Received 15/03/15

SOME FEJÉR TYPE INEQUALITIES FOR HARMONICALLY-CONVEX FUNCTIONS WITH APPLICATIONS TO SPECIAL MEANS

M. A. LATIF, S. S. DRAGOMIR[1,2], AND E. MOMONIAT

Abstract. In this paper, the notion of harmonic symmetricity of functions is introduced. A new identity involving harmonically symmetric functions is established and some new Fejér type integral inequalities are presented for the class of harmonically convex functions. The results presented in this paper are better than those established in recent literature concerning harmonically convex functions. Applications of our results to special means of positive real numbers are given as well.

1. Introduction

The theory of convexity has been subject to extensive research during the past few years due it its utility in various branches of pure and applied mathematics. Many inequalities have been established by a number of researchers for convex functions but one of the most interesting inequalities is the Hermite-Hadamard inequality which provides a necessary and sufficient condition for a functions to be convex.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if and only if f is convex. The inequalities (1.1) hold in reversed direction if f is concave.

Many researchers have generalized the classical convexity in a number of ways and the inequality (1.1) has been generalized or extended for many classes of convex functions in numerous ways, see for instance [3]-[22] and the references therein.

Let us recall some known concepts which will be used in the sequel of the paper.

Definition 1. [10] *Let $I \subset \mathbb{R}\backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq tf\,(y) + (1-t)\,f\,(x) \tag{1.2}$$

for all x, $y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Date: November 15, 2012.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26A51, 26E60, 41A55.

Key words and phrases. Hermite-Hadamard's inequality, Fejér's inequality, convex function, harmonically-convex function, Hölder's inequality, power mean inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Proposition 1. [10] *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval and $f : I \to \mathbb{R}$ is function, then:*

- *if $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
- *if $I \subset (0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
- *if $I \subset (-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.*
- *if $I \subset (-\infty,0)$ and f is convex and nonincreasing function then f is harmonically convex.*

In [10], İşcan has also proved the following results for harmonically convex functions.

Theorem 1. [10] *Let $I \subset \mathbb{R}\backslash\{0\}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L([a,b])$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

Theorem 2. [10] *Let $f : (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L([a,b])$. If $\left|f'\right|^q$ is harmonically convex $[a,b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\le \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)$$
$$= \lambda_1 - \lambda_2.$$

Theorem 3. [10] *Let $f : (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L([a,b])$. If $\left|f'\right|^q$ is harmonically convex $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\mu_1\left|f'(a)\right|^q + \mu_1\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$\mu_1 = \frac{a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2 = \frac{b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]}{2(b-a)^2(1-q)(1-2q)}.$$

Some applications of the above results can also be found in [10].

Chen and Wu [3] established the following Fejér type inequality for harmonically convex functions which provides a weighted generalization of the result given in Theorem 1.

Theorem 4. [3] *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and a, $b \in I$ with $a < b$. If $f \in L([a,b])$, them one has be continuous*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \le \int_a^b \frac{f(x)g(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx, \tag{1.3}$$

$g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and satisfies

$$g\left(\frac{ab}{x}\right) = g\left(\frac{ab}{a+b-x}\right).$$

The main goal of this paper is to introduce a new notion of harmonically symmetric functions and to establish an identity involving a harmonically symmetric function and a differentiable function. We will prove some Fejér type inequalities by using this identity and hence our results will provide a better weighted generalization of the results proved in Theorem 2 and Theorem 3. Some applications of our results to special means of positive real numbers will also be provided in Section 3. We believe that our findings are novel, new and better than those already exist and will open new ways for further research in this filed.

2. Main Results

Throughout this section we take $U(t) = \frac{2ab}{(1-t)a+(1+t)b}$ and $L(t) = \frac{2ab}{(1+t)a+(1-t)b}$. The Beta function, the Gamma function and the integral from of the hypergeometric function are defined as follows to be used in the sequel of the paper

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \alpha > 0, \beta > 0,$$
$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \alpha > 0$$

and

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$$

for $|z| < 1, \gamma > \beta > 0$.

The notion of harmonically symmetric functions is given in following definition.

Definition 2. *A function $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if*

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

Now we prove a weighted integral identity which will be used in establishing our main results.

Lemma 1. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$. If $f^{'} \in L([a,b])$, then the following equality holds*

$$\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx = \left(\frac{b-a}{4ab}\right) \times \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)\left[(U(t))^2 f^{'}(U(t)) - (L(t))^2 f^{'}(L(t))\right]dt. \quad (2.1)$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)(U(t))^2 f^{'}(U(t))\,dt$$

and

$$I_2 = \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)(L(t))^2 f^{'}(L(t))\,dt.$$

Since $g : [a,b] \to [0,\infty)$ is harmonically symmetric to $\frac{2ab}{a+b}$, then $g(U(t)) = g(L(t))$ for all $t \in [0,1]$. Hence, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)(U(t))^2 f^{'}(U(t))\,dt \\ &= \frac{-2ab}{b-a}\int_0^1 \left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)d\left[f(U(t))\right] \\ &= \frac{-2ab}{b-a}\left(\int_{L(t)}^{U(t)} \frac{g(x)}{x^2}dx\right)f(U(t))\Bigg|_0^1 - \int_0^1 \left[g(U(t)) + g(L(t))\right]f(U(t))\,dt \\ &= \frac{2ab}{b-a}f(a)\int_a^b \frac{g(x)}{x^2}dx - 2\int_0^1 g(U(t))f(U(t))\,dt \\ &= \frac{2ab}{b-a}f(a)\int_a^b \frac{g(x)}{x^2}dx - \frac{4ab}{b-a}\int_a^{\frac{2ab}{a+b}} \frac{g(x)f(x)}{x^2}dx. \quad (2.2)\end{aligned}$$

Analogously, we have

$$-I_2 = \frac{2ab}{b-a}f(b)\int_a^b \frac{g(x)}{x^2}dx - \frac{4ab}{b-a}\int_{\frac{2ab}{a+b}}^b \frac{g(x)f(x)}{x^2}dx. \quad (2.3)$$

Adding (2.2) and (2.3) and multiplying the result by $\frac{b-a}{4ab}$, we get the required identity. This completes the proof of the Lemma. □

Lemma 2. *For $v > u > 0$, we have*

$$\int_0^1 t\left[\frac{2uv}{(1-t)u + (1+t)v}\right]^2 dt = \left(\frac{2uv}{v+u}\right)^2 \lambda_1(u,v),$$

$$\int_0^1 t\left[\frac{2uv}{(1+t)u + (1-t)v}\right]^2 dt = \left(\frac{2uv}{v+u}\right)^2 \lambda_1(v,u),$$

$$\int_0^1 t^2 \left[\frac{2uv}{(1-t)\,u+(1+t)\,v}\right]^2 dt = \left(\frac{2uv}{v+u}\right)^2 \lambda_2\,(u,v)\,,$$

$$\int_0^1 t^2 \left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^2 dt = \left(\frac{2uv}{v+u}\right)^2 \lambda_2\,(v,u)\,,$$

where

$$\lambda_1\,(u,v) \triangleq \ln\left(\frac{2v}{u+v}\right) + \frac{u-v}{2v}$$

and

$$\lambda_2\,(u,v) \triangleq \left(\frac{v+u}{v-u}\right)\left[\frac{2v}{u+v} - \frac{u+v}{2v} - 2\ln\left(\frac{2v}{u+v}\right)\right].$$

Proof. The proof follows from a straightforward computation. □

Lemma 3. *For $v > u > 0$ and $p > 1$, we have*

$$\int_0^1 (1+t)\left[\frac{2uv}{(1-t)\,u+(1+t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_1\,(u,v;p)\,,$$

$$\int_0^1 (1-t)\left[\frac{2uv}{(1-t)\,u+(1+t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_2\,(u,v;p)\,,$$

$$\int_0^1 (1+t)\left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_1\,(v,u;p)\,,$$

$$\int_0^1 (1-t)\left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_2\,(v,u;p)\,,$$

where

$$\zeta_1\,(u,v;p) \triangleq \frac{2^{1-2p} v \left(\frac{v}{u+v}\right)^{-2p} \left[(1-2p)\,(v-u)-u\right]}{(v-u)^2\,(p-1)\,(2p-1)} - \frac{(u+v)\left[(1-2p)\,(v-u)-2u\right]}{2\,(v-u)^2\,(p-1)\,(2p-1)}$$

and

$$\zeta_2\,(u,v;p)\,\frac{4^{1-q} v^2 \left(\frac{v}{u+v}\right)^{-2p} + (u+v)\left[(2p-1)\,(v-u)-2v\right]}{2\,(v-u)^2\,(p-1)\,(2p-1)}.$$

Proof. The proof follows from a straightforward computation. □

Now we present new Fejér type inequalities for harmonically-convex functions, which provide weighted generalization of some of the results established in recent literature.

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and* *harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f^{'} \in L([a,b])$. *If* $\left|f^{'}\right|^{q}$ *is harmonically-convex on* $[a,b]$ *for* $q \geq 1$, *then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right| \leq \left(\frac{b-a}{b+a}\right)^2 \left(\frac{1}{2}\right)^{1/q} \|g\|_\infty$$
$$\times \left\{[\lambda_1(a,b)]^{1-1/q}\left[\xi_1(a,b)\left|f^{'}(a)\right|^q + \xi_2(a,b)\left|f^{'}(b)\right|^q\right]^{1/q}\right.$$
$$\left. + [\lambda_1(b,a)]^{1-1/q}\left[\xi_2(b,a)\left|f^{'}(a)\right|^q + \xi_1(b,a)\left|f^{'}(b)\right|^q\right]^{1/q}\right\}, \quad (2.4)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$,

$$\xi_1(a,b) \triangleq \lambda_1(a,b) + \lambda_2(a,b), \ \xi_2(a,b) \triangleq \lambda_1(a,b) - \lambda_2(a,b)$$

and $\lambda_1(\cdot,\cdot)$, $\lambda_2(\cdot,\cdot)$ *are defined in Lemma 2.*

Proof. From Lemma 1, we get

$$\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx$$
$$\leq -\left(\frac{b-a}{2ab}\right)^2 \|g\|_\infty \int_0^1 \left[t(U(t))^2 f^{'}(U(t)) - t(L(t))^2 f^{'}(L(t))\right]dt. \quad (2.5)$$

Now taking modulus on both sides of (2.5) and using Hölder's inequality, we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_\infty \left\{\left(\int_0^1 t(U(t))^2\right)^{1-1/q}\left(\int_0^1 t(U(t))^2\left|f^{'}(U(t))\right|^q\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 t(L(t))^2\right)^{1-1/q}\left(\int_0^1 t(L(t))^2\left|f^{'}(L(t))\right|^q\right)^{1/q}\right\}. \quad (2.6)$$

By the harmonic-convexity of $\left|f^{'}\right|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 2, we have

$$\int_0^1 t(U(t))^2\left|f^{'}(U(t))\right|^q = \int_0^1 t\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left|f^{'}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q$$
$$\leq \frac{1}{2}\left|f^{'}(a)\right|^q \int_0^1 tt(1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt$$
$$+ \frac{1}{2}\left|f^{'}(b)\right|^q \int_0^1 t(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt$$
$$= \frac{1}{2}\left(\frac{2ab}{b-a}\right)^2\left\{[\lambda_1(a,b)+\lambda_2(a,b)]\left|f^{'}(a)\right|^q + [\lambda_1(a,b)-\lambda_2(a,b)]\left|f^{'}(b)\right|^q\right\} \quad (2.7)$$

and

$$\int_0^1 t\,(L(t))^2 \left|f'(L(t))\right|^q = \int_0^1 t\left[\frac{2ab}{(1+t)\,a+(1-t)\,b}\right]^2 \left|f'\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)\right|^q$$
$$\le \frac{1}{2}\left|f'(a)\right|^q \int_0^1 t\,(1-t)\left[\frac{2ab}{(1+t)\,a+(1-t)\,b}\right]^2 dt$$
$$+\frac{1}{2}\left|f'(b)\right|^q \int_0^1 t\,(1+t)\left[\frac{2ab}{(1+t)\,a+(1-t)\,b}\right]^2 dt = \frac{1}{2}\left(\frac{2ab}{b-a}\right)^2$$
$$\times\left\{[\lambda_1(b,a)-\lambda_2(b,a)]\left|f'(a)\right|^q+[\lambda_1(b,a)+\lambda_2(b,a)]\left|f'(b)\right|^q\right\}. \quad (2.8)$$

A combination of (2.6), (2.7) and (2.8) gives the required result. This completes the proof of the theorem. □

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied. If $q=1$, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)\,g(x)}{x^2}dx\right| \le \left(\frac{1}{2}\right)\left(\frac{b-a}{b+a}\right)^2 \|g\|_\infty$$
$$\times\left\{[\xi_1(a,b)+\xi_2(b,a)]\left|f'(a)\right|+[\xi_2(a,b)+\xi_1(b,a)]\left|f'(b)\right|\right\}, \quad (2.9)$$

where $\|g\|_\infty=\sup_{x\in[a,b]} g(x)<\infty$ and $\xi_1(\cdot,\cdot)$, $\xi_2(\cdot,\cdot)$ are defined in Theorem 5.

Corollary 2. *If $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$ in Theorem 5, then*

$$\left|\frac{f(b)+f(a)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \left(\frac{1}{2}\right)^{1/q}\left(\frac{b-a}{ab}\right)\left(\frac{ab}{b+a}\right)^2$$
$$\times\left\{[\lambda_1(a,b)]^{1-1/q}\left[\xi_1(a,b)\left|f'(a)\right|^q+\xi_2(a,b)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+[\lambda_1(b,a)]^{1-1/q}\left[\xi_2(b,a)\left|f'(a)\right|^q+\xi_1(b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}. \quad (2.10)$$

Corollary 3. *If $q=1$ in Corollary 2, then we get the following inequality*

$$\left|\frac{f(b)+f(a)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \left(\frac{b-a}{2ab}\right)\left(\frac{ab}{b+a}\right)^2$$
$$\times\left\{[\xi_1(a,b)+\xi_2(b,a)]\left|f'(a)\right|+[\xi_2(a,b)+\xi_1(b,a)]\left|f'(b)\right|\right\}. \quad (2.11)$$

Theorem 6. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and a, $b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and*

harmonically symmetric to $\frac{2ab}{a+b}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|^q$ *is harmonically-convex on* $[a,b]$ *for* $q>1$, *then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right| \leq \|g\|_\infty \left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times\left\{\left[\zeta_1(a,b;q)\left|f'(a)\right|^q+\zeta_2(a,b;q)\left|f'(b)\right|^q\right]^{1/q} + \left[\zeta_2(b,a;q)\left|f'(a)\right|^q+\zeta_1(b,a;q)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (2.12)$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Proof. From (2.5) and Hölder's inequality, we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right| \leq \left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left(\int_0^1 t^{q/(q-1)}\right)^{1-1/q} \times\left\{\left(\int_0^1 (U(t))^{2q}\left|f'(U(t))\right|^q\right)^{1/q}+\left(\int_0^1 (L(t))^{2q}\left|f'(L(t))\right|^q\right)^{1/q}\right\}. \quad (2.13)$$

Since $\left|f'\right|^q$ is harmonically-convex on $[a,b]$, we obtain

$$\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q=\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q \leq \frac{1}{2}\left|f'(a)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt + \frac{1}{2}\left|f'(b)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt \quad (2.14)$$

and

$$\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q=\int_0^1\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}\left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q \leq \frac{1}{2}\left|f'(a)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt + \frac{1}{2}\left|f'(b)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt. \quad (2.15)$$

By applying Lemma 3 in inequalities (2.14) and (2.15) and then using the resulting inequalities in (2.13), we get the required inequality. □

Corollary 4. *If the assumptions of Theorem 6 are satisfied and if* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left[\zeta_1(a,b;q)\left|f'(a)\right|^q + \zeta_2(a,b;q)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left. + \left[\zeta_2(b,a;q)\left|f'(a)\right|^q + \zeta_1(b,a;q)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (2.16)$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Theorem 7. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* a, $b \in I^\circ$ *with* $a < b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|^q$ *is harmonically-convex on* $[a,b]$ *for* $q > 1$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{2/q-1}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\|g\|_\infty$$
$$\times\left\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(a)\right|^q + \left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(b)\right|^q\right\}^{1/q}, \quad (2.17)$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Proof. From the inequality 2.5 and Hölder's inequality, we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left(\int_0^1 t^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (2.18)$$

By the power-mean inequality ($a^r + b^r \leq 2^{1-r}(a+b)^r$ for $a > 0, b > 0$ and $r < 1$), we have

$$\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{2q} dt\right)^{1/q} + \left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{2q} dt\right)^{1/q}$$
$$\leq 2^{1-1/q}\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt + \int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt\right)^{1/q}. \quad (2.19)$$

Since $\left|f^{'}\right|^{q}$ is harmonically-convex on $[a,b]$ for $q>1$, we obtain

$$\begin{aligned}
&\int_0^1 [U(t)]^{2q}\left|f^{'}(U(t))\right|^{q}dt+\int_0^1 [L(t)]^{2q}\left|f^{'}(L(t))\right|^{q}dt\\
&\leq \frac{1}{2}\left|f^{'}(a)\right|^{q}\int_0^1 (1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt\\
&+\frac{1}{2}\left|f^{'}(b)\right|^{q}\int_0^1 (1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt\\
&+\frac{1}{2}\left|f^{'}(a)\right|^{q}\int_0^1 (1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt\\
&+\frac{1}{2}\left|f^{'}(b)\right|^{q}\int_0^1 (1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt\\
&=\frac{1}{2}\left(\frac{2ab}{b+a}\right)^{2q}\Big\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f^{'}(a)\right|^{q}\\
&\qquad+\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f^{'}(b)\right|^{q}\Big\}. \qquad (2.20)
\end{aligned}$$

Using (2.19) in (2.20), we get

$$\begin{aligned}
&\left(\int_0^1 [U(t)]^{2q}\left|f^{'}(U(t))\right|^{q}dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f^{'}(L(t))\right|^{q}dt\right)^{1/q}\\
&\leq 2^{1-2/q}\left(\frac{2ab}{b+a}\right)^{2}\Big\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f^{'}(a)\right|^{q}\\
&\qquad+\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f^{'}(b)\right|^{q}\Big\}^{1/q}. \qquad (2.21)
\end{aligned}$$

Applying (2.21) in (2.18), we obtain the required inequality (2.17). □

Corollary 5. *If the assumptions of Theorem 7 are satisfied and if* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$, *then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\
&\leq \frac{ab}{b-a}\left(\frac{b-a}{b+a}\right)^{2}\left(\frac{1}{2}\right)^{1-2/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\
&\times\Big\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f^{'}(a)\right|^{q}+\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f^{'}(b)\right|^{q}\Big\}^{1/q}, \qquad (2.22)
\end{aligned}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Theorem 8. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *and* a, $b\in I^{\circ}$ *with* $a<b$ *and let* $g:[a,b]\to[0,\infty)$ *be continuous positive mapping and*

harmonically symmetric to $\frac{2ab}{a+b}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|$ *is harmonically-convex on* $[a,b]$, *then the following inequality holds for* $q>1$

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\le \frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\|g\|_\infty\left([\varsigma(a,b)]^{1-1/q}\left\{[B(q+1,q+1)]^{1/q}\left|f'(b)\right|\right.\right.$$
$$\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(a)\right|\right\}+[\varsigma(b,a)]^{1-1/q}$$
$$\left.\times\left\{[B(q+1,q+1)]^{1/q}\left|f'(a)\right|+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(b)\right|\right\}\right), \tag{2.23}$$

where $B(\cdot,\cdot)$ *is the Beta function,* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function and*

$$\varsigma(a,b)\triangleq\frac{(q-1)\left[(a+b)^{-\frac{q+1}{q-1}}-(2b)^{-\frac{q+1}{q-1}}\right]}{(q+1)(b-a)}.$$

Proof. We continue from (2.5) and by using the harmonic-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\left|\frac{f(b)+f(a)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\int_0^1\left[t(U(t))^2\left|f'(U(t))\right|+t(L(t))^2\left|f'(L(t))\right|\right]dt$$
$$\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\int_0^1(U(t))^2\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right.$$
$$\left.+\int_0^1(L(t))^2\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt\right\}. \tag{2.24}$$

Using Hölder integral inequality, we have

$$\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt$$
$$\le\left(\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\varsigma(a,b)]^{1-1/q}\left\{[B(q+1,q+1)]^{1/q}\left|f'(b)\right|\right.$$
$$\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(a)\right|\right\}. \tag{2.25}$$

Similarly, one has

$$\int_0^1 \left[\frac{2ab}{(1+t)\,a+(1-t)\,b}\right]^2 \left[t\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+t\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|\right]dt$$
$$\leq \left(\int_0^1 \left[\frac{2ab}{(1+t)\,a+(1-t)\,b}\right]^{2q/(q-1)} dt\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(a)\right|+\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(b)\right|\right\}$$
$$=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2\left[\varsigma(b,a)\right]^{1-1/q}\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f^{'}(a)\right|\right.$$
$$\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f^{'}(b)\right|\right\}. \quad (2.26)$$

Using (2.25) and (2.26) in (2.24), we obtain the required inequality (2.23). □

Corollary 6. *Under the assumptions of Theorem 8, if* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\frac{ab}{b-a}\left(\left[\varsigma(a,b)\right]^{1-1/q}\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f^{'}(b)\right|\right.\right.$$
$$\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f^{'}(a)\right|\right\}$$
$$+\left[\varsigma(b,a)\right]^{1-1/q}\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f^{'}(a)\right|\right.$$
$$\left.\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f^{'}(b)\right|\right\}\right), \quad (2.27)$$

where $B(\cdot,\cdot)$ *is the Beta function,* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function and* $\varsigma_1(\cdot,\cdot)$ *is defined in Theorem 8.*

Theorem 9. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *and* a, $b\in I^\circ$ *with* $a<b$ *and let* $g:[a,b]\to[0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f^{'}\in L([a,b])$*. If* $\left|f^{'}\right|$ *is harmonically-convex on* $[a,b]$*, then the following inequality holds for* $q>1$

$$\left|\frac{f(b)+f(a)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)\,g(x)}{x^2}dx\right|\leq\frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\|g\|_\infty$$
$$\times\left\{\left[\nu(a,b)\right]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|\right]\right.$$
$$\left.+\left[\nu(b,a)\right]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|\right]\right\}, \quad (2.28)$$

where where $\Gamma(\cdot)$ *is the Gamma function,* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function and*

$$\nu(a,b)=\frac{\Gamma\left(\frac{2q-1}{q-1}\right)}{\Gamma\left(\frac{3q-2}{q-1}\right)}\left[b^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{b(a-b)}{a+b}\right)\right.$$
$$\left.-a^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{a(a-b)}{a+b}\right)\right].$$

Proof. From (2.5) and by using the harmonic-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\\
&\quad\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\int_0^1\left[t(U(t))^2\left|f'(U(t))\right|+t(L(t))^2\left|f'(L(t))\right|\right]dt\\
&\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\int_0^1(U(t))^2\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right.\\
&\qquad\left.+\int_0^1(L(t))^2\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt\right\}.
\end{aligned}\tag{2.29}$$

Application of Hölder integral inequality yields

$$\begin{aligned}
&\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\\
&\quad\le\left(\int_0^1 t^{q/(q-1)}\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}\\
&=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\nu(a,b)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(b)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(a)\right|\right].
\end{aligned}\tag{2.30}$$

Similarly, one has

$$\begin{aligned}
&\int_0^1\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt\\
&\quad\le\left(\int_0^1 t^{q/(q-1)}\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}\\
&=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\nu(b,a)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(b)\right|\right].
\end{aligned}\tag{2.31}$$

Using (2.30) and (2.31) in (2.29), we obtain the required inequality (2.28). □

Corollary 7. *Suppose the assumptions of Theorem 8 are satisfied and if* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left|\frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\left(\frac{ab}{b-a}\right)$$
$$\times\left\{[\nu(a,b)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(b)\right| + \left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(a)\right|\right]\right.$$
$$\left.+[\nu(b,a)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(a)\right| + \left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(b)\right|\right]\right\}, \quad (2.32)$$

where $\nu(\cdot,\cdot)$ *is defined in Theorem 9.*

Remark 1. *Some further results can be obtained from (2.24) but we omit the details for the interested readers.*

3. Applications to Special Means

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a harmonically convex function and a harmonically symmetric function to construct inequalities for special means.

For positive numbers $a > 0$ and $b > 0$ with $a \neq b$

$$A(a,b) = \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2ab}{a+b}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ L(a,b), & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively. For further information on means, we refer the readers to [1] and the references therein.

Let $g : [a,b] \to \mathbb{R}_0$ be defined as

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a,b].$$

It is obvious that

$$g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right) = g(x)$$

for all $x \in [a,b]$. Hence $g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2$, $x \in [a,b]$ is harmonically symmetric with respect to $x = \frac{2ab}{a+b}$.

Throughout in this section we will also assume that

$$\mu(a,b) = \frac{b-a}{2ab}.$$

Now applications of our results are given in the following theorems to come.

Theorem 10. *Let $0 < a < b$. Then the following inequality holds*

$$\left|\frac{A^2(a,b)+2G^2(a,b)}{3G^2(a,b)}-\frac{A(a,b)}{L(a,b)}\right| \leq (b-a)^2\mu(a,b)H^2(a,b)\left[\ln\left(\frac{G(a,b)}{A(a,b)}\right)+\mu^2(a,b)G^2(a,b)\right]. \quad (3.1)$$

Proof. Applying Theorem 5 to the functions

$$f(x)=x \text{ for } x>0$$

and

$$g(x)=\left(\frac{a+b}{2ab}-\frac{1}{x}\right)^2, x\in[a,b]$$

we get the desired result. □

Theorem 11. *Let $0 < a < b$. Then for $q \geq 1$, we have the following inequality holds*

$$\left|A\left(a^2,b^2\right)-G^2(a,b)\right| \leq \left(\frac{1}{2}\right)^{1/q}\mu(a,b)H^2(a,b)$$
$$\times\left\{[\lambda_1(a,b)]^{1-1/q}\left[2\lambda_1(a,b)A(a^q,b^q)-q(b-a)\lambda_2(a,b)L_{q-1}^{q-1}(a,b)\right]^{1/q}\right.$$
$$\left.+[\lambda_1(b,a)]^{1-1/q}\left[2\lambda_1(b,a)A(a^q,b^q)+q(b-a)\lambda_2(b,a)L_{q-1}^{q-1}(a,b)\right]^{1/q}\right\}. \quad (3.2)$$

where $\lambda_1(\cdot,\cdot)$ and $\lambda_2(\cdot,\cdot)$ are defined in are defined in Lemma 2.

Proof. The assertion follows from the inequality proved in Corollary 2 for $f(x)=x^2$ for $x>0$. □

Corollary 8. *If we take $q=1$ in Corollary 8, then the following inequality holds valid*

$$\left|A\left(a^2,b^2\right)-G^2(a,b)\right| \leq 2\mu(a,b)H^2(a,b)A(a,b)\left[3\ln\left(\frac{G(a,b)}{A(a,b)}\right)+2\mu^2(a,b)G^2(a,b)\right]. \quad (3.3)$$

Theorem 12. *Let $0 < a < b$ and $q > 1$. Then*

$$\left|A(a,b)-\frac{G^2(a,b)}{L(a,b)}\right| \leq \frac{(2q-2)^{1/q-1}\mu(a,b)}{(2q-1)(b-a)^{1/q}}$$
$$\times\left\{\left[A(a,b)H^{2q}(a,b)-a^{2q}b\right]^{1/q}+\left[ab^{2q}-A(a,b)H^{2q}(a,b)\right]^{1/q}\right\}. \quad (3.4)$$

Proof. Applying Corollary 4 to the function

$$f(x)=x \text{ for } x>0,$$

we get the desired result. □

Theorem 13. *Let* $0 < a < b$ *and* $r \in (-1, \infty) \setminus \{0\}$. *Then*

$$\begin{aligned} \left| A\left(a^{r+2}, b^{r+2}\right) - G^2(a,b) L_r^r(a,b) \right| &\le (r+2)\,\mu(a,b)\, H^2(a,b) \\ &\times \left\{ A\left(a^{r+2}, b^{r+2}\right) \left[\ln\left(\frac{G(a,b)}{A(a,b)}\right) + G^2(a,b)\,\mu^2(a,b) \right] \right. \\ &\left. + (r+1)\, A(a,b)\, L_r^r(a,b) \left[2\ln\left(\frac{G(a,b)}{A(a,b)}\right) + G^2(a,b)\,\mu^2(a,b) \right] \right\}. \quad (3.5) \end{aligned}$$

Proof. Applying Corollary 3 to the function

$$f(x) = x^{r+2} \text{ for } x > 0, r \in r \in (-1, \infty) \setminus \{0\},$$

we get the required result. □

Theorem 14. *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$\begin{aligned} &\left| \frac{A^2(a,b) + 2G^2(a,b)}{3G^2(a,b)} - \frac{A(a,b)}{L(a,b)} \right| \\ &\le \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(\frac{b-a}{b+a}\right)^3 \frac{G^{2/q}(a,b)\, L_{2q-q}^{2-2/q}(a,b)}{H^2(a,b)}. \quad (3.6) \end{aligned}$$

Proof. Applying Theorem 7 to the functions

$$f(x) = x \text{ for } x > 0$$

and

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a, b]$$

we get the desired result. □

References

[1] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers,Dordrecht/Boston/London, 2003.

[2] F. Chen and S. Wu, Fejér and Hermite-Hadamard type inequalities for harmonically convex functions, Journal of Applied Mathematics 2014, Article ID 386806, 6 pages.

[3] F. Chen and S. Wu, Hermite-Hadamard type inequalities for harmonically s-convex functions, Sci. World J. 2014 (2014) 7, Article ID 279158.

[4] S. S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (5) (1998) 91–95.

[5] S. S. Dragomir, C.E.M. Pearce, Selected topics on Hermite–Hadamard type inequalities and applications, RGMIA Monographs, Victoria University, 2000.

[6] V. N. Huy and N. T. Chung, Some generalizations of the Fejér and Hermite-Hadamard inequalities in Hölder spaces, J. Appl. Math. Inform. 29 (2011), no. 3-4, 859–868.

[7] J. Hua, B.-Y. Xi, and F. Qi, Hemite-Hadamard type inequalities for geometrically-arithmetically s-convex functions, Commun. Korean Math. Soc. 29 (2014), No. 1, pp. 51–63.

[8] J. Hua, B. -Y. Xi and F. Qi, Inequalities of Hermite–Hadamard type involving an s-convex function with applications, Applied Mathematics and Computation, 246 (2014), 752-760.

[9] İ. İşcan, Hemite-Hadamard type inequalities for GA-s-convex functions, Le Matematiche, LXIX (2014)-Fasc. II, pp. 129–146.

[10] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics 43 (6) (2014), 935-942.

[11] İ. İşcan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, Journal of Mathematics, 2014, Article ID 346305, 10 pages.

[12] İ. İşcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Applied Mathematica and Computation, 238 (2014), 237-244.

[13] A. P. Ji, T. Y. Zhang, F. Qi, Integral Inequalities of Hermite-Hadamard Type for (α, m)-GA-Convex Functions, Journal of Function Spaces and Applications, 2013 (2013), Article ID 823856, 8 pages.
[14] M. A. Latif, New Hermite-Hadamard type integral inequalities for GA-convex functions with applications, Volume 34, Issue 4 (Nov 2014).
[15] M. V. Mihai, M. A. Noor, K. I. Noor and M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Applied Mathematics and Computation 252 (2015) 257-262.
[16] M. A. Noor, K. I. Noor and M. U. Awana, Integral inequalities for coordinated harmonically convex functions, , Complex Var. Elliptic Eqn. (2014).
[17] M. A. Noor, K. I. Noor, M. U. Awana and S. Costache, Some integral inequalities for harmonically h-convex functions, U.P.B Sci. Bull. Serai A. (2015).
[18] M. Z. Sarikaya, On new Hermite Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57 (2012), no. 3, 377–386.
[19] Y. Shuang, H. P. Yin, F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions, Analysis 33, (2013), 1001–1010.
[20] B. -Y. Xi and F. Qi, Hemite-Hadamard type inequalities for geometrically r-convex functions, Studia Scientiarum Mathematicarum Hungarica 51 (4), 530–546 (2014).
[21] T. Y. Zhang, A. P. Ji, F. Qi, Some inequalities of Hermite-Hadamard type for GA-Convex functions with applications to means. Le Matematiche, 48 (2013), no. 1, 229–239.
[22] T. -Y. Zhang, A. -P. Ji and F. Qi, Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions, Proceedings of the Jangjeon Mathematical Society, 16(3) (2013), 399-407.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: m_amer_latif@hotmail.com

[1]School of Engineering and Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: sever.dragomir@vu.edu.au

Center for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: ebrahim.momoniat@wits.ac.za