonal* to K is defined by

$$K^{\perp} := \{x \in H : \langle x, k\rangle = 0 \text{ for all } k \in K\}.$$

We observe that $K^{\perp}$ is a *closed subspace* of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^{\perp}$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $K^{\perp}$. Therefore, H is the *internal Hilbert direct sum* of V and $V^{\perp}$, and we denote that as $H = V \oplus V^{\perp}$.

The linear operator $P_V : H \to H$ that maps x to v is called *the orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all *bounded self-adjoint* operators P such that $P^2 = P$. Specifically, the orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of V, which minimizes the distance $\|x - v\|$. This provides the geometrical interpretation of $P_V(x)$: it is *the best approximation* to x by elements of V.

Projections P_U and P_V are called *mutually orthogonal* if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. The sum of the two projections P_U and P_V is a projection only if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$.The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U\cap V}$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J} |\langle x, e_j\rangle|^2 \leq \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j\in J} |\langle x, e_j\rangle|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j\in J}$ an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j, \; x \in H. \tag{1.4}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle, \; x, y \in H \text{ and } \langle P_{\mathcal{E}} x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2, \; x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle, \; x, y \in H \tag{1.5}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq |\langle P_e x, y\rangle| \tag{1.6}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

Motivated by the above results we establish in this paper some vector inequalities for an orthogonal projection P that generalizes amongst others the Buzano inequality (1.6). Applications for norm and numerical radius inequalities are provided as well.

2. Vector Inequalities for a Projection

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathcal{B}(H)$ that $0 \leq P \leq 1_H$.

The following result holds:

Theorem 1. *Let $P : H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} \geq |\langle x, y\rangle - \langle Px, y\rangle|. \tag{2.1}$$

and

$$\|x\|\,\|y\| - \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} \geq |\langle Px, y\rangle|. \tag{2.2}$$

Proof. Using the properties of projection, we have

$$\begin{aligned}\langle x - Px, y - Py\rangle &= \langle x, y\rangle - \langle Px, y\rangle - \langle x, Py\rangle + \langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, y\rangle + \left\langle P^2 x, y\right\rangle \\ &= \langle x, y\rangle - \langle Px, y\rangle\end{aligned} \tag{2.3}$$

for any $x, y \in H$.

By Schwarz inequality we have

$$\|x - Px\|^2\,\|y - Py\|^2 \geq |\langle x - Px, y - Py\rangle|^2 \tag{2.4}$$

for any $x, y \in H$.

Since, by (2.1), we have

$$\|x - Px\|^2 = \|x\|^2 - \langle Px, x\rangle, \quad \|y - Py\|^2 = \|y\|^2 - \langle Py, y\rangle,$$

then by (2.4) we have

$$\left(\|x\|^2 - \langle Px, x\rangle\right)\left(\|y\|^2 - \langle Py, y\rangle\right) \geq |\langle x, y\rangle - \langle Px, y\rangle|^2 \tag{2.5}$$

for any $x, y \in H$.

Using the elementary inequality that holds for any real numbers a, b, c, d

$$(ac - bd)^2 \geq \left(a^2 - b^2\right)\left(c^2 - d^2\right),$$

we have

$$\left(\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2}\right)^2 \geq \left(\|x\|^2 - \langle Px, x\rangle\right)\left(\|y\|^2 - \langle Py, y\rangle\right) \tag{2.6}$$

for any $x, y \in H$.

Since

$$\|x\| \geq \langle Px, x\rangle^{1/2}, \quad \|y\| \geq \langle Py, y\rangle^{1/2},$$

then

$$\|x\| \|y\| - \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \geq 0,$$

for any $x, y \in H$.

By (2.5) and (2.6) we get

$$\left(\|x\| \|y\| - \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2}\right)^2 \geq |\langle x, y\rangle - \langle Px, y\rangle|^2$$

for any $x, y \in H$, which, by taking the square root, is equivalent to the desired inequality (2.1).

Observe that, if P is an orthogonal projection, then $Q := 1_H - P$ is also a projection. Indeed we have

$$Q^2 = (1_H - P)^2 = 1_H - 2P + P^2 = 1_H - P = Q.$$

Now, if we write the inequality (2.1) for the projection Q we get the desired inequality (2.2). □

Corollary 1. *With the assumptions of Theorem 1, we have the following refinements of Schwarz inequality*

$$\begin{aligned} \|x\| \|y\| &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \\ &\geq |\langle Px, y\rangle| + |\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle| \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \|x\| \|y\| &\geq \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + |\langle Px, y\rangle| \\ &\geq |\langle x, y\rangle - \langle Px, y\rangle| + |\langle Px, y\rangle| \geq |\langle x, y\rangle| \end{aligned} \tag{2.8}$$

for any $x, y \in H$.

Remark 1. *Since*

$$|\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle| - |\langle Px, y\rangle|$$

then by the first inequality in (2.7) we have

$$\|x\| \|y\| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle| - |\langle Px, y\rangle|$$

that produces the inequality

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} - |\langle Px, y\rangle| \geq 0 \tag{2.9}$$

for any $x, y \in H$.

We notice that the second inequality follows by Schwarz's inequality for the nonnegative selfadjoint operator P.

Since

$$|\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle Px, y\rangle| - |\langle x, y\rangle|$$

then by (2.7) we have

$$\begin{aligned} \|x\| \|y\| &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \\ &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle Px, y\rangle| - |\langle x, y\rangle|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x\| \|y\| + |\langle x, y\rangle| &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle Px, y\rangle| \\ &\geq 2 |\langle Px, y\rangle| \end{aligned}$$

and is equivalent to

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq\frac{1}{2}\left[\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}+|\langle Px,y\rangle|\right] \geq|\langle Px,y\rangle| \tag{2.10}$$

for any $x,y\in H$.

The inequality between the first and last term in (2.10), namely

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq|\langle Px,y\rangle| \tag{2.11}$$

for any $x,y\in H$, is a generalization of Buzano's inequality (1.3).

From the inequality (2.8) we can state that

$$\|x\|\,\|y\|-|\langle Px,y\rangle|\geq\left(\|x\|^{2}-\langle Px,x\rangle\right)^{1/2}\left(\|y\|^{2}-\langle Py,y\rangle\right)^{1/2} \geq|\langle x,y\rangle-\langle Px,y\rangle| \tag{2.12}$$

for any $x,y\in H$.

From the inequality (2.8) we also have

$$\begin{aligned}\|x\|\,\|y\|&\geq\left(\|x\|^{2}-\langle Px,x\rangle\right)^{1/2}\left(\|y\|^{2}-\langle Py,y\rangle\right)^{1/2}+|\langle Px,y\rangle|\\&\geq|\langle x,y\rangle-\langle Px,y\rangle|+|\langle Px,y\rangle|\geq|\langle Px,y\rangle|-|\langle x,y\rangle|+|\langle Px,y\rangle|\\&=2\,|\langle Px,y\rangle|-|\langle x,y\rangle|,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]&\geq\frac{1}{2}\left[\left(\|x\|^{2}-\langle Px,x\rangle\right)^{1/2}\left(\|y\|^{2}-\langle Py,y\rangle\right)^{1/2}\right]\\&+\frac{1}{2}\left[|\langle Px,y\rangle|+|\langle x,y\rangle|\right]\geq|\langle Px,y\rangle|\end{aligned} \tag{2.13}$$

for any $x,y\in H$.

The case of orthonormal families which is related to Bessel's inequality is of interest.

Let $\mathcal{E}=\{e_j\}_{j\in J}$ be an othonormal family in H. The for any $x,y\in H$ we have from (2.7) and (2.8) the inequalities

$$\begin{aligned}\|x\|\,\|y\|&\geq\left(\sum_{j\in J}|\langle x,e_j\rangle|^{2}\right)^{1/2}\left(\sum_{j\in J}|\langle y,e_j\rangle|^{2}\right)^{1/2}\\&+\left|\langle x,y\rangle-\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|\\&\geq\left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|+\left|\langle x,y\rangle-\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|\geq|\langle x,y\rangle|\end{aligned} \tag{2.14}$$

and

$$\|x\|\,\|y\| \geq \left(\|x\|^2 - \sum_{j\in J}|\langle x, e_j\rangle|^2\right)^{1/2}\left(\|y\|^2 - \sum_{j\in J}|\langle y, e_j\rangle|^2\right)^{1/2} + \left|\left\langle \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right\rangle\right| \geq \left|\langle x, y\rangle - \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right| + \left|\sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right| \geq |\langle x, y\rangle|\,. \tag{2.15}$$

By (2.9) and (2.10) we have

$$\|x\|\,\|y\| - |\langle x, y\rangle| \geq \left(\sum_{j\in J}|\langle x, e_j\rangle|^2\right)^{1/2}\left(\sum_{j\in J}|\langle y, e_j\rangle|^2\right)^{1/2} - \left|\left\langle \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right\rangle\right| \geq 0 \tag{2.16}$$

and

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq \frac{1}{2}\left(\sum_{j\in J}|\langle x, e_j\rangle|^2\right)^{1/2}\left(\sum_{j\in J}|\langle y, e_j\rangle|^2\right)^{1/2} + \frac{1}{2}\left|\left\langle\left\langle \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right\rangle\right\rangle\right| \geq \left|\left\langle \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right\rangle\right| \tag{2.17}$$

for any $x, y \in H$.

The inequality between the first and last term in (2.17) provides a generalization of Buzano's inequality for orthonormal families $\mathcal{E} = \{e_j\}_{j\in J}$.

The following result holds:

Theorem 2. *Let $P : H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$|\langle x, y\rangle - 2\langle Px, y\rangle| \leq \|x\|\,\|y\|\,, \tag{2.18}$$

$$\begin{aligned} &|\langle x, y\rangle - \langle Px, y\rangle| \\ &\leq \min\left\{\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right\} \\ &\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right] \\ &\leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\|x\|^2 + \|y\|^2 - \langle Py, y\rangle - \langle Px, x\rangle\right)^{1/2} \end{aligned} \tag{2.19}$$

and

$$|\langle Px,y\rangle| \le \min\left\{\|x\|\langle Py,y\rangle^{1/2}, \|y\|\langle Px,x\rangle^{1/2}\right\} \le \frac{1}{2}\left[\|x\|\langle Py,y\rangle^{1/2}+\|y\|\langle Px,x\rangle^{1/2}\right] \le \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\langle Px,x\rangle+\langle Py,y\rangle\right)^{1/2}. \tag{2.20}$$

Proof. Observe that

$$\|x-2Px\|^2 = \|x\|^2 - 4\operatorname{Re}\langle x,Px\rangle + 4\langle Px,Px\rangle = \|x\|^2 - 4\langle x,Px\rangle + 4\langle P^2x,x\rangle = \|x\|^2 - 4\langle x,Px\rangle + 4\langle Px,x\rangle = \|x\|^2$$

for any $x \in H$.

Using Schwarz's inequality we have

$$\|x\|\|y\| = \|x-2Px\|\|y\| \ge |\langle x-2Px,y\rangle| = |\langle x,y\rangle - 2\langle Px,y\rangle|$$

for any $x,y \in H$ and the inequality (2.18) is proved.

By Schwarz's inequality we also have

$$\|x-Px\|\|y\| \ge |\langle x-Px,y\rangle| = |\langle x,y\rangle - \langle Px,y\rangle|$$

and

$$\|x\|\|y-Py\| \ge |\langle x,y-Py\rangle| = |\langle x,y\rangle - \langle x,Py\rangle| = |\langle x,y\rangle - \langle Px,y\rangle|$$

for any $x,y \in H$, which implies the first inequality in (2.19).

The second and the third inequalities are obvious by the elementary inequalities

$$\min\{a,b\} \le \frac{1}{2}(a+b), \ a,b \in \mathbb{R}_+$$

and

$$ac+bd \le \left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2}, \ a,b,c,d \in \mathbb{R}_+.$$

The inequality (2.20) follows from (2.19) by replacing P with $1_H - P$. □

Remark 2. *By the triangle inequality we have*

$$\|x\|\|y\| + |\langle x,y\rangle| \ge |\langle x,y\rangle - 2\langle Px,y\rangle| + |\langle x,y\rangle| \ge 2|\langle Px,y\rangle|,$$

which implies that (see also (2.10) and (2.13))

$$\frac{1}{2}\left[\|x\|\|y\| + |\langle x,y\rangle|\right] \ge |\langle Px,y\rangle| \tag{2.21}$$

for any $x,y \in H$.

From (2.19) we also have

$$|\langle Px,y\rangle| \le |\langle x,y\rangle| + \min\left\{\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\right\} \tag{2.22}$$

and

$$|\langle x,y\rangle| \le |\langle Px,y\rangle| + \min\left\{\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\right\} \tag{2.23}$$

for any $x, y \in H$.

Now, if $\mathcal{E} = \{e_j\}_{j \in J}$ is an orthonormal family then by the inequalities (2.18) and (2.19) we have

$$\left| \langle x, y \rangle - 2 \sum_{j \in J} \langle x, e_j \rangle \langle e_j, y \rangle \right| \leq \|x\| \|y\|, \tag{2.24}$$

and

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{j \in J} \langle x, e_j \rangle \langle e_j, y \rangle \right| \\ &\leq \min \left\{ \|x\| \left(\|y\|^2 - \sum_{j \in J} |\langle y, e_j \rangle|^2 \right)^{1/2}, \|y\| \left(\|x\|^2 - \sum_{j \in J} |\langle x, e_j \rangle|^2 \right)^{1/2} \right\} \\ &\leq \frac{1}{2} \left[\|x\| \left(\|y\|^2 - \sum_{j \in J} |\langle y, e_j \rangle|^2 \right)^{1/2} + \|y\| \left(\|x\|^2 - \sum_{j \in J} |\langle x, e_j \rangle|^2 \right)^{1/2} \right] \\ &\leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - \sum_{j \in J} |\langle y, e_j \rangle|^2 - \sum_{j \in J} |\langle x, e_j \rangle|^2 \right)^{1/2} \end{aligned} \tag{2.25}$$

for any $x, y \in H$.

From (2.22) we also have

$$\begin{aligned} &\left| \sum_{j \in J} \langle x, e_j \rangle \langle e_j, y \rangle \right| \\ &\leq |\langle x, y \rangle| + \min \left\{ \|x\| \left(\|y\|^2 - \sum_{j \in J} |\langle y, e_j \rangle|^2 \right)^{1/2}, \|y\| \left(\|x\|^2 - \sum_{j \in J} |\langle x, e_j \rangle|^2 \right)^{1/2} \right\} \end{aligned} \tag{2.26}$$

for any $x, y \in H$.

3. Inequalities for Norm and Numerical Radius

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [26, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by [26, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T) \leq \|T\| \leq 2w(T), \text{ for any } T \in B(H).$$

Utilising Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [13] or [14]:

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $T:H\to H$ a bounded linear operator on H. Then*

$$w^2(T)\le\frac{1}{2}\left[w\left(T^2\right)+\|T\|^2\right]. \tag{3.1}$$

The constant $\frac{1}{2}$ is best possible in (3.1).

The following general result for the product of two operators holds [26, p. 37]:

Theorem 4. *If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then $w(AB)\le 4w(A)\,w(B)$. In the case that $AB=BA$, then $w(AB)\le 2w(A)\,w(B)$. The constant 2 is best possible here.*

The following results are also well known [26, p. 38].

Theorem 5. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB)\le w(B). \tag{3.2}$$

If A is an isometry and $AB=BA$, then (3.2) also holds true.

We say that A and B *double commute* if $AB=BA$ and $AB^*=B^*A$. The following result holds [26, p. 38].

Theorem 6. *If the operators A and B double commute, then*

$$w(AB)\le w(B)\,\|A\|. \tag{3.3}$$

As a consequence of the above, we have [26, p. 39]:

Corollary 2. *Let A be a normal operator commuting with B. Then*

$$w(AB)\le w(A)\,w(B). \tag{3.4}$$

A related problem with the inequality (3.3) is to find the best constant c for which the inequality

$$w(AB)\le cw(A)\,\|B\|$$

holds for any two commuting operators $A,B\in B(H)$. It is known that $1.064<c<1.169$, see [3], [34] and [35].

In relation to this problem, it has been shown in [24] that

Theorem 7. *For any $A,B\in B(H)$ we have*

$$w\left(\frac{AB+BA}{2}\right)\le\sqrt{2}w(A)\,\|B\|. \tag{3.5}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

The following result holds.

Theorem 8. *Let $P:H\to H$ be an orthogonal projection on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$. If A,B are two bounded linear operators on H, then*

$$|\langle BPAx,x\rangle|\le\frac{1}{2}\left[\|Ax\|\,\|B^*x\|+|\langle BAx,x\rangle|\right] \tag{3.6}$$

and

$$\|BPAx\|\le\frac{1}{2}\left[\|Ax\|\,\|B\|+\|BAx\|\right] \tag{3.7}$$

for any $x \in H$.

Moreover, we have

$$w(BPA) \leq \frac{1}{2}\left[\|A\|\|B\| + w(BA)\right] \tag{3.8}$$

and

$$\|BPA\| \leq \frac{1}{2}\left[\|A\|\|B\| + \|BA\|\right]. \tag{3.9}$$

Proof. From the inequality (2.11) we have

$$|\langle PAx, B^*y\rangle| \leq \frac{1}{2}\left[\|Ax\|\|B^*y\| + |\langle Ax, B^*y\rangle|\right]$$

that is equivalent to

$$|\langle BPAx, y\rangle| \leq \frac{1}{2}\left[\|Ax\|\|B^*y\| + |\langle BAx, y\rangle|\right] \tag{3.10}$$

for any $x, y \in H$.

If we take $y = x$ in (3.10), then we get (3.6).

Taking the supremum over $y \in H$ with $\|y\| = 1$ in (3.10) we have

$$\begin{aligned}\|BPAx\| &= \sup_{\|y\|=1}|\langle BPAx, y\rangle| \leq \frac{1}{2}\sup_{\|y\|=1}\left[\|Ax\|\|B^*y\| + |\langle BAx, y\rangle|\right]\\ &\leq \frac{1}{2}\left[\|Ax\|\sup_{\|y\|=1}\|B^*y\| + \sup_{\|y\|=1}|\langle BAx, y\rangle|\right]\\ &= \frac{1}{2}\left[\|Ax\|\|B\| + \|BAx\|\right]\end{aligned}$$

for any $x \in H$.

The inequalities (3.8) and (3.9) follow from (3.6) and (3.7) by taking the supremum over $x \in H$ with $\|x\| = 1$. □

Corollary 3. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle APAx, x\rangle| \leq \frac{1}{2}\left[\|Ax\|\|A^*x\| + \left|\left\langle A^2x, x\right\rangle\right|\right] \tag{3.11}$$

and

$$\|APAx\| \leq \frac{1}{2}\left[\|Ax\|\|A\| + \left\|A^2x\right\|\right] \tag{3.12}$$

for any $x \in H$.

Moreover, we have

$$w(APA) \leq \frac{1}{2}\left[\|A\|^2 + w\left(A^2\right)\right] \tag{3.13}$$

and

$$\|APA\| \leq \frac{1}{2}\left[\|A\|^2 + \left\|A^2\right\|\right]. \tag{3.14}$$

Remark 3. *Let $e \in H$, $\|e\| = 1$. If we write the inequalities (3.6) and (3.7) for the projector P_e defined by $P_ex = \langle x, e\rangle e$, $x \in H$, we have*

$$|\langle Ax, e\rangle||\langle Be, x\rangle| \leq \frac{1}{2}\left[\|Ax\|\|B^*x\| + |\langle BAx, x\rangle|\right] \tag{3.15}$$

and

$$|\langle Ax, e\rangle| \, \|Be\| \leq \frac{1}{2} [\|Ax\| \, \|B\| + \|BAx\|] \tag{3.16}$$

for any $x \in H$.

Now, if we take the supremum over $x \in H$, $\|x\| = 1$ in (3.16), then we get

$$\|A^* e\| \, \|Be\| \leq \frac{1}{2} [\|A\| \, \|B\| + \|BA\|] \tag{3.17}$$

for any $e \in H$, $\|e\| = 1$.

If in (3.17) we take $B = A$, we have

$$\|A^* e\| \, \|Ae\| \leq \frac{1}{2} \left[\|A\|^2 + \left\|A^2\right\| \right] \tag{3.18}$$

for any $e \in H$, $\|e\| = 1$.

If in (3.15) we take $B = A$, then we get

$$|\langle Ax, e\rangle| \, |\langle e, A^* x\rangle| \leq \frac{1}{2} \left[\|Ax\| \, \|A^* x\| + \left|\left\langle A^2 x, x\right\rangle\right| \right] \tag{3.19}$$

for any $x \in H$ and $e \in H$, $\|e\| = 1$, and in particular

$$|\langle Ae, e\rangle|^2 \leq \frac{1}{2} \left[\|Ae\| \, \|A^* e\| + \left|\left\langle A^2 e, e\right\rangle\right| \right] \tag{3.20}$$

for any $e \in H$, $\|e\| = 1$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (3.20) we recapture the result in Theorem 3.

For a given operator T we consider the modulus of T defined as $|T| := (T^* T)^{1/2}$.

Corollary 4. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If A, B are two bounded linear operators on H, then*

$$w(BPA) \leq \frac{1}{2} w(BA) + \frac{1}{4} \left\| |A|^2 + |B^*|^2 \right\|. \tag{3.21}$$

In particular, we have

$$w(APA) \leq \frac{1}{2} w\left(A^2\right) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\|. \tag{3.22}$$

Proof. From the inequality (3.6) we have

$$\begin{aligned} |\langle BPAx, x\rangle| &\leq \frac{1}{2} [\|Ax\| \, \|B^* x\| + |\langle BAx, x\rangle|] \\ &\leq \frac{1}{2} |\langle BAx, x\rangle| + \frac{1}{4} \left[\|Ax\|^2 + \|B^* x\|^2 \right] \end{aligned} \tag{3.23}$$

for any $x \in H$, where for the second inequality we used the elementary inequality

$$ab \leq \frac{1}{2} \left(a^2 + b^2\right), \ a, b \in \mathbb{R}. \tag{3.24}$$

Since

$$\begin{aligned} \|Ax\|^2 + \|B^* x\|^2 &= \langle Ax, Ax\rangle + \langle B^* x, B^* x\rangle = \langle A^* Ax, x\rangle + \langle BB^* x, x\rangle \\ &= \left\langle \left(|A|^2 + |B^*|^2 \right) x, x \right\rangle \end{aligned}$$

for any $x \in H$, then from (3.23) we have

$$|\langle BPAx, x\rangle| \leq \frac{1}{2} |\langle BAx, x\rangle| + \frac{1}{4} \left\langle \left(|A|^2 + |B^*|^2 \right) x, x \right\rangle \tag{3.25}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.25) we get the desired result (3.21). □

Remark 4. *We observe that, by (3.20) we have*

$$|\langle Ae, e\rangle|^2 \leq \frac{1}{2}\left[\|Ae\| \|A^*e\| + \left|\langle A^2e, e\rangle\right|\right] \leq \frac{1}{2}\left|\langle A^2e, e\rangle\right| + \frac{1}{4}\left[\|Ae\|^2 + \|A^*e\|^2\right] = \frac{1}{2}\left|\langle A^2e, e\rangle\right| + \frac{1}{4}\left\langle\left(|A|^2 + |A^*|^2\right)e, e\right\rangle \tag{3.26}$$

for any $e \in H$, $\|e\| = 1$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (3.26) we get

$$w^2(A) \leq \frac{1}{2}w\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|, \tag{3.27}$$

for any bounded linear operator A.

Since

$$\left\||A|^2 + |A^*|^2\right\| \leq \left\||A|^2\right\| + \left\||A^*|^2\right\| = 2\|A\|^2,$$

then the inequality (3.27) is better than the inequality in Theorem 3.

The following result also holds:

Theorem 9. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w\left(B\left(\frac{1}{2}1_H - P\right)A\right) \leq \frac{1}{4}\left\||A|^2 + |B^*|^2\right\|. \tag{3.28}$$

In particular, we have

$$w\left(A\left(\frac{1}{2}1_H - P\right)A\right) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|. \tag{3.29}$$

Proof. From the inequality (2.18) we have

$$|\langle(1_H - 2P)Ax, B^*x\rangle| \leq \|Ax\| \|B^*x\|,$$

that is equivalent to

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \leq \frac{1}{2}\|Ax\| \|B^*x\| \tag{3.30}$$

for any $x \in H$.

Using the elementary inequality (3.24) we have

$$\frac{1}{2}\|Ax\| \|B^*x\| \leq \frac{1}{4}\left(\|Ax\|^2 + \|B^*x\|^2\right) = \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle$$

and by (3.30) we get

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \leq \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle \tag{3.31}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.31) we get the desired result (3.28). □

Remark 5. *If we take in (3.28) $P = 1_H$, then we get* [18, p. 6]

$$w(BA) \leq \frac{1}{2} \left\| |A|^2 + |B^*|^2 \right\| \tag{3.32}$$

for any A, B bounded linear operators on H.

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