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GRÜSS TYPE INEQUALITIES FOR A PROJECTION IN HILBERT SPACES AND APPLICATIONS

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Abstract. In this paper we establish some vector inequalities related to Grüss' inequality in inner product spaces. We show amongst others that, if $P : H \to H$ is an *orthogonal projection* on H, $x, y \in H$ and $\alpha, \beta \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$, $r, s > 0$ are such that

$$\|x - \alpha Px\| \le r \|x\| \text{ and } \|y - \beta Py\| \le s \|y\|,$$

then

$$|\langle x, y\rangle - \langle Px, y\rangle| \le rs \|x\| \|y\|.$$

Applications for norm and numerical radius inequalities of two bounded operators are given as well.

1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \ge |\langle x, y\rangle| \text{ for any } x, y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

We observe that the following two conditions are equivalent [11], [12]

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \ge 0, \ \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \ge 0 \tag{1.2}$$

and

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \le \frac{1}{2} |\Phi - \varphi|, \ \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \le \frac{1}{2} |\Gamma - \gamma| \tag{1.3}$$

if $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y, e are vectors in H with $\|e\| = 1$.

In [6], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the condition (1.2) holds, then we have the inequality*

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \le \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \tag{1.4}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

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For other Schwarz and Grüss related inequalities in inner product spaces, see [1]-[4], [5]-[14], [22]-[26], [30]-[39], and the monographs [17], [18] and [19].

Now, let us recall some basic facts on *orthogonal projection* that will be used in the sequel.

If K is a subset of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, the set of *vectors orthogonal* to K is defined by

$$K^{\perp} := \{x \in H : \langle x, k \rangle = 0 \text{ for all } k \in K\}.$$

We observe that $K^{\perp}$ is a *closed subspace* of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^{\perp}$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $K^{\perp}$. Therefore, H is the *internal Hilbert direct sum* of V and $V^{\perp}$, and we denote that as $H = V \oplus V^{\perp}$.

The linear operator $P_V : H \to H$ that maps x to v is called *the orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all *bounded self-adjoint* operators P such that $P^2 = P$. Specifically, the orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of V, which minimizes the distance $\|x - v\|$. This provides the geometrical interpretation of $P_V(x)$: it is *the best approximation* to x by elements of V.

Projections P_U and P_V are called *mutually orthogonal* if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. The sum of the two projections P_U and P_V is a projection only if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$.The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U \cap V}$.

A family $\{e_j\}_{j \in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j \in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j \in J}$ we have *Bessel's inequality*

$$\sum_{j \in J} |\langle x, e_j \rangle|^2 \leq \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j \in J} |\langle x, e_j \rangle|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j \in J}$ is an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j \in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j \in J} \langle x, e_j \rangle e_j, \; x \in H. \tag{1.5}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y \rangle = \sum_{j \in J} \langle x, e_j \rangle \langle e_j, y \rangle, \; x, y \in H \text{ and } \langle P_{\mathcal{E}} x, x \rangle = \sum_{j \in J} |\langle x, e_j \rangle|^2, \; x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle, \ x, y \in H \tag{1.6}$$

and the inequality (1.4) can be written as

$$|\langle x, y\rangle - \langle P_e x, y\rangle| \le \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| \tag{1.7}$$

provided that the condition (1.2) holds.

Motivated by the above results we establish in this paper some vector inequalities for an orthogonal projection P that generalizes amongst others the Grüss inequality (1.7). Applications for norm and numerical radius inequalities are provided as well.

2. Vector Inequalities for a Projection

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathcal{B}(H)$ that $0 \le P \le 1_H$.

The following result holds:

Theorem 2. *Let $P : H \to H$ be an orthogonal projection on H. If $x, y \in H$ and $\alpha, \beta \in \mathbb{K}$, $r, s > 0$ are such that*

$$\|x - \alpha P x\| \le r \|x\| \ \text{ and } \ \|y - \beta P y\| \le s \|y\|, \tag{2.1}$$

then

$$|\langle x, y\rangle - \langle P x, y\rangle| \le rs \|x\| \|y\|. \tag{2.2}$$

Proof. Using the properties of projection, we have

$$\begin{aligned} \langle x - Px, y - Py\rangle &= \langle x, y\rangle - \langle Px, y\rangle - \langle x, Py\rangle + \langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, y\rangle + \langle P^2 x, y\rangle \\ &= \langle x, y\rangle - \langle Px, y\rangle \end{aligned} \tag{2.3}$$

for any $x, y \in H$.

By Schwarz inequality we have

$$\|x - Px\|^2 \|y - Py\|^2 \ge |\langle x - Px, y - Py\rangle|^2 \tag{2.4}$$

for any $x, y \in H$, which is equivalent to

$$\|x - Px\| \|y - Py\| \ge |\langle x, y\rangle - \langle Px, y\rangle| \tag{2.5}$$

for any $x, y \in H$.

For any $z \in H$ and $\lambda \in \mathbb{K}$ we also have that

$$\begin{aligned} \langle z - \lambda Pz, z - Pz\rangle &= \|z\|^2 - \lambda \langle Pz, z\rangle - \langle z, Pz\rangle + \lambda \langle Pz, Pz\rangle \\ &= \|z\|^2 - \lambda \langle Pz, z\rangle - \langle z, Pz\rangle + \lambda \langle P^2 z, z\rangle \\ &= \|z\|^2 - \lambda \langle Pz, z\rangle - \langle z, Pz\rangle + \lambda \langle Pz, z\rangle \\ &= \|z\|^2 - \langle z, Pz\rangle = \|z - Pz\|^2. \end{aligned}$$

By Schwarz inequality we get

$$\|z - Pz\|^2 = |\langle z - \lambda Pz, z - Pz\rangle| \le \|z - \lambda Pz\| \|z - Pz\|,$$

which implies that

$$\|z - Pz\| \le \|z - \lambda Pz\| \tag{2.6}$$

for any $z \in H$ and $\lambda \in \mathbb{K}$.

By using (2.6) and (2.1) we have

$$\|x - Px\| \le \|x - \alpha Px\| \le r\|x\|$$

and

$$\|y - Py\| \le \|y - \beta Py\| \le s\|y\|$$

and by (2.5) we get the desired result (2.2). □

Corollary 1. *Let $P : H \to H$ be an orthogonal projection on H. If $x, y \in H$ and $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ with $\Phi \ne -\varphi$ and $\Gamma \ne -\gamma$ are such that either*

$$\operatorname{Re}\langle \Phi x - Px, Px - \varphi x\rangle \ge 0 \text{ and } \operatorname{Re}\langle \Gamma y - Py, Py - \gamma y\rangle \ge 0 \tag{2.7}$$

or, equivalently

$$\left\| Px - \frac{\Phi + \varphi}{2} x \right\| \le \frac{1}{2} |\Phi - \varphi| \|x\| \text{ and } \left\| Py - \frac{\Gamma + \gamma}{2} y \right\| \le \frac{1}{2} |\Gamma - \gamma| \|y\|, \tag{2.8}$$

then we have the inequality

$$|\langle x, y\rangle - \langle Px, y\rangle| \le \frac{|\Phi - \varphi|}{|\Phi + \varphi|} \frac{|\Gamma - \gamma|}{|\Gamma + \gamma|} \|x\| \|y\|. \tag{2.9}$$

Proof. Let a, z, A be vectors in the inner product space $(H, \langle \cdot, \cdot \rangle)$ over $\mathbb{K}$ with $a \ne A$. Then

$$\operatorname{Re}\langle A - z, z - a\rangle \ge 0$$

if and only if

$$\left\| z - \frac{a + A}{2} \right\| \le \frac{1}{2} \|A - a\|.$$

Indeed, if

$$I_1 := \operatorname{Re}\langle A - z, z - a\rangle \text{ and } I_2 := \frac{1}{4}\|A - a\|^2 - \left\| z - \frac{a + A}{2} \right\|^2,$$

then simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}[\langle z, a\rangle + \langle A, z\rangle] - \operatorname{Re}\langle A, a\rangle - \|z\|^2$$

and thus, obviously, $I_1 \ge 0$ iff $I_2 \ge 0$ showing the required equivalence.

We observe that if we take $z = Px$, $a = \varphi x$ and $A = \Phi x$, then we get from the above statement the equivalence of the first two conditions in (2.7) and (2.8). The second part goes likewise.

If $\Phi \ne -\varphi$ and $\Gamma \ne -\gamma$ then from (2.8) we get

$$\left\| \frac{2}{\Phi + \varphi} Px - x \right\| \le \frac{|\Phi - \varphi|}{|\Phi + \varphi|} \|x\|$$

and

$$\left\| \frac{2}{\Gamma + \gamma} Py - y \right\| \le \frac{|\Gamma - \gamma|}{|\Gamma + \gamma|} \|y\|.$$

If we use Theorem 2 for

$$\alpha = \frac{2}{\Phi + \varphi}, \beta = \frac{2}{\Gamma + \gamma}, r = \frac{|\Phi - \varphi|}{|\Phi + \varphi|} \text{ and } s = \frac{|\Gamma - \gamma|}{|\Gamma + \gamma|},$$

then we obtain the desired inequality (2.9). □

The following result holds:

Theorem 3. *Let $P : H \to H$ be an orthogonal projection on H. If $x, y \in H$ and $\alpha, \beta \in \mathbb{K}$, $r, s > 0$ are such that (2.1) holds true, then*

$$|\langle x, y\rangle - \langle Px, y\rangle| \leq \min\{s, r\} \|x\| \|y\| . \tag{2.10}$$

Proof. By Schwarz's inequality we have

$$\|x - Px\| \|y\| \geq |\langle x - Px, y\rangle| = |\langle x, y\rangle - \langle Px, y\rangle|$$

and

$$\|x\| \|y - Py\| \geq |\langle x, y - Py\rangle| = |\langle x, y\rangle - \langle x, Py\rangle| = |\langle x, y\rangle - \langle Px, y\rangle|$$

for any $x, y \in H$, which implies that

$$|\langle x, y\rangle - \langle Px, y\rangle| \leq \min\{\|x - Px\| \|y\| , \|x\| \|y - Py\|\} . \tag{2.11}$$

By (2.1) and (2.6) we have

$$\|x - Px\| \leq \|x - \alpha Px\| \leq r \|x\|$$

and

$$\|y - Py\| \leq \|y - \beta Py\| \leq s \|y\|$$

and by (2.11) we get the desired result (2.10). □

Corollary 2. *With the assumptions of Corollary 1 we have*

$$|\langle x, y\rangle - \langle Px, y\rangle| \leq \min\left\{\frac{|\Phi - \varphi|}{|\Phi + \varphi|}, \frac{|\Gamma - \gamma|}{|\Gamma + \gamma|}\right\} \|x\| \|y\| . \tag{2.12}$$

The proof is similar to the one from Corollary 1 and the details are omitted.

Remark 1. *Since, in general $\min\{r, s\}$ can be either smaller or bigger than rs, then both bounds for $|\langle x, y\rangle - \langle Px, y\rangle|$ provided by (2.2) and (2.10) are of interest.*

The following Grüss' type result also holds.

Theorem 4. *Let $e, x, y \in H$ with $\|e\| = 1$. If $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\operatorname{Re}\left(\alpha\overline{\beta}\right), \operatorname{Re}\left(\gamma\overline{\delta}\right) > 0$ are such that*

$$\left\|e - \frac{\alpha + \beta}{2} x\right\| \leq \frac{1}{2} |\alpha - \beta| \|x\| \text{ and } \left\|e - \frac{\gamma + \delta}{2} y\right\| \leq \frac{1}{2} |\gamma - \delta| \|y\| \tag{2.13}$$

or, equivalently

$$\operatorname{Re}\langle \alpha x - e, e - \beta x\rangle \geq 0 \text{ and } \operatorname{Re}\langle \gamma y - e, e - \delta y\rangle \geq 0, \tag{2.14}$$

then we have the inequalities

$$\begin{aligned} &|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\leq \min\left\{\frac{|\alpha - \beta|}{|\alpha + \beta|} \frac{|\gamma - \delta|}{|\gamma + \delta|}, \min\left\{\frac{|\alpha - \beta|}{|\alpha + \beta|}, \frac{|\gamma - \delta|}{|\gamma + \delta|}\right\}\right\} \|x\| \|y\| . \end{aligned} \tag{2.15}$$

Proof. We observe that, since

$$0 \leq \operatorname{Re}\langle \alpha x - e, e - \beta x\rangle = \operatorname{Re}\left[\alpha \langle x, e\rangle + \overline{\beta} \langle e, x\rangle\right] - 1 - \operatorname{Re}\left(\alpha\overline{\beta}\right) \|x\|^2 \tag{2.16}$$

then, if we assume that $\langle x, e\rangle = 0$, we get

$$1 + \operatorname{Re}\left(\alpha\overline{\beta}\right) \|x\|^2 \leq 0,$$

which is impossible since $\operatorname{Re}\left(\alpha\overline{\beta}\right) > 0$.

Therefore $\langle x, e\rangle \neq 0$ and similarly $\langle y, e\rangle \neq 0$.

Also, if $\operatorname{Re}\left(\alpha\overline{\beta}\right) > 0$, then $|\alpha+\beta|^2 = |\alpha|^2 + 2\operatorname{Re}\left(\alpha\overline{\beta}\right) + |\beta|^2 > 0$. Therefore $\alpha \neq -\beta$ and similarly $\gamma \neq -\delta$.

Now by multiplying the first inequality in (2.13) by $|\langle x,e\rangle| > 0$ and the second inequality by $|\langle y,e\rangle| > 0$ we get

$$\left\|\langle x,e\rangle e - \frac{\alpha\langle x,e\rangle + \beta\langle x,e\rangle}{2}x\right\| \leq \frac{1}{2}\left|\alpha\langle x,e\rangle - \beta\langle x,e\rangle\right|\|x\|$$

and

$$\left\|\langle y,e\rangle e - \frac{\gamma\langle y,e\rangle + \delta\langle y,e\rangle}{2}y\right\| \leq \frac{1}{2}\left|\gamma\langle y,e\rangle - \delta\langle y,e\rangle\right|\|y\|$$

namely,

$$\left\|P_e x - \frac{\alpha\langle x,e\rangle + \beta\langle x,e\rangle}{2}x\right\| \leq \frac{1}{2}\left|\alpha\langle x,e\rangle - \beta\langle x,e\rangle\right|\|x\|$$

and

$$\left\|P_e y - \frac{\gamma\langle y,e\rangle + \delta\langle y,e\rangle}{2}y\right\| \leq \frac{1}{2}\left|\gamma\langle y,e\rangle - \delta\langle y,e\rangle\right|\|y\|,$$

where $P_e z = \langle z,e\rangle e$, $z \in H$.

By utilizing (2.9) for the projection $P = P_e$, $\Phi = \alpha\langle x,e\rangle$, $\varphi = \beta\langle x,e\rangle$, $\Gamma = \gamma\langle y,e\rangle$ and $\gamma = \delta\langle y,e\rangle$ we get the first part of (2.15).

The second part of (2.15) follows by (2.12). □

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ consider the projector $P_{\mathcal{E}} : H \to H$

$$P_{\mathcal{E}}x := \sum_{j\in J}\langle x,e_j\rangle e_j\ ,\ x \in H.$$

If $x,y \in H$ and $\zeta,\eta,\theta,\vartheta \in \mathbb{K}$ with $\zeta \neq \eta$ and $\theta \neq \vartheta$ are such that either

$$\operatorname{Re}\left\langle \zeta x - \sum_{j\in J}\langle x,e_j\rangle e_j\ , \sum_{j\in J}\langle x,e_j\rangle e_j\ - \eta x\right\rangle \geq 0$$

and

$$\operatorname{Re}\left\langle \theta y - \sum_{j\in J}\langle y,e_j\rangle e_j\ , \sum_{j\in J}\langle y,e_j\rangle e_j\ - \vartheta y\right\rangle \geq 0$$

or, equivalently

$$\left\|\sum_{j\in J}\langle x,e_j\rangle e_j\ - \frac{\zeta+\eta}{2}x\right\| \leq \frac{1}{2}|\zeta-\eta|\|x\|$$

and

$$\left\|\sum_{j\in J}\langle y,e_j\rangle e_j\ - \frac{\theta+\vartheta}{2}y\right\| \leq \frac{1}{2}|\theta-\vartheta|\|y\|,$$

then by (2.9) and (2.12) we have the inequalities

$$\left|\langle x,y\rangle - \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right| \leq \min\left\{\frac{|\zeta-\eta|}{|\zeta+\eta|}\frac{|\theta-\vartheta|}{|\theta+\vartheta|}, \min\left\{\frac{|\zeta-\eta|}{|\zeta+\eta|}, \frac{|\theta-\vartheta|}{|\theta+\vartheta|}\right\}\right\}\|x\|\|y\|. \tag{2.17}$$

3. Inequalities for Norm and Numerical Radius

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [27, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by [27, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T)\le\|T\|\le 2w(T),\ \text{for any } T\in B(H).$$

The following general result for the product of two operators holds [27, p. 37]:

Theorem 5. *If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then $w(AB)\le 4w(A)w(B)$. In the case that $AB=BA$, then $w(AB)\le 2w(A)w(B)$. The constant 2 is best possible here.*

The following results are also well known [27, p. 38].

Theorem 6. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB)\le w(B). \tag{3.1}$$

If A is an isometry and $AB=BA$, then (3.1) also holds true.

We say that A and B *double commute* if $AB=BA$ and $AB^*=B^*A$. The following result holds [27, p. 38].

Theorem 7. *If the operators A and B double commute, then*

$$w(AB)\le w(B)\|A\|. \tag{3.2}$$

As a consequence of the above, we have [27, p. 39]:

Corollary 3. *Let A be a normal operator commuting with B. Then*

$$w(AB)\le w(A)w(B). \tag{3.3}$$

A related problem with the inequality (3.2) is to find the best constant c for which the inequality

$$w(AB)\le cw(A)\|B\|$$

holds for any two commuting operators $A,B\in B(H)$. It is known that $1.064<c<1.169$, see [3], [35] and [36].

In relation to this problem, it has been shown in [25] that

Theorem 8. *For any $A,B\in B(H)$ we have*

$$w\left(\frac{AB+BA}{2}\right)\le\sqrt{2}w(A)\|B\|. \tag{3.4}$$

For other numerical radius inequalities see the recent monograph [19] and the references therein.

We have the following new result:

Theorem 9. *Let $P : H \to H$ be an orthogonal projection on H. If $\alpha, \beta \in \mathbb{K}$, $r, s > 0$ are such that*

$$\|1_H - \alpha P\| \le r \text{ and } \|1_H - \beta P\| \le s, \tag{3.5}$$

then for any two bounded operators A and B we have the norm inequality

$$\|B(1_H - P)A\| \le \min\{rs, \min\{r,s\}\} \|A\| \|B\| \tag{3.6}$$

and the numerical radius inequality

$$w(B(1_H - P)A) \le \frac{1}{2}\min\{rs, \min\{r,s\}\} \|A^*A + BB^*\|. \tag{3.7}$$

Proof. Let $x, y \in H$. By (3.5) we have

$$\|x - \alpha Px\| \le \|1_H - \alpha P\| \|x\| \le r\|x\|$$

and

$$\|y - \beta Py\| \le \|1_H - \beta P\| \|y\| \le s\|y\|.$$

Using Theorem 2 we have

$$|\langle (1_H - P)x, y\rangle| \le rs\|x\|\|y\|$$

for any $x, y \in H$.

Taking in this inequality $x = Au$, $y = B^*v$, $u, v \in H$, we get

$$|\langle B(1_H - P)Au, v\rangle| \le rs\|Au\|\|B^*v\| \tag{3.8}$$

for any $u, v \in H$.

Taking the supremum over $u, v \in H$, $\|u\| = \|v\| = 1$ we have

$$\begin{aligned}\|B(1_H - P)A\| &= \sup_{\|u\|=\|v\|=1} |\langle B(1_H - P)Au, v\rangle| \\ &\le rs \sup_{\|u\|=\|v\|=1} \{\|Au\|\|B^*v\|\} \\ &= rs \sup_{\|u\|=1}\|Au\| \sup_{\|v\|=1}\|B^*v\| = rs\|A\|\|B\|\end{aligned}$$

and the first part of (3.6) is proved.

The second part follows by Theorem 3.

From (3.8) and the arithmetic mean - geometric mean inequality we have

$$\begin{aligned}|\langle B(1_H - P)Au, u\rangle| &\le rs\|Au\|\|B^*u\| \\ &\le \frac{1}{2}rs\left(\|Au\|^2 + \|B^*u\|^2\right) \\ &= \frac{1}{2}rs\langle (A^*A + BB^*)u, u\rangle\end{aligned} \tag{3.9}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.9) we get the first part of (3.7). The second part goes likewise. □

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform, [15] or [19, p. 99]

$$C_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = C_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [15] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re}\langle C_{\alpha,\beta}(T,U)\,x, x\rangle = \operatorname{Re}\langle C_{\beta,\alpha}(T,U)\,x, x\rangle = \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2, \tag{3.10}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following [19, p. 100]:

Lemma 1. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}(T,U)$ is accretive;*
(ii) *We have the norm inequality*

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|Ux\| \tag{3.11}$$

for any $x \in H$.

As a consequence of the above lemma we can state:

Corollary 4. *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $C_{\alpha,\beta}(T,U)$ is accretive, then*

$$\left\|T - \frac{\alpha+\beta}{2}\cdot U\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|U\|. \tag{3.12}$$

Remark 2. *In order to give examples of linear operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operators S and V and the complex numbers z, w $(w \neq 0)$ with the property that $\|Sx - zVx\| \leq |w|\,\|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T and U satisfy (3.11), i.e., $C_{\alpha,\beta}(T,U)$ is accretive.*

Using some Grüss' type inequalities in inner product spaces we have the following results, see [16] or [19, p. 104 -106]:

Theorem 10. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$w(BA) \leq w(A)\,w(B) + \frac{1}{4}|\beta-\alpha|\,|\gamma-\delta|. \tag{3.13}$$

and

Theorem 11. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ and the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then*

$$\frac{w(BA)}{w(A)\,w(B)} \leq 1 + \frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}}. \tag{3.14}$$

We have the following result:

Theorem 12. *Let $P : H \to H$ be an orthogonal projection on H and A, B two bounded linear operators on H. If $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq -\beta$ and $\gamma \neq -\delta$ are such that $C_{\alpha,\beta}(PA, A)$ and $C_{\gamma,\delta}(PB^*, B^*)$ are accretive, then we have the norm inequalities*

$$\begin{aligned} &\|B(1_H - P)A\| \\ &\leq \min\left\{\frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}, \min\left\{\frac{|\Phi-\varphi|}{|\Phi+\varphi|}, \frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\right\}\right\}\|A\|\|B\|. \end{aligned} \tag{3.15}$$

Moreover, we have the numerical radius inequalities

$$\begin{aligned} &w(B(1_H - P)A) \\ &\leq \frac{1}{2}\min\left\{\frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}, \min\left\{\frac{|\Phi-\varphi|}{|\Phi+\varphi|}, \frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\right\}\right\}\|A^*A + BB^*\|. \end{aligned} \tag{3.16}$$

Proof. If $C_{\alpha,\beta}(PA, A)$ and $C_{\gamma,\delta}(PB, B)$ are accretive, then by Lemma 1 we have

$$\left\|PAu - \frac{\alpha+\beta}{2}Au\right\| \leq \frac{1}{2}|\beta-\alpha|\|Au\|$$

and

$$\left\|PB^*v - \frac{\gamma+\delta}{2}B^*v\right\| \leq \frac{1}{2}|\gamma-\delta|\|B^*v\|$$

for any $u, v \in H$.

Now, if we apply Corollary 1 for $x = Au$ and $y = B^*v$ we get

$$|\langle Au, B^*v\rangle - \langle PAu, B^*v\rangle| \leq \frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\|Au\|\|B^*v\|$$

for any $u, v \in H$.

This inequality is equivalent to

$$|\langle B(1_H - P)Au, v\rangle| \leq \frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\|Au\|\|B^*v\| \tag{3.17}$$

for any $u, v \in H$.

Taking the supremum over $u, v \in H$, $\|u\| = \|v\| = 1$ we get the first part of inequality (3.15).

The second part of inequality (3.15) follows by (2.12).

From (3.17) and by the elementary inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right), \ a, b \in \mathbb{R}$$

we have

$$\begin{aligned} |\langle B(1_H - P)Au, u\rangle| &\leq \frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\|Au\|\|B^*u\| \\ &\leq \frac{1}{2}\frac{|\Phi-\varphi|}{|\Phi+\varphi|}\frac{|\Gamma-\gamma|}{|\Gamma+\gamma|}\langle(A^*A + BB^*)u, u\rangle \end{aligned} \tag{3.18}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.18) we get the first part of inequality (3.16).

The second part is similar. □

Finally, we have:

Theorem 13. *Let $A,B\in B(H)$ and $\alpha,\beta,\gamma,\delta\in\mathbb{K}$ with $\operatorname{Re}\left(\alpha\overline{\beta}\right),\ \operatorname{Re}\left(\gamma\overline{\delta}\right)>0$ be such that the transforms $C_{\alpha,\beta}\left(1_H,A\right)$ and $C_{\gamma,\delta}\left(1_H,B^*\right)$ are accretive, then*

$$\begin{aligned}(3.19)\qquad w(BA) &\leq w(B)\,w(A)\\ &+\frac{1}{2}\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|},\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|},\frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\\ &\times\left\|A^*A+BB^*\right\|.\end{aligned}$$

Proof. Since the transforms $C_{\alpha,\beta}\left(1_H,A\right)$ and $C_{\gamma,\delta}\left(1_H,B\right)$ are accretive, then

$$\left\|x-\frac{\alpha+\beta}{2}Ax\right\|\leq\frac{1}{2}|\beta-\alpha|\,\|Ax\|$$

and

$$\left\|x-\frac{\gamma+\delta}{2}B^*x\right\|\leq\frac{1}{2}|\gamma-\delta|\,\|B^*x\|$$

for any $x\in H$ with $\|x\|=1$.

Making use of Theorem 4 we have

$$\begin{aligned}(3.20)\qquad &\left|\langle Ax,B^*x\rangle-\langle Ax,x\rangle\langle x,B^*x\rangle\right|\\ &\leq\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|},\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|},\frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\|Ax\|\,\|B^*x\|,\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

By the triangle inequality we have

$$\left|\langle Ax,B^*x\rangle\right|-\left|\langle Ax,x\rangle\langle x,B^*x\rangle\right|\leq\left|\langle Ax,B^*x\rangle-\langle Ax,e\rangle\langle e,B^*x\rangle\right|$$

and then by (3.20) we get

$$\begin{aligned}&\left|\langle BAx,x\rangle\right|\\ &\leq\left|\langle Ax,x\rangle\right|\left|\langle Bx,x\rangle\right|\\ &+\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|},\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|},\frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\|Ax\|\,\|B^*x\|\\ &\leq\left|\langle Ax,x\rangle\right|\left|\langle Bx,x\rangle\right|\\ &+\frac{1}{2}\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|},\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|},\frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\\ &\times\left\langle\left(A^*A+BB^*\right)x,x\right\rangle\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Taking the supremum over $x\in H$ with $\|x\|=1$ we deduce the desired result (3.19). □

Remark 3. *We observe that, by (3.20), if $C_{\alpha,\beta}\left(1_H,A\right)$ and $C_{\gamma,\delta}\left(1_H,A^*\right)$ are accretive, then*

$$\begin{aligned}&\left|\langle Ax,x\rangle\right|^2-\left|\left\langle A^2x,x\right\rangle\right|\\ &\leq\left|\langle Ax,A^*x\rangle-\langle Ax,x\rangle\langle x,A^*x\rangle\right|\\ &\leq\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|}\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|},\frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\|Ax\|\,\|A^*x\|,\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, which, in a similar fashion, produces the following numerical radius inequality for one operator

$$0 \le w^2(A) - w(A^2) \le \frac{1}{2}\min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}\frac{|\gamma-\delta|}{|\gamma+\delta|}, \min\left\{\frac{|\alpha-\beta|}{|\alpha+\beta|}, \frac{|\gamma-\delta|}{|\gamma+\delta|}\right\}\right\}\|A^*A + AA^*\|. \tag{3.21}$$

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