

**INEQUALITIES OF HERMITE-HADAMARD TYPE FOR
GA-CONVEX FUNCTIONS**

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ABSTRACT. Some inequalities of Hermite-Hadamard type for *GA*-convex functions defined on positive intervals are given.

1. INTRODUCTION

Let $I \subset (0, \infty)$ be an interval; a real-valued function $f : I \rightarrow \mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$(1.1) \quad f(x^{1-\lambda}y^\lambda) \leq (\geq) (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Since the condition (1.1) can be written as

$$(1.2) \quad f \circ \exp((1-\lambda)\ln x + \lambda\ln y) \leq (\geq) (1-\lambda)f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y),$$

then we observe that $f : I \rightarrow \mathbb{R}$ is *GA-convex* (concave) on I if and only if $f \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a, b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is *GA-convex* on $(0, \infty)$ [4].

For real and positive values of x , the *Euler gamma* function Γ and its *logarithmic derivative* ψ , the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [54] that the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is *GA-concave* on $(0, \infty)$ while the function $g : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is *GA-convex* on $(0, \infty)$.

If $[a, b] \subset (0, \infty)$ and the function $g : [\ln a, \ln b] \rightarrow \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $f : [a, b] \rightarrow \mathbb{R}$, $f(t) = g(\ln t)$ is *GA-convex* (concave) on $[a, b]$.

Indeed, if $x, y \in [a, b]$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} f(x^{1-\lambda}y^\lambda) &= g(\ln(x^{1-\lambda}y^\lambda)) = g[(1-\lambda)\ln x + \lambda\ln y] \\ &\leq (\geq) (1-\lambda)g(\ln x) + \lambda g(\ln y) = (1-\lambda)f(x) + \lambda f(y) \end{aligned}$$

showing that f is *GA-convex* (concave) on $[a, b]$.

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We recall that the classical Hermite-Hadamard inequality that states that

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}$$

for any convex function $f : [a, b] \rightarrow \mathbb{R}$.

For related results, see [1]-[20], [23]-[25], [26]-[35] and [36]-[46].

In [54] the authors obtained the following Hermite-Hadamard type inequality.

Theorem 1. *If $b > a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable GA-convex (concave) function on $[a, b]$, then*

$$(1.4) \quad f(I(a, b)) \leq (\geq) \frac{1}{b-a} \int_a^b f(t) dt \leq (\geq) \frac{b-L(a, b)}{b-a} f(b) + \frac{L(a, b)-a}{b-a} f(a).$$

The *identric mean* $I(a, b)$ is defined by

$$I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}.$$

The differentiability of the function is not necessary in Theorem 1 for the first inequality (1.4) to hold. A proof of this fact is proved below after some short preliminaries. The second inequality in (1.4) has been proved in [54] without differentiability assumption.

2. PRELIMINARIES

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior \mathring{I} and $f : I \rightarrow \mathbb{R}$ is a convex function on I . Then f is continuous on \mathring{I} and has finite left and right derivatives at each point of \mathring{I} . Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on \mathring{I} . It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \rightarrow \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \rightarrow [-\infty, \infty]$ such that $\varphi(\mathring{I}) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I , then ∂f is nonempty, $f'_-, f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on \mathring{I} , then $\partial f = \{f'\}$.

Now, since $f \circ \exp$ is convex on $[\ln a, \ln b]$ it follows that f has finite lateral derivatives on $(\ln a, \ln b)$ and by gradient inequality for convex functions we have

$$(2.1) \quad f \circ \exp(x) - f \circ \exp(y) \geq (x-y)\varphi(\exp y) \exp y$$

where $\varphi(\exp y) \in [f'_-(\exp y), f'_+(\exp y)]$ for any $x, y \in (\ln a, \ln b)$.

If $s, t \in (a, b)$ and we take in (2.1) $x = \ln t, y = \ln s$, then we get

$$(2.2) \quad f(t) - f(s) \geq (\ln t - \ln s) \varphi(s) s$$

where $\varphi(s) \in [f'_-(s), f'_+(s)]$.

Now, if we take the integral mean on $[a, b]$ in the inequality (2.2) we get

$$\frac{1}{b-a} \int_a^b f(t) dt - f(s) \geq \left(\frac{1}{b-a} \int_a^b \ln t dt - \ln s \right) \varphi(s) s$$

and since

$$\frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b)$$

then we get

$$(2.3) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(s) + (\ln I(a, b) - \ln s) \varphi(s) s$$

for any $s \in (a, b)$ and $\varphi(s) \in [f'_-(s), f'_+(s)]$. This is an inequality of interest in itself.

Now, if we take in (2.3) $s = I(a, b) \in (a, b)$ then we get the first inequality in (1.4) for GA-convex functions.

If f is differentiable and GA-convex on (a, b) , then we have from (2.3) the inequality

$$(2.4) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(s) + (\ln I(a, b) - \ln s) f'(s) s$$

for any $s \in (a, b)$.

If we take in (2.4) $s = \frac{a+b}{2} = A(a, b)$, then we get

$$(2.5) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(A(a, b)) - f'(A(a, b)) A(a, b) \ln \left(\frac{A(a, b)}{I(a, b)} \right).$$

If we assume that $f'(A(a, b)) \leq 0$, then, since $I(a, b) \leq A(a, b)$, we get

$$(2.6) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(A(a, b))$$

provided that f is differentiable and GA-convex on (a, b) .

Also, if we take in (2.4) $s = L(a, b)$, then we get

$$(2.7) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(L(a, b)) + f'(L(a, b)) L(a, b) \ln \left(\frac{I(a, b)}{L(a, b)} \right).$$

If we assume that $f'(L(a, b)) \geq 0$, then we get from (2.7) that

$$(2.8) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(L(a, b))$$

provided that f is differentiable and GA-convex on (a, b) .

Now, if we take in (2.4) $s = \sqrt{ab} = G(a, b)$, then we get

$$(2.9) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(G(a, b)) + f'(G(a, b)) G(a, b) \ln \left(\frac{I(a, b)}{G(a, b)} \right).$$

Since

$$\begin{aligned} \ln \left(\frac{I(a,b)}{G(a,b)} \right) &= \ln I(a,b) - \ln G(a,b) \\ &= \frac{b \ln b - a \ln a}{b-a} - 1 - \frac{\ln a + \ln b}{2} \\ &= \frac{a+b}{2} \frac{\ln b - \ln a}{b-a} - 1 = \frac{A(a,b) - L(a,b)}{L(a,b)}, \end{aligned}$$

then (2.9) is equivalent to

$$(2.10) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(G(a,b)) + f'(G(a,b)) G(a,b) \frac{A(a,b) - L(a,b)}{L(a,b)}.$$

If $f'(G(a,b)) \geq 0$, then we have

$$(2.11) \quad \frac{1}{b-a} \int_a^b f(t) dt \geq f(G(a,b))$$

provided that f is differentiable and GA-convex on (a,b) .

Motivated by the above results we establish in this paper other inequalities of Hermite-Hadamard type for GA-convex functions. Applications for special means are also provided.

3. NEW RESULTS

We start with the following result that provide in the right side of (1.4) a bound in terms of the identric mean.

Theorem 2. *Let $f : [a,b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$(3.1) \quad \begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &\leq (\geq) \frac{(\ln b - \ln I(a,b)) f(a) + (\ln I(a,b) - \ln a) f(b)}{\ln b - \ln a} \\ &= \frac{b - L(a,b)}{b-a} f(b) + \frac{L(a,b) - a}{b-a} f(a). \end{aligned}$$

Proof. Since is a GA-convex (concave) function on $[a,b]$ then $f \circ \exp$ is convex (concave) and we have

$$(3.2) \quad \begin{aligned} f(t) &= f \circ \exp(\ln t) = f \circ \exp \left(\frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a} \right) \\ &\leq (\geq) \frac{(\ln b - \ln t) f \circ \exp(\ln a) + (\ln t - \ln a) f \circ \exp(\ln b)}{\ln b - \ln a} \\ &= \frac{(\ln b - \ln t) f(a) + (\ln t - \ln a) f(b)}{\ln b - \ln a} \end{aligned}$$

for any $t \in [a,b]$.

This inequality is of interest in itself as well.

If we take the integral mean in (3.2) we get

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(t) dt \\ &\leq (\geq) \frac{\left(\ln b - \frac{1}{b-a} \int_a^b \ln t dt \right) f(a) + \left(\frac{1}{b-a} \int_a^b \ln t dt - \ln a \right) f(b)}{\ln b - \ln a} \end{aligned}$$

and since

$$\frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b),$$

then we obtain the desired result (3.1).

Now, we observe that

$$\begin{aligned} \frac{\ln b - \ln I(a, b)}{\ln b - \ln a} &= \frac{\ln b - \frac{b \ln b - a \ln a}{b-a} + 1}{\ln b - \ln a} \\ &= \frac{(b-a) \ln b - b \ln b + a \ln a + b - a}{(b-a)(\ln b - \ln a)} \\ &= \frac{b-a-a(\ln b - \ln a)}{(b-a)(\ln b - \ln a)} \\ &= \frac{L(a, b) - a}{b-a} \end{aligned}$$

and, similarly

$$\frac{\ln I(a, b) - \ln a}{\ln b - \ln a} = \frac{b - L(a, b)}{b - a},$$

which proves the last part of (3.1). \square

If $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is a GA -convex (concave) on I then we have the inequality

$$(3.3) \quad f(\sqrt{xy}) \leq (\geq) \frac{f(x) + f(y)}{2}$$

for any $x, y \in I$.

The following refinement of (3.3), which is an inequality of Hermite-Hadamard type, holds (see [44] for an extension for GA h -convex functions). For the sake of completeness we give here a short proof.

Lemma 1. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex (concave) function on $[a, b]$. Then we have*

$$(3.4) \quad f(\sqrt{ab}) \leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq (\geq) \frac{f(a) + f(b)}{2}.$$

Proof. By the definition of GA -convex (concave) functions on $[a, b]$ we have

$$(3.5) \quad f(a^{1-\lambda}b^\lambda) \leq (\geq) (1-\lambda)f(a) + \lambda f(b)$$

for any $\lambda \in [0, 1]$.

Integrating the inequality (3.5) on $[0, 1]$ we get

$$(3.6) \quad \int_0^1 f(a^{1-\lambda}b^\lambda) d\lambda \leq (\geq) f(a) \int_0^1 (1-\lambda) d\lambda + f(b) \int_0^1 \lambda d\lambda.$$

Since

$$\int_0^1 (1-\lambda) d\lambda = \int_0^1 \lambda d\lambda = \frac{1}{2}$$

and, by changing the variable $t = a^{1-\lambda}b^\lambda$, $\lambda \in [0, 1]$, we have

$$\int_0^1 f(a^{1-\lambda}b^\lambda) d\lambda = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt$$

then by (3.6) we get the second inequality in (3.4).

By the inequality (3.3) we have

$$(3.7) \quad f(\sqrt{ab}) = f\left(\sqrt{a^{1-\lambda}b^\lambda a^\lambda b^{1-\lambda}}\right) \leq (\geq) \frac{1}{2} [f(a^{1-\lambda}b^\lambda) + f(a^\lambda b^{1-\lambda})]$$

for any $\lambda \in [0, 1]$.

Integrating the inequality (3.7) $[0, 1]$ we get

$$(3.8) \quad f(\sqrt{ab}) \leq (\geq) \frac{1}{2} \left[\int_0^1 f(a^{1-\lambda}b^\lambda) d\lambda + \int_0^1 f(a^\lambda b^{1-\lambda}) d\lambda \right].$$

Since

$$\int_0^1 f(a^\lambda b^{1-\lambda}) d\lambda = \int_0^1 f(a^{1-\lambda}b^\lambda) d\lambda = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt,$$

then by (3.8) we get the first inequality in (3.4). \square

Remark 1. The inequality (3.3) can be also written for any $d > c > 0$ with $c, d \in I$ as

$$(3.9) \quad f(\sqrt{cd}) \leq (\geq) \int_0^1 f(c^{1-\lambda}d^\lambda) d\lambda \leq (\geq) \frac{f(c) + f(d)}{2},$$

provided GA-convex (concave) function on I .

We have the following representation result:

Lemma 2. Let $g : [x, y] \rightarrow \mathbb{C}$ be a Lebesgue integrable function on $[x, y]$. Then for any $\lambda \in [0, 1]$ we have the representation

$$(3.10) \quad \int_0^1 g[(1-t)x + ty] dt = (1-\lambda) \int_0^1 g[(1-t)((1-\lambda)x + \lambda y) + ty] dt \\ + \lambda \int_0^1 g[(1-t)x + t((1-\lambda)x + \lambda y)] dt.$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (3.3) is obvious.

Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 g[(1-t)(\lambda y + (1-\lambda)x) + ty] dt \\ = \int_0^1 g[((1-t)\lambda + t)y + (1-t)(1-\lambda)x] dt$$

and

$$\int_0^1 g[t(\lambda y + (1-\lambda)x) + (1-t)x] dt = \int_0^1 g[t\lambda y + (1-\lambda t)x] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda) dt$. Then

$$\int_0^1 g[((1-t)\lambda + t)y + (1-t)(1-\lambda)x] dt = \frac{1}{1-\lambda} \int_\lambda^1 g[uy + (1-u)x] du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 g[t\lambda y + (1-\lambda t)x] dt = \frac{1}{\lambda} \int_0^\lambda g[uy + (1-u)x] du.$$

Therefore

$$\begin{aligned}
 & (1 - \lambda) \int_0^1 g [(1 - t) (\lambda y + (1 - \lambda) x) + ty] dt \\
 & + \lambda \int_0^1 g [t (\lambda y + (1 - \lambda) x) + (1 - t) x] dt \\
 & = \int_\lambda^1 g [uy + (1 - u) x] du + \int_0^\lambda g [uy + (1 - u) x] du \\
 & = \int_0^1 g [uy + (1 - u) x] du
 \end{aligned}$$

and the identity (3.3) is proved. \square

Corollary 1. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{C}$ be a Lebesgue integrable function on $[a, b]$. Then for any $\lambda \in [0, 1]$ we have the representation*

$$\begin{aligned}
 (3.11) \quad & \int_0^1 f (a^{1-s} b^s) ds \\
 & = (1 - \lambda) \int_0^1 f \left(\left[a^{(1-\lambda)} b^\lambda \right]^{1-s} b^s \right) ds + \lambda \int_0^1 f \left(a^{1-s} \left[a^{(1-\lambda)} b^\lambda \right]^s \right) ds.
 \end{aligned}$$

Proof. Using (3.10) we have

$$\begin{aligned}
 & \int_0^1 f (a^{1-s} b^s) ds \\
 & = \int_0^1 f \circ \exp ((1 - s) \ln a + s \ln b) ds \\
 & = (1 - \lambda) \int_0^1 f \circ \exp [(1 - t) ((1 - \lambda) \ln a + \lambda \ln b) + t \ln b] dt \\
 & + \lambda \int_0^1 f \circ \exp [(1 - t) \ln a + t ((1 - \lambda) \ln a + \lambda \ln b)] dt \\
 & = (1 - \lambda) \int_0^1 f \circ \exp \left[(1 - t) \ln \left[a^{(1-\lambda)} b^\lambda \right] + t \ln b \right] dt \\
 & + \lambda \int_0^1 f \circ \exp \left[(1 - t) \ln a + t \ln \left[a^{(1-\lambda)} b^\lambda \right] \right] dt \\
 & = (1 - \lambda) \int_0^1 f \left(\left[a^{(1-\lambda)} b^\lambda \right]^{1-t} b^t \right) dt + \lambda \int_0^1 f \left(a^{1-t} \left[a^{(1-\lambda)} b^\lambda \right]^t \right) dt \\
 & = (1 - \lambda) \int_0^1 f \left(\left[a^{(1-\lambda)} b^\lambda \right]^{1-s} b^s \right) ds + \lambda \int_0^1 f \left(a^{1-s} \left[a^{(1-\lambda)} b^\lambda \right]^s \right) ds
 \end{aligned}$$

and the identity (3.11) is proved. \square

We are able now to provide a refinement of (3.4) as follows:

Theorem 3. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA -convex (concave) function on $[a, b]$. Then for any $\lambda \in [0, 1]$ we have

$$(3.12) \quad \begin{aligned} f\left(\sqrt{ab}\right) &\leq (\geq) (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq (\geq) \frac{1}{2} [f(a^{1-\lambda} b^\lambda) + (1-\lambda) f(b) + \lambda f(a)] \\ &\leq (\geq) \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. We prove the inequalities only for the GA -convex case.

Using the inequality (3.9) we have

$$f\left(\sqrt{a^{1-\lambda} b^\lambda}\right) \leq \int_0^1 f\left([a^{1-\lambda} b^\lambda]^{1-s} b^s\right) ds \leq \frac{f(a^{1-\lambda} b^\lambda) + f(b)}{2},$$

that is equivalent to

$$(3.13) \quad f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) \leq \int_0^1 f\left([a^{1-\lambda} b^\lambda]^{1-s} b^s\right) ds \leq \frac{f(a^{1-\lambda} b^\lambda) + f(b)}{2},$$

for any $\lambda \in [0, 1]$.

We also have

$$f\left(\sqrt{a a^{1-\lambda} b^\lambda}\right) \leq \int_0^1 f\left(a^{1-s} [a^{1-\lambda} b^\lambda]^s\right) ds \leq \frac{f(a) + f(a^{1-\lambda} b^\lambda)}{2}$$

that is equivalent to

$$(3.14) \quad f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \leq \int_0^1 f\left(a^{1-s} [a^{1-\lambda} b^\lambda]^s\right) ds \leq \frac{f(a) + f(a^{1-\lambda} b^\lambda)}{2}$$

for any $\lambda \in [0, 1]$.

If we multiply (3.13) by $1-\lambda$ and (3.14) by λ and add the obtained inequalities we get, by the identity (3.11), that

$$\begin{aligned} &(1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \int_0^1 f(a^{1-s} b^s) ds \\ &\leq (1-\lambda) \frac{f(a^{1-\lambda} b^\lambda) + f(b)}{2} + \lambda \frac{f(a) + f(a^{1-\lambda} b^\lambda)}{2} \\ &= \frac{1}{2} [f(a^{1-\lambda} b^\lambda) + (1-\lambda) f(b) + \lambda f(a)] \end{aligned}$$

for any $\lambda \in [0, 1]$.

This proves the second and third inequalities in (3.12).

By the GA -convexity we have

$$\begin{aligned} (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) &\geq f\left[\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)^{1-\lambda} \left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right)^\lambda\right] \\ &= f\left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right), \end{aligned}$$

which proves the first inequality in (3.12).

By the GA-convexity we also have

$$\begin{aligned} & \frac{1}{2} [f(a^{1-\lambda}b^\lambda) + (1-\lambda)f(b) + \lambda f(a)] \\ & \leq \frac{1}{2} [(1-\lambda)f(a) + \lambda f(b) + (1-\lambda)f(b) + \lambda f(a)] \\ & = \frac{f(a) + f(b)}{2}, \end{aligned}$$

which proves the last inequality in (3.12). \square

Corollary 2. *With the assumptions of Theorem 3 we have*

$$\begin{aligned} (3.15) \quad f(\sqrt{ab}) & \leq (\geq) \frac{1}{2} \left[f\left(a^{\frac{1}{4}}b^{\frac{3}{4}}\right) + f\left(a^{\frac{3}{4}}b^{\frac{1}{4}}\right) \right] \\ & \leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ & \leq (\geq) \frac{1}{2} \left[f(\sqrt{ab}) + \frac{f(b) + f(a)}{2} \right] \leq (\geq) \frac{f(a) + f(b)}{2}. \end{aligned}$$

4. RELATED RESULTS

The following result also holds:

Theorem 4. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex (concave) function on $[a, b]$. Then for any $t \in [a, b]$ we have*

$$\begin{aligned} (4.1) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ & \leq (\geq) \frac{1}{2} \left[f(t) + \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a} \right] \\ & \leq (\geq) \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. We give a proof only for the GA-convex case.

From the inequality (2.2) we have that

$$(4.2) \quad f(t) - f(s) \geq (\ln t - \ln s) f'_+(s) s$$

for any $s \in (a, b)$ and $t \in [a, b]$.

We divide by $s > 0$ and integrate on $[a, b]$ over s to get

$$(4.3) \quad f(t) \int_a^b \frac{1}{s} ds - \int_a^b \frac{f(s)}{s} ds \geq \left(\int_a^b f'_+(s) ds \right) \ln t - \int_a^b f'_+(s) \ln s ds$$

for any $t \in [a, b]$.

However

$$\int_a^b \frac{1}{s} ds = \ln b - \ln a, \quad \int_a^b f'_+(s) ds = f(b) - f(a)$$

and

$$\int_a^b f'_+(s) \ln s ds = f(s) \ln s \Big|_a^b - \int_a^b \frac{f(s)}{s} ds = f(b) \ln b - f(a) \ln a - \int_a^b \frac{f(s)}{s} ds.$$

Therefore, by (4.3) we get

$$\begin{aligned} & f(t)(\ln b - \ln a) - \int_a^b \frac{f(s)}{s} ds \\ & \geq (f(b) - f(a)) \ln t - f(b) \ln b + f(a) \ln a + \int_a^b \frac{f(s)}{s} ds \end{aligned}$$

namely

$$f(t)(\ln b - \ln a) - (f(b) - f(a)) \ln t + f(b) \ln b - f(a) \ln a \geq 2 \int_a^b \frac{f(s)}{s} ds,$$

which can be written as

$$f(t)(\ln b - \ln a) + f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a) \geq 2 \int_a^b \frac{f(s)}{s} ds$$

and the first inequality in (4.1) is proved.

Using (3.2) we have

$$\begin{aligned} & f(t) + \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a} \\ & \leq \frac{(\ln b - \ln t)f(a) + (\ln t - \ln a)f(b)}{\ln b - \ln a} \\ & + \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a} \\ & = f(a) + f(b) \end{aligned}$$

for any $t \in [a, b]$ that proves the last part of (4.1). \square

By taking the integral mean in the inequality (4.1) we have:

Corollary 3. *With the assumptions in Theorem 4 we have*

$$\begin{aligned} (4.4) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ & \leq (\geq) \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(t) dt + \frac{f(b)(\ln b - \ln I(a, b)) + f(a)(\ln I(a, b) - \ln a)}{\ln b - \ln a} \right] \\ & \leq (\geq) \frac{f(a) + f(b)}{2}. \end{aligned}$$

Since a simple calculation reveals (see the proof of Theorem 2) that

$$\begin{aligned} & \frac{f(b)(\ln b - \ln I(a, b)) + f(a)(\ln I(a, b) - \ln a)}{\ln b - \ln a} \\ & = \frac{L(a, b) - a}{b-a} f(b) + \frac{b - L(a, b)}{b-a} f(a), \end{aligned}$$

then the inequality (4.4) is equivalent to

$$\begin{aligned}
 (4.5) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 & \leq (\geq) \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(t) dt + \frac{L(a,b) - a}{b-a} f(b) + \frac{b - L(a,b)}{b-a} f(a) \right] \\
 & \leq (\geq) \frac{f(a) + f(b)}{2}.
 \end{aligned}$$

Remark 2. Taking specific values for $t \in [a, b]$ in (4.1) we get the following results

$$\begin{aligned}
 (4.6) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 & \leq (\geq) \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(b)(\ln b - \ln \frac{a+b}{2}) + f(a)(\ln \frac{a+b}{2} - \ln a)}{\ln b - \ln a} \right] \\
 & \leq (\geq) \frac{f(a) + f(b)}{2},
 \end{aligned}$$

$$\begin{aligned}
 (4.7) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \leq (\geq) \frac{1}{2} \left[f(\sqrt{ab}) + \frac{f(a) + f(b)}{2} \right] \\
 & \leq (\geq) \frac{f(a) + f(b)}{2}
 \end{aligned}$$

$$\begin{aligned}
 (4.8) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 & \leq (\geq) \frac{1}{2} \left[f(I(a,b)) + \frac{f(b)(\ln b - \ln I(a,b)) + f(a)(\ln I(a,b) - \ln a)}{\ln b - \ln a} \right] \\
 & = \frac{1}{2} \left[f(I(a,b)) + \frac{L(a,b) - a}{b-a} f(b) + \frac{b - L(a,b)}{b-a} f(a) \right] \\
 & \leq (\geq) \frac{f(a) + f(b)}{2},
 \end{aligned}$$

and

$$\begin{aligned}
 (4.9) \quad & \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 & \leq (\geq) \frac{1}{2} \left[f(L(a,b)) + \frac{f(b)(\ln b - \ln L(a,b)) + f(a)(\ln L(a,b) - \ln a)}{\ln b - \ln a} \right] \\
 & \leq (\geq) \frac{f(a) + f(b)}{2}.
 \end{aligned}$$

Now, observe that

$$f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a) = 0$$

iff

$$\ln t = \frac{f(b) \ln b - f(a) \ln a}{f(b) - f(a)} = \ln \left(\frac{b^{f(b)}}{a^{f(a)}} \right)^{\frac{1}{f(b) - f(a)}},$$

which is equivalent to

$$t = \left(\frac{bf(b)}{af(a)} \right)^{\frac{1}{f(b)-f(a)}}.$$

Therefore, if

$$t = \left(\frac{bf(b)}{af(a)} \right)^{\frac{1}{f(b)-f(a)}} \in [a, b]$$

then by (4.1) we get

$$(4.10) \quad \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \leq (\geq) \frac{1}{2} f \left(\left(\frac{bf(b)}{af(a)} \right)^{\frac{1}{f(b)-f(a)}} \right) \leq (\geq) \frac{f(a) + f(b)}{2}.$$

The following result also holds

Theorem 5. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex (concave) function on $[a, b]$. Then for any $t \in [a, b]$ we have

$$(4.11) \quad \frac{1}{2} \left[f(t) + \frac{f(b)b(\ln b - \ln t) + af(a)(\ln t - \ln a)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(s) ds \\ \geq (\leq) \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \ln t \right].$$

Proof. We give a proof only for the GA-convex case.

Integrate over s in the inequality (4.2) to get

$$(4.12) \quad f(t)(b-a) - \int_a^b f(s) ds \geq \ln t \int_a^b f'_+(s) ds - \int_a^b f'_+(s) s \ln s ds$$

for any $t \in [a, b]$.

Observe that, integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_+(s) ds = bf(b) - af(a) - \int_a^b f(s) ds$$

and

$$\int_a^b f'_+(s) s \ln s ds = f(b)b \ln b - f(a)a \ln a - \int_a^b (s \ln s)' f(s) ds \\ = f(b)b \ln b - f(a)a \ln a - \int_a^b (\ln s + 1) f(s) ds \\ = f(b)b \ln b - f(a)a \ln a - \int_a^b f(s) \ln s ds - \int_a^b f(s) ds.$$

Using the inequality (4.12) we get

$$\begin{aligned}
 & f(t)(b-a) - \int_a^b f(s) ds \\
 & \geq \ln t \left(bf(b) - af(a) - \int_a^b f(s) ds \right) \\
 & - f(b)b \ln b + f(a)a \ln a + \int_a^b f(s) \ln s ds + \int_a^b f(s) ds \\
 & = bf(b) \ln t - af(a) \ln t - \ln t \int_a^b f(s) ds \\
 & - f(b)b \ln b + f(a)a \ln a + \int_a^b f(s) \ln s ds + \int_a^b f(s) ds
 \end{aligned}$$

that is equivalent to

$$\begin{aligned}
 & f(t)(b-a) - bf(b) \ln t + af(a) \ln t + f(b)b \ln b - f(a)a \ln a - 2 \int_a^b f(s) ds \\
 & \geq \int_a^b f(s) \ln s ds - \ln t \int_a^b f(s) ds,
 \end{aligned}$$

namely

$$\begin{aligned}
 & f(t)(b-a) + f(b)b(\ln b - \ln t) + af(a)(\ln t - \ln a) - 2 \int_a^b f(s) ds \\
 & \geq \int_a^b f(s) \ln s ds - \ln t \int_a^b f(s) ds,
 \end{aligned}$$

for any $t \in [a, b]$ and the inequality (4.11) is proved. \square

Corollary 4. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function on $[a, b]$. Then*

$$\begin{aligned}
 (4.13) \quad & \frac{bf(b)(\ln b - \ln I(a, b)) + af(a)(\ln I(a, b) - \ln a)}{b-a} - \frac{1}{b-a} \int_a^b f(s) ds \\
 & \geq \frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \ln I(a, b).
 \end{aligned}$$

Moreover, if f is monotonic nondecreasing, then

$$\begin{aligned}
 (4.14) \quad & \frac{bf(b)(\ln b - \ln I(a, b)) + af(a)(\ln I(a, b) - \ln a)}{b-a} - \frac{1}{b-a} \int_a^b f(s) ds \\
 & \geq \frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \ln I(a, b) \geq 0.
 \end{aligned}$$

Proof. Integrating over t on $[a, b]$ and dividing by $b - a$ in (4.11) we get

$$\begin{aligned} & \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(s) ds + \frac{f(b)b \left(\ln b - \frac{1}{b-a} \int_a^b \ln t dt \right) + af(a) \left(\frac{1}{b-a} \int_a^b \ln t dt - \ln a \right)}{b-a} \right] \\ & - \frac{1}{b-a} \int_a^b f(s) ds \\ & \geq (\leq) \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \frac{1}{b-a} \int_a^b \ln t dt \right], \end{aligned}$$

that is equivalent to (4.13).

Now, since f is monotonic nondecreasing on $[a, b]$, then by Čebyšev inequality for synchronous functions, we have

$$\frac{1}{b-a} \int_a^b f(s) \ln s ds \geq \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \frac{1}{b-a} \int_a^b \ln t dt$$

that proves (4.14). \square

Corollary 5. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function on $[a, b]$. Then*

$$\begin{aligned} (4.15) \quad & \frac{1}{2} \left[f(\exp(\mu_f)) + \frac{f(b)b(\ln b - \mu_f) + af(a)(\mu_f - \ln a)}{b-a} \right] \\ & \geq \frac{1}{b-a} \int_a^b f(s) ds, \end{aligned}$$

where

$$\mu_f := \frac{\int_a^b f(s) \ln s ds}{\int_a^b f(s) ds} \in [\ln a, \ln b].$$

Proof. Follows by (4.11) on taking

$$\ln t = \frac{\int_a^b f(s) \ln s ds}{\int_a^b f(s) ds} \in [\ln a, \ln b].$$

\square

Remark 3. *If we take $t = \sqrt{ab}$ in (4.11), then we get*

$$\begin{aligned} (4.16) \quad & \frac{1}{2} \left[f(\sqrt{ab}) + \frac{f(b)b + af(a)}{2L(a, b)} \right] - \frac{1}{b-a} \int_a^b f(s) ds \\ & \geq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \ln \sqrt{ab} \right]. \end{aligned}$$

If we take $t = I(a, b)$ in (4.11), then we get

$$(4.17) \quad \begin{aligned} & \frac{1}{2} \left[f(I(a, b)) + \frac{f(b)b(\ln b - \ln I(a, b)) + af(a)(\ln I(a, b) - \ln a)}{b-a} \right] \\ & - \frac{1}{b-a} \int_a^b f(s) ds \\ & \geq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(s) \ln s ds - \left(\frac{1}{b-a} \int_a^b f(s) ds \right) \ln I(a, b) \right]. \end{aligned}$$

We use the following results obtained by the author in [21] and [22]

Lemma 3. *Let $h : [\alpha, \beta] \rightarrow \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then we have the inequalities*

$$(4.18) \quad \begin{aligned} & \frac{1}{8} \left[h'_+ \left(\frac{\alpha + \beta}{2} \right) - h'_- \left(\frac{\alpha + \beta}{2} \right) \right] (\beta - \alpha) \\ & \leq \frac{h(\alpha) + h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_\alpha^\beta h(t) dt \\ & \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha) \end{aligned}$$

and

$$(4.19) \quad \begin{aligned} & \frac{1}{8} \left[h'_+ \left(\frac{\alpha + \beta}{2} \right) - h'_- \left(\frac{\alpha + \beta}{2} \right) \right] (\beta - \alpha) \\ & \leq \frac{1}{\beta - \alpha} \int_\alpha^\beta h(t) dt - h \left(\frac{\alpha + \beta}{2} \right) \\ & \leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha). \end{aligned}$$

The constant $\frac{1}{8}$ is best possible in (4.18) and (4.19).

Finally, we have

Theorem 6. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex (concave) function on $[a, b]$. Then we have*

$$(4.20) \quad \begin{aligned} & \frac{1}{8} \left[f'_+ \left(\sqrt{ab} \right) - f'_- \left(\sqrt{ab} \right) \right] (\ln b - \ln a) \\ & \leq (\geq) \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ & \leq (\geq) \frac{1}{8} [f'_-(b)b - f'_+(a)a] (\ln b - \ln a) \end{aligned}$$

and

$$(4.21) \quad \begin{aligned} & \frac{1}{8} \left[f'_+ \left(\sqrt{ab} \right) - f'_- \left(\sqrt{ab} \right) \right] (\ln b - \ln a) \\ & \leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds - f \left(\sqrt{ab} \right) \\ & \leq (\geq) \frac{1}{8} [f'_-(b)b - f'_+(a)a] (\ln b - \ln a). \end{aligned}$$

Proof. Consider the function $h : [\ln a, \ln b] \rightarrow \mathbb{R}$ defined by $h(t) = f \circ \exp(t)$. Since f is a GA -convex (concave) function on $[a, b]$, then we have the lateral derivatives

$$h'_{\pm}(t) = (f'_{\pm} \circ \exp(t)) \exp t, \quad t \in [\ln a, \ln b].$$

If we apply the inequality (4.18) for the convex function $f \circ \exp$ on the interval $[\ln a, \ln b]$, then we have

$$\begin{aligned} & \frac{1}{8} \left[f'_{+} \circ \exp \left(\frac{\ln a + \ln b}{2} \right) - f'_{-} \circ \exp \left(\frac{\ln a + \ln b}{2} \right) \right] (\ln b - \ln a) \\ & \leq \frac{f \circ \exp(\ln a) + f \circ \exp(\ln b)}{2} - \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} f \circ \exp(t) dt \\ & \leq \frac{1}{8} [(f'_{-} \circ \exp(\ln b)) \exp(\ln b) - (f'_{+} \circ \exp(\ln a)) \exp(\ln a)] (\ln b - \ln a) \end{aligned}$$

that is equivalent to

$$\begin{aligned} (4.22) \quad & \frac{1}{8} \left[f'_{+}(\sqrt{ab}) - f'_{-}(\sqrt{ab}) \right] (\ln b - \ln a) \\ & \leq \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} f \circ \exp(t) dt \\ & \leq \frac{1}{8} [f'_{-}(b)b - f'_{+}(a)a] (\ln b - \ln a). \end{aligned}$$

If we change the variable $s = \exp t$, then $t = \ln s$ and $dt = \frac{ds}{s}$. Therefore

$$\int_{\ln a}^{\ln b} f \circ \exp(t) dt = \int_a^b \frac{f(s)}{s} ds$$

and by (4.22) we get the desired inequality (4.20).

The inequality (4.21) follows by (4.19). \square

Remark 4. *If the function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is differentiable and a GA -convex function on $[a, b] \subset \hat{I}$ then we have the following inequalities*

$$\begin{aligned} (4.23) \quad & 0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ & \leq \frac{1}{8} [f'(b)b - f'(a)a] (\ln b - \ln a) \end{aligned}$$

and

$$\begin{aligned} (4.24) \quad & 0 \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds - f(\sqrt{ab}) \\ & \leq \frac{1}{8} [f'(b)b - f'(a)a] (\ln b - \ln a). \end{aligned}$$

5. SOME APPLICATIONS

Let $p \neq 0$ and consider the convex function $g(t) = \exp(pt)$, $t \in \mathbb{R}$. Then the function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = g(\ln t) = \exp(p \ln t) = t^p$ is a GA -convex function

on $(0, \infty)$. We observe that for $0 < a < b$ we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b t^p dt &= \begin{cases} \frac{1}{p+1} \frac{b^{p+1} - a^{p+1}}{b-a}, & p \neq -1 \\ \frac{\ln b - \ln a}{b-a}, & p = -1 \end{cases} \\ &= \begin{cases} L_p^p(a, b), & p \neq -1 \\ L^{-1}(a, b), & p = -1 \end{cases} \end{aligned}$$

where $L_p(a, b)$ ($p \neq -1$) is the p -Logarithmic mean and L is the logarithmic mean defined in the introduction.

Using the inequality

$$\frac{1}{b-a} \int_a^b f(t) dt \leq \frac{b-L(a,b)}{b-a} f(b) + \frac{L(a,b)-a}{b-a} f(a)$$

for $f(t) = t^p$ ($p \neq 0$), we get

$$(5.1) \quad L_p^p(a, b) \leq \frac{b-L(a,b)}{b-a} b^p + \frac{L(a,b)-a}{b-a} a^p$$

for $p \neq 0$.

Observe that

$$\begin{aligned} \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt &= \frac{1}{b-a} \int_a^b t^{p-1} dt \\ &= \frac{1}{p} \frac{b^p - a^p}{b-a} = L_{p-1}^{p-1}(a, b), \quad p \neq 0. \end{aligned}$$

If we use the inequality

$$\begin{aligned} f(\sqrt{ab}) &\leq (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2} [f(a^{1-\lambda} b^\lambda) + (1-\lambda) f(b) + \lambda f(a)] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned}$$

for $\lambda \in [0, 1]$ and $f(t) = t^p$ ($p \neq 0$), then we get

$$\begin{aligned} (5.2) \quad G^p(a, b) &\leq (1-\lambda) G^p(a^{1-\lambda}, b^{\lambda+1}) + \lambda G^p(a^{2-\lambda}, b^\lambda) \\ &\leq L(a, b) L_{p-1}^{p-1}(a, b) \\ &\leq \frac{1}{2} [G^p(a^{2(1-\lambda)}, b^{2\lambda}) + (1-\lambda) b^p + \lambda a^p] \leq \frac{a^p + b^p}{2} \end{aligned}$$

for $\lambda \in [0, 1]$.

If we use the inequalities (4.23) and (4.24) for $f(t) = t^p$ ($p \neq 0$), then we get

$$(5.3) \quad 0 \leq \frac{a^p + b^p}{2} - L(a, b) L_{p-1}^{p-1}(a, b) \leq \frac{1}{8} p^2 \frac{L_{p-1}^{p-1}(a, b)}{L(a, b)} (b-a)^2$$

and

$$(5.4) \quad 0 \leq L(a, b) L_{p-1}^{p-1}(a, b) - G^p(a, b) \leq \frac{1}{8} p^2 \frac{L_{p-1}^{p-1}(a, b)}{L(a, b)} (b-a)^2.$$

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