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SOME NEW WILKER AND CUSA TYPE INEQUALITIES FOR GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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ABSTRACT. We obtain some new Wilker and Cusa type inequalities for generalized trigonometric and hyperbolic functions by element inequalities theory. The results generalize some known inequalities.

1. INTRODUCTION

It is well known from basic calculus that

$$\arcsin x = \int_0^x \frac{1}{(1-t^2)^{1/2}}\,\mathrm{d}t \tag{1.1}$$

for $0 \le x \le 1$ and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}}\,\mathrm{d}t. \tag{1.2}$$

For $1 < p < \infty$ and $0 \le x \le 1$, the arc sine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{1/p}}\,\mathrm{d}t \tag{1.3}$$

and

$$\frac{\pi_p}{2} = \arcsin_p 1 = \int_0^1 \frac{1}{(1-t^p)^{1/p}}\,\mathrm{d}t. \tag{1.4}$$

The inverse of $\arcsin_p$ on $\left[0, \frac{\pi_p}{2}\right]$ is called the generalized sine function, denoted by $\sin_p$, and may be extended to $(-\infty, \infty)$. In the same way, we can define the generalized cosine function, the generalized tangent function and their inverses, and also the corresponding hyperbolic functions. Their definitions and formulas may see recent references [4, 5, 10].

In [5], some classical inequalities for generalized trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality, Huygens' inequality, and Wilker's inequality were generalized. In [10], some new second Wilker type inequalities for generalized trigonometric and hyperbolic functions were established. In [9], some Turán type inequalities for generalized trigonometric and hyperbolic functions were presented. Very recently, a conjecture posed in [3] was verified in [4].

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In this paper, we will establish some new Wilker and Cusa type inequalities for the generalized trigonometric and hyperbolic functions. Some known inequalities in [10] are the special cases of our results.

2. Lemmas

Lemma 2.1 ([10, Lemma 2.7]). *For all $p \in (1, \infty)$, we have*

$$\cos_p^\alpha x < \frac{\sin_p x}{x} < 1, \quad x \in \left(0, \frac{\pi_p}{2}\right) \tag{2.1}$$

and

$$\cosh_p^\alpha x < \frac{\sinh_p x}{x} < \cosh_p^\beta x, \quad x > 0, \tag{2.2}$$

where the constants $\alpha = \frac{1}{p+1}$ and $\beta = 1$ are the best possible.

Lemma 2.2 ([10, Theorem 3.5]). *For $p \in (1, 2]$, then*

$$\left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x} > 2, \quad x \in \left(0, \frac{\pi_p}{2}\right). \tag{2.3}$$

Lemma 2.3. [6] *Let $a > 0, b > 0$ and $r \geq 1$, then*

$$(a+b)^r \leq 2^{r-1}\left(a^r + b^r\right). \tag{2.4}$$

Lemma 2.4. [7] *Let $a_k > 0, k = 1, 2, \ldots, n$, then*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{(1+a_1)(1+a_2)\cdots(1+a_n)} - 1 \geq \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{2.5}$$

Lemma 2.5 ([5, Theorem 3.4]). *For $p \in [2, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, then*

$$\frac{\sin_p x}{x} < \frac{x}{\sinh_p x}. \tag{2.6}$$

Lemma 2.6. *For $p \in [2, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$\left(\frac{\sin_p x}{x}\right)^p < \frac{x}{\sinh_p x}. \tag{2.7}$$

Proof. Using Lemma 2.5 and $\frac{\sin_p x}{x} < 1$, we have

$$\frac{x}{\sinh_p x} > \frac{\sin_p x}{x} > \left(\frac{\sin_p x}{x}\right)^p. \tag{2.8}$$

This implies inequality (2.7). □

Lemma 2.7 ([5, Corollary 3.10]). *For $p \in [2, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, then*

$$\left(\frac{x}{\sinh_p x}\right)^{p+1} < \frac{\sin_p x}{x}. \tag{2.9}$$

Lemma 2.8 ([5, Theorem 3.22]). *For $p \in (1, 2]$, the double inequality*

$$\frac{\sin_p x}{x} < \frac{\cos_p x + p}{1+p} \leq \frac{\cos_p x + 2}{3} \tag{2.10}$$

holds for all $x \in \left(0, \frac{\pi_p}{2}\right]$.

3. Main Results

Theorem 3.1. *For $x \in \left(0, \frac{\pi_p}{2}\right)$, $p \in (1,\infty)$ and $\alpha - p\beta \le 0$, $\beta > 0$, we have*

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} > 2. \tag{3.1}$$

Proof. Arithmetic-geometric means inequality and Lemma 2.1 yield

$$\begin{aligned}
\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} &\ge 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha}{2}}\left(\frac{\tan_p x}{x}\right)^{\frac{\beta}{2}} \\
&= 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha+\beta}{2}}\left(\frac{1}{\cos_p x}\right)^{\frac{\beta}{2}} \\
&> 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha+\beta}{2}}\left(\frac{\sin_p x}{x}\right)^{-\frac{(p+1)\beta}{2}} \\
&= 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha-p\beta}{2}} \ge 2.
\end{aligned}$$

□

Remark 3.1. If $p = \alpha = 2, \beta = 1$, the inequality (3.1) turns into

$$\left(\frac{\sin x}{x}\right)^{2} + \frac{\tan x}{x} > 2. \tag{3.2}$$

Inequality (3.2) is called the first Wilker inequality in [8].

Remark 3.2. If $\alpha = 2p, \beta = p$ and $p \ge 2$, then $\alpha - p\beta = 2p - p^2 \le 0$. So, inequality (3.1) reduces to

$$\left(\frac{\sin_p x}{x}\right)^{2p} + \left(\frac{\tan_p x}{x}\right)^{p} > 2. \tag{3.3}$$

Theorem 3.2. *For $p \in (1,2], x \in \left(0, \frac{\pi_p}{2}\right)$, and $\alpha - p\beta \le 0, \beta \le -1$, we have*

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} > 2. \tag{3.4}$$

Proof. Using $\frac{x}{\sin_p x} \ge 1$ and $\alpha - p\beta \le 0$, we have

$$\begin{aligned}
\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} &= \left(\frac{x}{\sin_p x}\right)^{-\alpha} + \left(\frac{x}{\tan_p x}\right)^{-\beta} \\
&= \left(\frac{x}{\sin_p x}\right)^{-p\beta}\left(\frac{x}{\sin_p x}\right)^{p\beta-\alpha} + \left(\frac{x}{\tan_p x}\right)^{-\beta} \\
&\ge \left[\left(\frac{x}{\sin_p x}\right)^{p}\right]^{-\beta} + \left(\frac{x}{\tan_p x}\right)^{-\beta}.
\end{aligned}$$

Applying Lemma 2.2 and 2.3, we obtain

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta}$$

$$\geq 2^{1+\beta}\left[\left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x}\right]^{-\beta} > 2.$$

This completes the proof. □

Same method to Theorem 3.1, we easily obtain following Theorem 3.3 by Lemma 2.1 and arithmetic and geometric means inequality. We omit the proof for the sake of simplicity.

Theorem 3.3. *For $p \in (1, \infty)$, $x \in \big(0, \infty\big)$ and $\alpha - p\beta \leq 0, \beta > 0$, then*

$$\left(\frac{\sinh_p x}{x}\right)^\alpha + \left(\frac{\tanh_p x}{x}\right)^\beta > 2. \tag{3.5}$$

Remark 3.3. Taking $\alpha = 2, \beta = 1$ and $p = 2$, inequality (3.5) becomes

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{3.6}$$

Inequality (3.6) is called the first hyperbolic Wilker inequality.

Remark 3.4. Taking $\alpha = 2p, \beta = p$ and $p \in [2, \infty)$, we have

$$\left(\frac{\sinh_p x}{x}\right)^{2p} + \left(\frac{\tanh_p x}{x}\right)^p > 2. \tag{3.7}$$

Theorem 3.4. *For all $x \in \big(0, \frac{\pi_p}{2}\big)$ and $\alpha - p\beta \leq 0, \beta > 0$, we have*

$$\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right] > 4 \tag{3.8}$$

and

$$\left(\frac{\sin_p x}{x}\right)^\alpha + \left(\frac{\tan_p x}{x}\right)^\beta > 2\sqrt{\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right]} - 2 > 2. \tag{3.9}$$

Proof. Setting $n = 2, a_1 = \big(\frac{\sin_p x}{x}\big)^\alpha$ and $a_2 = \big(\frac{\tan_p x}{x}\big)^\beta$ in Lemma 2.4, we have

$$\begin{aligned}&\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right]\\ \geq&\left[\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha}{2}}\left(\frac{\tan_p x}{x}\right)^{\frac{\beta}{2}} + 1\right]^2\\ >&\left[\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha - p\beta}{2}} + 1\right]^2 > 4\end{aligned}$$

and

$$\left(\frac{\sin_p x}{x}\right)^\alpha + \left(\frac{\tan_p x}{x}\right)^\beta > 2\sqrt{\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right] + \left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right]} - 2 > 2$$

by using Lemma 2.1. □

Remark 3.5. If $n = 3$ and $a_1 = a_2 = \left(\frac{\sin_p x}{x}\right)^\alpha, a_3 = \left(\frac{\tan_p x}{x}\right)^\beta$ in Lemma 2.4, it is easily obtain

$$\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right]^2 \left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right] > 8 \tag{3.10}$$

and

$$2\left(\frac{\sin_p x}{x}\right)^\alpha + \left(\frac{\tan_p x}{x}\right)^\beta > 3\sqrt[3]{\left[1 + \left(\frac{\sin_p x}{x}\right)^\alpha\right]^2 \left[1 + \left(\frac{\tan_p x}{x}\right)^\beta\right]} - 3 > 3, \tag{3.11}$$

by similar method to Theorem 3.4.

Theorem 3.5. *For $p \in [2, \infty), t > 0$ and $x \in \left(0, \frac{\pi_p}{2}\right]$, then*

$$\left(\frac{x}{\sin_p x}\right)^{pt} + \left(\frac{x}{\sinh_p x}\right)^t > 2. \tag{3.12}$$

Proof. Applying AGM inequality $a+b \geq 2\sqrt{ab}$ and Lemma 2.6 for $a = \left(\frac{x}{\sin_p x}\right)^{pt}, b = \left(\frac{x}{\sinh_p x}\right)^t$, we easily obtain

$$a + b \geq 2\sqrt{\left(\frac{x}{\sin_p x}\right)^{pt} \left(\frac{x}{\sinh_p x}\right)^t} > 2.$$

The proof is complete. □

Theorem 3.6. *For $p \in [2, \infty), t > 0$ and $x \in \left(0, \frac{\pi_p}{2}\right]$, then*

$$p\left(\frac{x}{\sin_p x}\right)^t + \left(\frac{x}{\sinh_p x}\right)^t > p + 1. \tag{3.13}$$

Proof. AGM inequality $(p + 1)a + b \geq (p + 1) \sqrt[p+1]{a^p b}$ and Lemma 2.6 for $a = \left(\frac{x}{\sin_p x}\right)^{pt}, b = \left(\frac{x}{\sinh_p x}\right)^t$ yield inequality (3.13). □

Applying AGM inequality and Lemma 2.7, we easily obtain following theorems by similar method to Theorem 3.7 and Theorem 3.8.

Theorem 3.7. *For $p \in [2, \infty), t > 0$ and $x \in \left(0, \frac{\pi_p}{2}\right]$, then*

$$\left(\frac{\sinh_p x}{x}\right)^{(p+1)t} + \left(\frac{\sin_p x}{x}\right)^t > 2. \tag{3.14}$$

Theorem 3.8. *For $p \in [2, \infty), t > 0$ and $x \in \left(0, \frac{\pi_p}{2}\right]$, then*

$$(p + 1)\left(\frac{\sinh_p x}{x}\right)^t + \left(\frac{\sin_p x}{x}\right)^t > p + 2. \tag{3.15}$$

Finally, we give a Cusa type inequality.

Theorem 3.9. *For $p \in (1, 2]$ and $x \in (0, \frac{\pi_p}{2}]$, then the function $f(x) = \frac{\ln\left(\frac{\sin_p x}{x}\right)}{\ln\left(\frac{p+\cos_p x}{p+1}\right)}$ is strictly increasing. As a result, we have following inequality*

$$\left(\frac{p + \cos_p x}{p + 1}\right)^\alpha < \frac{\sin_p x}{x} < \left(\frac{p + \cos_p x}{p + 1}\right)^\beta \tag{3.16}$$

with the best constants $\alpha = \dfrac{\ln\left(\frac{2\sin_p \frac{\pi_p}{2}}{\pi_p}\right)}{\ln\left(\frac{p+\cos_p \frac{\pi_p}{2}}{p+1}\right)}$ *and* $\beta = 1$.

Proof. A simple computation yields

$$\begin{aligned}
&\ln^2\left(\frac{p+\cos_p x}{p+1}\right) f'(x) = \frac{x\cos_p x - \sin_p x}{x\sin_p x}\ln\left(\frac{p+\cos_p x}{p+1}\right) + \frac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\ln\left(\frac{\sin_p x}{x}\right)\\
&> \ln\left(\frac{\sin_p x}{x}\right)\left(\frac{x\cos_p x - \sin_p x}{x\sin_p x} + \frac{\cos_p x\tan_p^{p-1} x}{p+\cos_p x}\right)\\
&= \ln\left(\frac{\sin_p x}{x}\right)\left(\frac{(x\cos_p x - \sin_p x)(p+\cos_p x) + x\sin_p x\cos_p x\tan_p^{p-1} x}{x\sin_p x(p+\cos_p x)}\right)\\
&= \frac{\ln\left(\frac{\sin_p x}{x}\right)}{x\sin_p x(p+\cos_p x)} g(x)
\end{aligned}$$

where

$$g(x) = x\cos_p^2 x\sec_p^p x + px\cos_p x - p\sin_p x - \sin_p x\cos_p x.$$

Since

$$g'(x) = \cos_p x\tan_p^{p-1} x h(x)$$

where

$$h(x) = 2\sin_p x - px - (2-p)x\sec_p^{p-1} x,$$

with

$$\begin{aligned}
&h'(x) = 2\cos_p x - p - (2-p)\sec_p^{p-1} x\\
&-(2-p)(p-1)x\sec_p^{p-1} x\tan_p^{p-1} x
\end{aligned}$$

and

$$\begin{aligned}
&h''(x) = -2\cos_p x\tan_p^{p-1} x - 2(2-p)(p-1)\sec_p^{p-1} x\tan_p^{p-1} x\\
&-(2-p)(p-1)^2 x\sec_p^{p-1} x\tan_p^{p-1} x(\tan_p^{p-1} x + \sec_p^{p-1}) < 0.
\end{aligned}$$

So $h'(x)$ is decreasing on $\left(0, \frac{\pi_p}{2}\right]$. From $h'(x) < h'(0) = 0$, it is immediately deduced that $h(x) < h(0) = 0$. Due to $g'(x) < 0$, we derive that the function $g(x)$ is decreasing on $\left(0, \frac{\pi_p}{2}\right]$. Inequality $g(x) < g(0) = 0$ indicates that $f'(x) > 0$. So $f(x)$ is strictly increasing for $x \in (0, \frac{\pi_p}{2})$. As a result, we have $f(0) < f(x) \le f(\frac{\pi_p}{2})$.

Using Höspital rule, we easily obtain

$$\begin{aligned}
&f(0^+) = \lim_{x\to 0^+}\frac{\ln\left(\frac{\sin_p x}{x}\right)}{\ln\left(\frac{p+\cos_p x}{p+1}\right)} = \lim_{x\to 0^+} -\frac{x\cos_p x - \sin_p x}{x\sin_p x}\frac{p+\cos_p x}{\cos_p x\tan_p^{p-1} x}\\
&= -(p+1)\lim_{x\to 0^+}\frac{x\cos_p x - \sin_p x}{x^{p+1}} = 1
\end{aligned}$$

and

$$f(\frac{\pi_p}{2}) = \frac{\ln\left(\frac{2\sin_p \frac{\pi_p}{2}}{\pi_p}\right)}{\ln\left(\frac{p+\cos_p \frac{\pi_p}{2}}{p+1}\right)}.$$

The proof is complete. □

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