Received 10/03/15

SOME INEQUALITIES ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITIES FOR OPERATOR h-CONVEX FUNCTIONS

A. TAGHAVI, V. DARVISH, H. M. NAZARI, S. S. DRAGOMIR

Abstract. In this paper, we introduce the concept of operator h-convex functions for positive linear maps and prove some Hermite-Hadamard type inequalities for these functions. As applications, we obtain several singular value and trace inequalities for operators which provide refinements of previous results.

1. Introduction and preliminaries

Let $B(H)$ stand for the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is positive and write $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. Let $B(H)^+$ stand for all positive operators in $B(H)$.

A linear map $\Phi : B(H) \to B(H)$ is *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$ and Φ is said to be *unital* if $\Phi(I) = I$.
Let $\Phi : B(H) \to B(H)$ then $\Phi(A) = X^*AX$ where X is an operator in $B(H)$ and $\Phi(A) = A^*$ for $A \in B(H)$ are examples of positive linear maps.

We say that a linear map is *invertible preserving* if $\Phi(A)$ is invertible whenever A is invertible.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [17, p.3]):

For any $f, g \in C(\mathrm{Sp}(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

2010 *Mathematics Subject Classification.* 47A63, 15A60, 47B05, 47B10, 26D15.
Key words and phrases. Hermite-Hadamard inequality, Operator h-convex function, Linear map, Singular value inequality, Trace inequality.

with this notation we define

$$f(A) = \Phi(f) \text{ for all } \ f \in C(\mathrm{Sp}(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \ \text{ for any } \ t \in \mathrm{Sp}(A) \ \text{ implies that } \ f(A) \geq g(A), \tag{1.1}$$

in the operator order of $B(H)$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1.2}$$

It was firstly discovered by Hermite in 1881 in the journal Mathesis (see [22]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [25].

Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [3]. In 1974, Mitrinovič found Hermite's note in Mathesis [22]. Since (1.2) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [25].

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty, \quad t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R}, \quad g(x, y)(t) := f[(1-t)x + ty], \quad t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the Hermite-Hadamard integral inequality (see [12, p.2], [13, p.2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{1.3}$$

which can be derived from the classical Hermite-Hadamard inequality (1.2) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, we have the following norm inequality from (1.3) (see [24]):

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2},$$

for any $x, y \in X$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A+\lambda B) \leq (\geq)(1-\lambda)f(A)+\lambda f(B), \tag{1.4}$$

in the operator order, for all $\lambda \in [0,1]$ and for every self-adjoint operator A and B on a Hilbert space H whose spectra are contained in I (see [14]).

As an example of such functions, we note that $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$ (see [5, p.147]).

Motivated by above results, Dragomir in [14] investigated the operator version of the Hermite-Hadamard inequality for operator convex functions asserts that if $f : I \to \mathbb{R}$ is an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I the following inequalities hold

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\left((1-t)A+tB\right)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right] \\ &\leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.5}$$

To prove above inequalities, he considered the convex function $\varphi(t)=\langle f(tA+(1-t)B)x,x\rangle$ on $[0,1]$ for any $x \in H$ with $\|x\|=1$, self-adjoint operators A and B with spectra in I and operator convex function f.
By considering $\varphi(t)$ on $[\frac{1}{4},\frac{3}{4}]$, we can give a refinement for (1.5) as follows

$$\varphi\left(\frac{\frac{1}{4}+\frac{3}{4}}{2}\right) \leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} \varphi(t)dt \leq \frac{\varphi(\frac{1}{4})+\varphi(\frac{3}{4})}{2}. \tag{1.6}$$

So,

$$\begin{aligned} \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} \langle f(tA+(1-t)B)x,x\rangle dt \\ &\leq \frac{1}{2}\left[\left\langle f\left(\frac{A+3B}{4}\right)x,x\right\rangle+\left\langle f\left(\frac{3A+B}{4}\right)x,x\right\rangle\right]. \end{aligned}$$

The continuity of f implies that

$$\int_{\frac{1}{4}}^{\frac{3}{4}} \langle f(tA+(1-t)B)x,x\rangle dt = \left\langle \int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dtx,x\right\rangle,$$

for any $x \in H$ with $\|x\| = 1$.
Therefore,

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+3B}{4}\right)+f\left(\frac{3A+B}{4}\right)\right], \end{aligned}$$

for self-adjoint operators A and B with spectra in I.

Another class of functions considered by Hudzik and Maligranda [21] are s-convex functions and are defined as follows:
Let $f : \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted by K_s^2.

In [15], Dragomir and Fitzpatrick proved the following Hermite-Hadamard type inequality for s-convex functions in the second sense states that let $f : [0, \infty) \to [0, \infty)$ be an s-convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$ such that $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

In order to extend this class of functions to operators, Ghazanfari defined operator s-convex function in [18] as follows:
Let I be an interval in $[0, \infty)$. A continuous function $f : I \to \mathbb{R}$ is said to be operator s-convex on I for operators in $B(H)^+$ if

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B),$$

in the operator order in B(H), for all $\lambda \in [0, 1]$ and for every positive operators A and B in $B(H)^+$ whose spectra are contained in I and for fixed $s \in (0, 1]$.

By above definition of operator s-convex function, he proved that if $f : I \to \mathbb{R}$ is an operator s-convex function on the interval $I \subseteq [0, \infty)$, then the following inequalities hold

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A)+f(B)}{s+1}.$$

In this paper, we introduce the concept of operator h-convex functions and obtain some Hermite-Hadamard type inequalities for these class of functions for positive linear maps. These results lead us further to obtain some inequalities for singular values and trace functional of operators. Some of these inequalities improve recent results.

2. Inequalities for operator h-convex functions of positive linear maps

In this section, we give Hermite-Hadamard type inequalities for operator h-convex functions of positive linear maps.

Let $I, J \subseteq \mathbb{R}$, $(0,1) \subseteq J$ and functions f, h are real non-negative on I and J.

Definition 2.1. [30] Let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{2.1}$$

If inequality (2.1) is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that if $h(\lambda) = \lambda$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(\lambda) = \lambda^s$, where $s \in (0,1)$ then $K_s^2 \subseteq SX(h, I)$.

The following inequalities due to Sarikaya [26], gives the Hermite-Hadamard type inequalities for h-convex functions. Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L^1([a,b])$. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (f(a)+f(b))\int_0^1 h(t)dt. \tag{2.2}$$

Here, we define operator h-convex function.

Definition 2.2. A continuous function $f : I \to \mathbb{R}$ is said to be operator h-convex on I if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda)f(A) + h(1-\lambda)f(B), \tag{2.3}$$

for all $\lambda \in (0,1)$ and self-adjoint $A, B \in B(H)$ whose spectra are contained in I.

Lemma 2.3. *If f is an operator h-convex function then $\varphi_{x,A,B}(t) = \langle f(tA + (1-t)B)x, x\rangle$ for any $x \in H$ with $\|x\| = 1$, is an h-convex function on $(0,1)$.*

Proof. Let f be an operator h-convex function, then for $u, v \in [0,1]$ we have

$$\begin{aligned} \varphi_{x,A,B}(tu+(1-t)v) &= \langle f\left[(tu+(1-t)v)A + (1-(tu+(1-t)v)B)\right]x, x\rangle \\ &= \langle f(t[uA+(1-u)B] + (1-t)[(vA+(1-v)B)]x, x\rangle \\ &\leq h(t)\langle f(uA+(1-u)B)x, x\rangle \\ &\quad + h(1-t)\langle f(vA+(1-v)B)x, x\rangle \\ &= h(t)\varphi_{x,A,B}(u) + h(1-t)\varphi_{x,A,B}(v). \end{aligned}$$

So, $\varphi_{x,A,B}$ is an h-convex function on $[0,1]$.

□

Theorem 2.4. *Let f be an operator h-convex function. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA+(1-t)B)dt \leq (f(A)+f(B))\int_0^1 h(t)dt. \quad (2.4)$$

Proof. Since f is operator h-convex function, by Lemma 2.3, we have $\varphi_{x,A,B}(t) = \langle f(tA+(1-t)B)x, x\rangle$ is h-convex function on $[0,1]$. So, by (2.2) we obtain

$$\begin{aligned} \frac{\varphi_{x,A,B}(\frac{1}{2})}{2h(\frac{1}{2})} &\leq \int_0^1 \varphi_{x,A,B}(t)dt \\ &\leq (\varphi_{x,A,B}(0)+\varphi_{x,A,B}(1))\int_0^1 h(t)dt. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})}\left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle &\leq \int_0^1 \langle f(tA+(1-t)B)x, x\rangle dt \\ &\leq (\langle f(A)x, x\rangle + \langle f(B)x, x\rangle)\int_0^1 h(t)dt, \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and self-adjoint operators A and B with spectra in I.
By the continuity of f we have

$$\int_0^1 \langle f(tA+(1-t)B)x, x\rangle dt = \left\langle \int_0^1 f(tA+(1-t)B)dtx, x\right\rangle.$$

So, the proof is complete. □

Let $I = [0,\infty)$ in above theorem. Since Φ is positive map and spectrum of positive operator is in $[0,\infty)$, we have the following result.

Corollary 2.5. *Let $f : [0,\infty) \to \mathbb{R}$ be an operator h-convex function for operators in $B(H)^+$ and $\Phi : B(H) \to B(H)$ be a positive linear map. Then*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})}f\left(\Phi\left(\frac{A+B}{2}\right)\right) &\leq \int_0^1 f(\Phi(tA+(1-t)B))dt \\ &\leq [f(\Phi(A))+f(\Phi(B))]\int_0^1 h(t)dt. \end{aligned}$$

Here, we can obtain some results for positive linear operators when $h(t) = t^s$ and $h(t) = t$.

Let $h(t) = t^s$ for $s \in (0,1)$ in Corollary 2.5, then we have

$$2^{s-1}f\left(\Phi\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 f(\Phi(tA+(1-t)B))dt \leq \frac{f(\Phi(A))+f(\Phi(B))}{s+1}. \quad (2.5)$$

Also, let $h(t) = t$ in Corollary 2.5, then we have

$$f\left(\Phi\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 f(\Phi(tA+(1-t)B))dt \leq \left(\frac{f(\Phi(A))+f(\Phi(B))}{2}\right). \quad (2.6)$$

Example 2.6. [18] Let $AB + BA \geq 0$ for $A, B \in B(H)^+$, (AB+BA is called symmetrized product of A and B) then the continuous function $f(t) = t^s$, $0 < s \leq 1$ is an operator s-convex function on $[0, \infty)$.

It should be mentioned here that $f(t) = t^s$ is not necessarily operator s-convex function for $s \in (0, 1]$ without $AB + BA \geq 0$. For showing this, let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. One can check that $AB + BA \ngeq 0$ and $(A+B)^{\frac{1}{2}} \nleq A^{\frac{1}{2}} + B^{\frac{1}{2}}$.
It is obvious $f(t) = t^s$, for $s \in (0, \infty)$ is operator s-convex function on $[0, \infty)$ if C^*-algebra $B(H)$ is commutative. In [29, Theorem 1] Uchiyama showed the relation between commutativity and symmetrized product of A and B. Later the authors of [23, Theorem 2] gave a weaker condition for two commuting operators. They proved an unital C^*-algebra $\mathcal{A}$ is commutative if and only if positive operators A and B in $\mathcal{A}$ satisfy $AB + BA \geq 0$ and $AB^2 + BA^2 \geq 0$.

By inequalities (2.5) and Example 2.6, we have the following

$$2^{s-1}\left(\Phi\left(\frac{A+B}{2}\right)\right)^s \leq \int_0^1 (\Phi(tA+(1-t)B))^s dt \leq \frac{(\Phi(A))^s + (\Phi(B))^s}{s+1}, \quad (2.7)$$

for $A, B \in B(H)^+$ such that $AB + BA \geq 0$.

The following Jensen type inequality is due to C. Davis [11].

Lemma 2.7. *Let $\Phi : B(H) \to B(H)$ be a unital positive linear map and f be an operator convex function on $[0, \infty)$. Then, for every $A \geq 0$*

$$f(\Phi(A)) \leq \Phi(f(A)). \quad (2.8)$$

Now, by applying above lemma and inequalities (2.6), we obtain the following inequalities for unital positive linear map

$$f\left(\Phi\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 f(\Phi(tA+(1-t)B))dt \leq \Phi\left(\frac{f(A)+f(B)}{2}\right). \quad (2.9)$$

where f is an operator convex function on $[0, \infty)$.

Also, by making use of following lemma we can improve above inequalities for a specific interval $I \subseteq [0, \infty)$. The following lemma is well known however for reader's convenience we provide a short proof.

Lemma 2.8. *Let $\Phi : B(H) \to B(H)$ be a unital and invertible preserving map, then Φ is spectrum compressing (i.e.* $\mathrm{Sp}(\Phi(A)) \subseteq \mathrm{Sp}(A)$*).*

Proof. If $A \in B(H)$ and $\lambda \in \mathbb{C}$, then $\Phi(\lambda I - A) = \lambda I - \Phi(A)$. Since Φ is invertible preserving map, $\lambda \notin \mathrm{Sp}(\Phi(A))$ whenever $\lambda \notin \mathrm{Sp}(A)$. Hence, $\mathrm{Sp}(\Phi(A)) \subseteq \mathrm{Sp}(A)$. □

By Corollary 2.5 and above lemma we have:

Corollary 2.9. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be an operator s-convex functions for operators in $B(H)^+$ with* $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$ *and $\Phi : B(H) \to B(H)$ be a unital positive invertible preserving map. Then*

$$2^{s-1} f\left(\Phi\left(\frac{A+B}{2}\right)\right) \le \int_0^1 f(\Phi(tA + (1-t)B))dt \le \frac{f(\Phi(A)) + f(\Phi(B))}{s+1}. \tag{2.10}$$

Let $s = 1$ in above corollary, then we have

$$f\left(\Phi\left(\frac{A+B}{2}\right)\right) \le \int_0^1 f(\Phi(tA + (1-t)B))dt \le \Phi\left(\frac{f(A) + f(B)}{2}\right), \tag{2.11}$$

for operator convex function f.

3. Some singular value inequalities for operators

In this section we give some inequalities for singular values of operators. First we recall some preliminaries.

Let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $A \in B(H)$, let $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$ denote the usual operator norm of A and $|A| = (A^*A)^{1/2}$ be the absolute value of A.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$. For $1 \le p < \infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p = (\mathrm{Tr}\,|A|^p)^{1/p}$, where Tr is the usual trace functional. Note that for $A \in K(H)$ we have, $\|A\| = s_1(A)$, and if A is a Hilbert-Schmidt operator, then $\|A\|_2 = (\sum_{j=1}^{\infty} s_j^2(A))^{1/2}$. These norms are special examples of the more general class of the Schatten p-norms, which are unitarily invariant [5].

The direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $H \oplus H$ (see [31]). It is easy to see that

$$\|A \oplus B\| = \max(\|A\|, \|B\|) \tag{3.1}$$

and

$$\|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{1/p}. \tag{3.2}$$

We denote the singular values of an operator $A \in K(H)$ as $s_1(A) \geq s_2(A) \geq \ldots$ are the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$ which repeated accordingly to multiplicity.

There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $|||\cdot|||$ is unitarily invariant norm, then there exists a unique symmetric gauge function Φ such that

$$|||A||| = \Phi(s_1(A), s_2(A), \ldots),$$

for every operator $A \in K(H)$. Let $A \in K(H)$, and if $U, V \in B(H)$ are unitary operators, then

$$s_j(UAV) = s_j(A),$$

for $j = 1, 2, \ldots$ and so unitarily invariant norms satisfies the invariance property

$$|||UAV||| = |||A|||.$$

The following inequality is due to Hirzallah and Kittaneh [19, Corollary 2.2] asserts that if $A, B \in K(H)$, then

$$s_j\left(\frac{A+B}{2}\right) \leq s_j(A \oplus B), \tag{3.3}$$

for $j = 1, 2, \ldots$.

We give a refinement of above inequality for positive operators. For this aim, we need the following lemma.

Lemma 3.1. [5, p. 75] *Let $A, B \in B(H)$ such that A is compact. Then*

$$s_j(AB) \leq \|B\| s_j(A),$$

for $j = 1, 2, \ldots$.

We noted in introduction that $\Phi(A) = X^*AX$ from $B(H)$ to $B(H)$ is positive map for an arbitrary operator $X \in B(H)$.

Theorem 3.2. *Let X be an arbitrary operator in $B(H)$. Then,*

(1) We have

$$\begin{aligned} \frac{1}{2} s_j\left((A+B)^{1/2}X\right)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \frac{2}{r+1}\|X\|^{2r} s_j(A \oplus B)^r, \end{aligned}$$

for $j = 1, 2, \ldots$ where $r \in [\frac{1}{2}, 1]$ and positive operators $A, B \in K(H)$ such that $AB + BA \geq 0$.

(2) We also have

$$\begin{aligned} \frac{1}{2^r} s_j((A+B)^{1/2}X)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \|X\|^{2r} s_j(A \oplus B)^r, \end{aligned}$$

for $j = 1, 2, \ldots$ *where* $r \in [-1, 0] \cup [1, 2]$ *and positive operators* $A, B \in K(H)$.

Proof. We just prove the first part of above theorem, the second part is similar by making use of inequalities (2.6).
By inequalities (2.5) for $\Phi(A) = X^*AX$, we have

$$\begin{aligned} 2^{r-1} f\left(X^*\left(\frac{A+B}{2}\right)X\right) &\leq \int_0^1 f(X^*(tA+(1-t)B)X)dt \\ &\leq \frac{f(X^*AX)+f(X^*BX)}{r+1}, \end{aligned} \tag{3.4}$$

for $A, B \in B(H)^+$ and $r \in (0, 1]$.

Let $f(t) = t^r$ be an operator s-convex function on $[0, \infty)$ for $r \in (0, 1]$, then we have

$$\begin{aligned} \frac{1}{2}\left(X^*(A+B)X\right)^r &\leq \int_0^1 (X^*(tA+(1-t)B)X)^r dt \\ &\leq \frac{(X^*AX)^r + (X^*BX)^r}{r+1}, \end{aligned} \tag{3.5}$$

for $A, B \in B(H)^+$ such that $AB + BA \geq 0$. Since A and B are positive, we have

$$\begin{aligned} \frac{1}{2}\left(X^*(A+B)X\right)^r &= \frac{1}{2}\left(X^*(A+B)^{1/2}(A+B)^{1/2}X\right)^r \\ &= \frac{1}{2}\left(\left|(A+B)^{1/2}X\right|^2\right)^r \\ &= \frac{1}{2}\left(\left|(A+B)^{1/2}X\right|\right)^{2r}. \end{aligned}$$

So,

$$\frac{1}{2}\left(X^*(A+B)X\right)^r = \frac{1}{2}\left(\left|(A+B)^{1/2}X\right|\right)^{2r}.$$

On the other hand, similar to above we have

$$\frac{(X^*AX)^r + (X^*BX)^r}{r+1} = \frac{|A^{1/2}X|^{2r} + |B^{1/2}X|^{2r}}{r+1}.$$

Therefore, by (3.5) we have

$$\begin{aligned}\frac{1}{2}\left(\left|(A+B)^{1/2}X\right|\right)^{2r} &\leq \int_0^1 (X^*(tA+(1-t)B)X)^r dt \\ &\leq \frac{|A^{1/2}X|^{2r}+|B^{1/2}X|^{2r}}{r+1}.\end{aligned}$$

By Weyl's monotonicity principle, we obtain

$$\begin{aligned}\frac{1}{2}s_j\left(\left|(A+B)^{1/2}X\right|\right)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \frac{s_j\left(|A^{1/2}X|^{2r}+|B^{1/2}X|^{2r}\right)}{r+1}, \end{aligned}\tag{3.6}$$

for $j = 1, 2, \ldots$.
By above inequalities and inequality (3.3), we have

$$\begin{aligned}\frac{1}{2}s_j\left((A+B)^{1/2}X\right)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \frac{2}{r+1}s_j\left(|A^{1/2}X|^{2r}\oplus|B^{1/2}X|^{2r}\right) \\ &= \frac{2}{r+1}s_j\left(\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2}\end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X\end{bmatrix}\right)^{2r} \\ &\leq \frac{2}{r+1}\|X\oplus X\|^{2r}s_j(A^{1/2}\oplus B^{1/2})^{2r} \\ &= \frac{2}{r+1}\|X\|^{2r}s_j(A\oplus B)^r,\end{aligned}$$

for $j = 1, 2, \ldots$ and $r \in [\frac{1}{2}, 1]$. In the last inequality above, we applied Lemma 3.1 and min-max principle [5, p. 58].
Finally, we have

$$\begin{aligned}\frac{1}{2}s_j\left((A+B)^{1/2}X\right)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \frac{2}{r+1}\|X\|^{2r}s_j(A\oplus B)^r,\end{aligned}$$

for $j = 1, 2, \ldots$, $r \in [\frac{1}{2}, 1]$ and positive operators A and B such that $AB + BA \geq 0$. □

Let $X = I$ in above theorem, we have

Corollary 3.3. *Let $A, B \in K(H)$ be two positive operators. Then,*

(1) *We have*

$$\frac{1}{2}s_j(A+B)^r \leq s_j\left(\int_0^1 (tA+(1-t)B)^r dt\right) \leq \frac{2}{r+1}s_j(A\oplus B)^r, \tag{3.7}$$

for $r\in[\frac{1}{2},1]$, $j=1,2,\dots$ *and additional assumption* $AB+BA\geq 0$.

(2) *We also have*

$$s_j\left(\frac{A+B}{2}\right)^r \leq s_j\left(\int_0^1 (tA+(1-t)B)^r dt\right) \leq s_j(A\oplus B)^r, \tag{3.8}$$

for $j=1,2,\dots$ *and* $r\in[-1,0]\cup[1,2]$.

We point out in what follows some known inequalities which can be achieved by the above results.
In [8] Bhatia and Kittaneh obtained the following inequality for $A,B\in K(H)$

$$2s_j(AB^*) \leq s_j(A^*A+B^*B), \tag{3.9}$$

for $j=1,2,\dots$. By min-max principle, we can write above inequality as follows for $n\in\mathbb{N}$

$$2^{\frac{1}{n}}s_j^{\frac{1}{n}}(AB^*) \leq s_j(A^*A+B^*B)^{\frac{1}{n}}, \tag{3.10}$$

for $j=1,2,\dots$.
It is clear that if A and B are positive then (3.9) reduces to

$$2s_j(AB) \leq s_j(A^2+B^2), \tag{3.11}$$

for $j=1,2,\dots$.
Moreover, it has been shown by Ando [1], that if $A,B\in K(H)$ are positive and if $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$s_j(AB) \leq s_j\left(\frac{1}{p}A^p+\frac{1}{q}B^q\right),$$

for $j=1,2,\dots$.

The authors of [7] were wondered if the following inequality is true or not?

$$2s_j^{\frac{1}{2}}(AB) \leq s_j(A+B), \tag{3.12}$$

for $j=1,2,\dots$. Since the square function on self-adjoint is operator convex, i.e., $\left(\frac{A+B}{2}\right)^2 \leq \left(\frac{A^2+B^2}{2}\right)$. Therefore, the inequality (3.12) is stronger than (3.11).
In [16], Drury proved inequality (3.12) for positive operators A and B in $K(H)$.

Here, we prove similar inequality to (3.12) for arbitrary operators.
Let write inequalities (2.5) for $\Phi(A)=A$ and $f(t)=t^r$, where $r\in(0,1]$

$$2^{r-1}\left(\frac{A+B}{2}\right)^r \le \int_0^1 (tA+(1-t)B)^r dt \le \frac{A^r+B^r}{r+1}.$$

By Weyl's monotonicity principle, we have

$$s_j(A+B)^r \le 2s_j\left(\int_0^1 (tA+(1-t)B)^r dt\right) \le \frac{2}{r+1}s_j(A^r+B^r), \tag{3.13}$$

for $j=1,2,\ldots$, $r\in(0,1]$ and $A,B\in K(H)^+$ such that $AB+BA\ge 0$.

On the other hand, by making use of (3.10) and (3.13), we can write

$$\begin{aligned} 2^{\frac{1}{n}} s_j^{\frac{1}{n}}(AB^*) &\le s_j(A^*A+B^*B)^{\frac{1}{n}} \\ &\le 2s_j\left(\int_0^1 (tA^*A+(1-t)B^*B)^{\frac{1}{n}} dt\right) \\ &\le \frac{2n}{n+1}s_j((A^*A)^{\frac{1}{n}}+(B^*B)^{\frac{1}{n}}). \end{aligned}$$

It means that

$$\begin{aligned} 2^{\frac{1}{n}-1}\left(\frac{n+1}{n}\right)s_j^{\frac{1}{n}}(AB^*) &\le \frac{n+1}{n}s_j\left(\int_0^1 (tA^*A+(1-t)B^*B)^{\frac{1}{n}} dt\right) \\ &\le s_j((A^*A)^{\frac{1}{n}}+(B^*B)^{\frac{1}{n}}), \end{aligned} \tag{3.14}$$

for$n\in\mathbb{N}$, $j=1,2,\ldots$ and operators $A,B\in K(H)$ such that $A^*AB^*B+B^*BA^*A\ge 0$. The inequality (3.14) is therefore a generalization of (3.9).
Let $n=2$ in inequality (3.14), we have

$$\frac{3}{2\sqrt{2}}s_j^{\frac{1}{2}}(AB^*) \le \frac{3}{2}s_j\left(\int_0^1 (tA^*A+(1-t)B^*B)^{\frac{1}{2}} dt\right) \le s_j(|A|+|B|),$$

for $j=1,2,\ldots$ and operators $A,B\in K(H)$ such that $A^*AB^*B+B^*BA^*A\ge 0$.
Since every unitarily invariant norm is a monotone function of the singular values of an operator, we can write

$$\frac{3}{2\sqrt{2}}|||AB^*|^{\frac{1}{2}}||| \le \frac{3}{2}\left|\left|\left|\int_0^1 (tA^*A+(1-t)B^*B)^{\frac{1}{2}} dt\right|\right|\right| \le |||A|+|B|||.$$

The following inequality due to Bhatia and Kittaneh [6] asserts that if $A,B\in B(H)$ are positive operators and m is any positive integer, then

$$|||A^m+B^m||| \le |||(A+B)^m|||.$$

Moreover, they have proved the following inequalities for usual operator norm:

$$\|A^r+B^r\| \ge \|(A+B)^r\|, \quad \text{for } \ 0\le r\le 1 \tag{3.15}$$

and

$$\|A^r + B^r\| \leq \|(A+B)^r\|, \quad \text{for} \ \ 1 \leq r < \infty, \tag{3.16}$$

for $A, B \in B(H)^+$.
On the other hand, Ando [2] proved the following more general inequalities: For every non-negative operator monotone function $f(t)$ on $[0, \infty)$ and all positive operators A and B in $B(H)$, we have

$$|||f(A) + f(B)||| \geq |||f(A+B)|||$$

and for every non-negative increasing function $g(t)$ on $[0, \infty)$ with $g(0) = 0$ and $g(\infty) = \infty$, whose inverse function is operator monotone, we have

$$|||g(A) + g(B)||| \leq |||g(A+B)|||.$$

The authors of [4] obtained several inequalities for eigenvalues and unitarily invariant norm of Hermitian matrices. They proved the following inequality for non-negative convex function f and Hermitian matrices A and B.

$$|||A+B|^r||| \leq 2^{r-1}|||A|^r + |B|^r|||,$$

for $r \geq 1$. Above inequality is an upper bound for (3.16).

By the left side of inequalities (3.8) and convexity of $f(t) = t^r$ for $r \in [-1, 0] \cup [1, 2]$, we can obtain

$$\begin{aligned} s_j\left(\frac{A+B}{2}\right)^r &\leq s_j\left(\int_0^1 (tA + (1-t)B)^r dt\right) \\ &\leq s_j\left(\int_0^1 tA^r + (1-t)B^r dt\right) \\ &= \frac{1}{2}s_j(A^r + B^r), \end{aligned}$$

for $j = 1, 2, \ldots$. It follows that

$$s_j(A+B)^r \leq 2^r s_j\left(\int_0^1 (tA + (1-t)B)^r dt\right) \leq 2^{r-1}s_j(A^r + B^r), \tag{3.17}$$

for $j = 1, 2, \ldots$, $r \in [-1, 0] \cup [1, 2]$ and $A, B \in K(H)^+$.

Now we apply inequalities (3.17) to see that

$$|||(A+B)^r||| \leq 2^r \left|\left|\left|\int_0^1 (tA + (1-t)B)^r dt\right|\right|\right| \leq 2^{r-1}|||A^r + B^r|||,$$

for $r \in [-1, 0] \cup [1, 2]$.
Let $r \in [\frac{1}{2}, 1]$, then by above inequality we have

$$|||(A+B)^{\frac{1}{r}}||| \leq 2^{\frac{1}{r}} \left|\left|\left|\int_0^1 (tA + (1-t)B)^{\frac{1}{r}} dt\right|\right|\right| \leq 2^{\frac{1-r}{r}}|||A^{\frac{1}{r}} + B^{\frac{1}{r}}|||.$$

We replace A and B by A^r and B^r and knowing that $|||A|||^{\frac{1}{r}} \leq |||A^{\frac{1}{r}}|||$ when $|||.|||$ is normalized as $|||I||| = 1$ to obtain

$$|||A^r + B^r|||^{\frac{1}{r}} \leq 2^{\frac{1}{r}} \left|\left|\left| \int_0^1 (tA^r + (1-t)B^r)^{\frac{1}{r}} dt \right|\right|\right| \leq 2^{\frac{1-r}{r}} |||A + B|||.$$

It follows that

$$|||A^r + B^r||| \leq 2 \left|\left|\left| \int_0^1 (tA^r + (1-t)B^r)^{\frac{1}{r}} dt \right|\right|\right|^r \leq 2^{1-r} |||(A + B)^r|||,$$

for $r \in [\frac{1}{2}, 1]$ and $A, B \in B(H)^+$.

Remark 3.4. Let a and b be positive real numbers. Then,

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \tag{3.18}$$

for $r \geq 1$.
The following inequality

$$s_j(A + B)^r \leq 2^{r-1} s_j(A^r + B^r),$$

which obtained in (3.17), gives an operator version of (3.18) for $r \in [1, 2]$.

As our other result, we can write inequalities (3.13) as follows

$$|||(A + B)^{\frac{1}{2}}||| \leq 2 \left|\left|\left| \int_0^1 (tA + (1-t)B)^{\frac{1}{2}} dt \right|\right|\right| \leq \frac{4}{3} |||A^{\frac{1}{2}} + B^{\frac{1}{2}}|||, \tag{3.19}$$

for positive operators A and B such that $AB + BA \geq 0$.
Also, we know that $|||A \oplus B||| \leq |||A + B|||$ for positive operators A and B (see [9, Inequality (3.4)]. So, we have

$$|||A \oplus B|||^{\frac{1}{2}} \leq |||(A + B)^{\frac{1}{2}}|||. \tag{3.20}$$

Inequalities (3.19) and (3.20) imply that

$$\begin{aligned} |||A \oplus B|||^{\frac{1}{2}} &\leq |||(A + B)^{\frac{1}{2}}||| \\ &\leq 2 \left|\left|\left| \int_0^1 (tA + (1-t)B)^{\frac{1}{2}} dt \right|\right|\right| \\ &\leq \frac{4}{3} |||A^{\frac{1}{2}} + B^{\frac{1}{2}}|||. \end{aligned}$$

Let replace A and B by A^*A and B^*B, we obtain

$$\begin{aligned} |||A^*A \oplus B^*B|||^{\frac{1}{2}} &\leq 2 \left|\left|\left| \int_0^1 (tA^*A + (1-t)B^*B)^{\frac{1}{2}} dt \right|\right|\right| \\ &\leq \frac{4}{3} |||(A^*A)^{\frac{1}{2}} + (B^*B)^{\frac{1}{2}}|||. \end{aligned}$$

It follows that

$$\begin{aligned} |||\,|A| \oplus |B|\,||| &\le 2\left|\left|\left|\int_0^1 (tA^*A + (1-t)B^*B)^{\frac{1}{2}} dt\right|\right|\right| \\ &\le \frac{4}{3}|||\,|A| + |B|\,|||. \end{aligned}$$

Equivalently, we can write

$$|||A \oplus B||| \le 2\left|\left|\left|\int_0^1 (tA^*A + (1-t)B^*B)^{\frac{1}{2}} dt\right|\right|\right| \le \frac{4}{3}|||\,|A| + |B|\,|||, \tag{3.21}$$

for arbitrary operators $A, B \in K(H)$ such that $A^*AB^*B + B^*BA^*A \ge 0$.
For usual operator norm, we have

$$\max\{\|A\|, \|B\|\} \le 2\left\|\int_0^1 (tA^*A + (1-t)B^*B)^{\frac{1}{2}} dt\right\| \le \frac{4}{3}\|\,|A| + |B|\,\|.$$

4. Some trace inequalities for operators

In this section, we established some trace inequalities. Some of these are obtained by applying inequalities in Section 2.
We remind some basic properties of trace for operators.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H, we say that $A \in B(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{4.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $B_1(H)$ the set of trace class operators in $B(H)$.

We define the *trace* of a trace class operator $A \in B_1(H)$ to be

$$\operatorname{Tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{4.2}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4.1. *We have*

(i) If $A \in B_1(H)$ then $A^* \in B_1(H)$ and

$$\operatorname{Tr}(A^*) = \overline{\operatorname{Tr}(A)}; \tag{4.3}$$

(ii) If $A \in B_1(H)$ and $T \in B(H)$, then $AT, TA \in B_1(H)$ and

$$\operatorname{Tr}(AT) = \operatorname{Tr}(TA) \quad \textit{and} \quad |\operatorname{Tr}(AT)| \le \|A\|_1 \|T\|; \tag{4.4}$$

(iii) $\operatorname{Tr}(\cdot)$ is a bounded linear functional on $B_1(H)$ with $\|\operatorname{Tr}\| = 1$;

(iv) If $A, B \in B_1(H)$ then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

For the theory of trace functionals and their applications the reader is referred to [28].

Lemma 4.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an operator convex function. Then $g(t) = \operatorname{Tr}(f(tA + (1-t)B))$ is convex on $[0,1]$ for self-adjoint operators A and B such that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$.*

Proof. For showing $g(t) = \operatorname{Tr}(f(tA + (1-t)B))$ is convex on $[0,1]$, we should prove that

$$g(\alpha u + (1-\alpha)v) = \alpha g(u) + (1-\alpha)g(v),$$

for $u, v \in [0,1]$ and $0 \le \alpha \le 1$.
So, by convexity and monotonicity of trace functional and operator convexity of f, we have

$$\begin{aligned}
g(\alpha u + (1-\alpha)v) &= \operatorname{Tr}\left(f((\alpha u + (1-\alpha)v)A + (1 - \alpha u - (1-\alpha)v)B)\right) \\
&= \operatorname{Tr}\left(f(\alpha(uA + (1-u)B) + (1-\alpha)(vA + (1-v)B))\right) \\
&\le \operatorname{Tr}\left(\alpha f(uA + (1-u)B) + (1-\alpha)f(vA + (1-v)B)\right) \\
&= \alpha \operatorname{Tr}\left(f(uA + (1-u)B)\right) + (1-\alpha)\operatorname{Tr}\left(f(vA + (1-v)B)\right) \\
&= \alpha g(u) + (1-\alpha)g(v),
\end{aligned}$$

for any $u, v \in [0,1]$ and $0 \le \alpha \le 1$.

□

Theorem 4.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an operator convex function. Then we have*

$$\operatorname{Tr}\left(f\left(\frac{A+B}{2}\right)\right) \le \int_0^1 \operatorname{Tr}(f(tA + (1-t)B))dt \le \frac{\operatorname{Tr}(f(A) + f(B))}{2}, \tag{4.5}$$

for self-adjoint operators A and B such that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$.

Proof. Since $g(t) = \operatorname{Tr}\left(f(tA + (1-t)B)\right)$ is convex on $[0,1]$ for self-adjoint operators A and B such that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$. By Lemma 4.2, we have

$$g\left(\frac{0+1}{2}\right) \le \int_0^1 g(t)dt \le \frac{g(0) + g(1)}{2}.$$

Therefore, we have the desired results.

□

In [27, Theorem 3.1], Shebrawi and Albadawi proved the following inequality for positive matrices

$$\operatorname{Tr}(A+B)^r \le 2^{r-1}\operatorname{Tr}(A^r + B^r), \tag{4.6}$$

where $r \ge 1$.

Now, let $f(t) = t^r$ in inequalities (4.5) for $r \in [-1,0] \cup [1,2]$ on $(0,\infty)$, we have

$$\operatorname{Tr}\left(\frac{A+B}{2}\right)^r \leq \int_0^1 \operatorname{Tr}(tA+(1-t)B)^r dt \leq \frac{\operatorname{Tr}(A^r+B^r)}{2}, \tag{4.7}$$

for $A, B \in B_1(H)^+$ and $r \in [-1,0] \cup [1,2]$.
Hence, we obtain a refinement of inequality (4.6) for $A, B \in B_1(H)^+$ and $r \in [-1,0] \cup [1,2]$ as follows

$$\operatorname{Tr}(A+B)^r \leq 2^r \int_0^1 \operatorname{Tr}(tA+(1-t)B)^r dt \leq 2^{r-1} \operatorname{Tr}(A^r+B^r).$$

Moreover, inequalities (4.7) for $r = -1$ is a refinement of special case of [20, Theorem 7.6.10] which asserts that

$$4\operatorname{Tr}(A+B)^{-1} \leq \operatorname{Tr}(A)^{-1} + \operatorname{Tr}(B)^{-1},$$

for positive definite matrices A and B.

Example 4.4. Let $\Phi : B_1(H) \to \mathbb{R}^+$, we define $\Phi(A) = \operatorname{Tr}(A)$. This map is a positive linear map which preserves invertibility.

Let $\Phi(A) = \operatorname{Tr}(A)$ in (2.7), then we have

$$\frac{(\operatorname{Tr}(A+B))^s}{2} \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B))^s dt \leq \frac{(\operatorname{Tr}(A))^s + (\operatorname{Tr}(B))^s}{s+1},$$

for $s \in (0,1]$ and $A, B \in B_1(H)^+$ such that $AB + BA \geq 0$.
Similar to above, we can obtain the following inequalities, by making use of (2.6) for operator convex functions

$$f\left(\operatorname{Tr}\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 f(\operatorname{Tr}(tA+(1-t)B))dt \leq \frac{f(\operatorname{Tr}(A)) + f(\operatorname{Tr}(B))}{2}. \tag{4.8}$$

Let $f(t) = t^r$ for $-1 \leq r \leq 0$ and $1 \leq r \leq 2$. So, inequalities (4.8) change to

$$\left(\operatorname{Tr}\left(\frac{A+B}{2}\right)\right)^r \leq \int_0^1 (\operatorname{Tr}(tA+(1-t)B))^r dt \leq \frac{(\operatorname{Tr}(A))^r + (\operatorname{Tr}(B))^r}{2},$$

for $A, B \in B_1(H)^+$.

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Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, P. O. Box 47416-146, Babolsar, Iran.
E-mail address: taghavi@umz.ac.ir, vahid.darvish@vu.edu.au, m.nazari@stu.umz.ac.ir,

Mathematics, School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC 8001, Australia.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.
E-mail address: sever.dragomir@vu.edu.au