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HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the concept of operator geometrically convex functions for positive linear operators and prove some Hermite-Hadamard type inequalities for these functions. As applications, we obtain trace inequalities for operators which give some refinements of previous results.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a sub-algebra of $B(H)$ stand for the commutative C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in \mathcal{A}$ is positive and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Let $\mathcal{A}^+$ stand for all strictly positive operators in $\mathcal{A}$.

Let A be a self-adjoint operator in $\mathcal{A}$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

For any $f, g \in C(\mathrm{Sp}(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

with this notation we define

$$f(A) = \Phi(f) \text{ for all } \ f \in C(\mathrm{Sp}(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on

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$\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in \mathrm{Sp}(A) \quad \text{implies that} \quad f(A) \geq g(A), \tag{1.1}$$

in the operator order of $B(H)$, see [18].

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex function if

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + 1 - \lambda f(b)$$

for $a, b \in I$ and $\lambda \in [0, 1]$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1.2}$$

It was firstly discovered by Hermite in 1881 in the journal Mathesis (see [10]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [14].

Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [1]. In 1974, Mitrinovič found Hermites note in Mathesis [10]. Since (3.2) was known as Hadamards inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [14].

Definition 1.1. [12] A continuous function $f : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ is said to be geometrically convex function (or multiplicatively convex function) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda}$$

for $a, b \in I$ and $\lambda \in [0, 1]$.

The author of [8] established the Hermite-Hadamard type inequalities for geometrically convex functions as follows:

Theorem 1.2. *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ be a geometrically convex function and $a, b \in I$ with $a < b$. If $f \in L^1[a, b]$, then*

$$\begin{aligned} f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t) f\left(\frac{ab}{t}\right)}dt \\ &\leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \\ &\leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

By changing variables $t = a^\lambda b^{1-\lambda}$ we have

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt = \int_0^1 f(a^\lambda b^{1-\lambda})d\lambda.$$

Remark 1. It is well-known that for positive numbers a and b

$$\min\{a,b\} \le G(a,b) = \sqrt{ab} \le L(a,b) = \frac{b-a}{\ln b - \ln a} \le A(a,b) = \frac{a+b}{2} \le \max\{a,b\}.$$

The author of [9] mentioned the following inequality, but here we provide a short proof which gives a refinement for above theorem.

Theorem 1.3. *Let f be a geometrically convex function defined on I a subinterval of $\mathbb{R}^+$. Then, we have*

$$f(\sqrt{ab}) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt \le \sqrt{f(a)f(b)}$$

for $a, b \in I$.

Proof. Since f is geometrically convex function, we can write

$$\begin{aligned} f(\sqrt{ab}) &= f\left(\sqrt{(a^\lambda b^{1-\lambda})(a^{1-\lambda}b^\lambda)}\right) \\ &\le \sqrt{f(a^\lambda b^{1-\lambda})f(a^{1-\lambda}b^\lambda)} \\ &\le \sqrt{f(a)^\lambda f(b)^{1-\lambda} f(a)^{1-\lambda} f(b)^\lambda} \\ &= \sqrt{f(a)f(b)}. \end{aligned}$$

for all $\lambda \in [0,1]$.
So, we have

$$f(\sqrt{ab}) \le \sqrt{f(a^\lambda b^{1-\lambda})f(a^{1-\lambda}b^\lambda)} \le \sqrt{f(a)f(b)}. \tag{1.3}$$

Integrate (1.3) over $[0,1]$, we have

$$f(\sqrt{ab}) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt \le \sqrt{f(a)f(b)}.$$

□

Lemma 1.4. [12, Page. 156] *Suppose that I is a subinterval of $\mathbb{R}^+$ and $f : I \to (0,\infty)$ is a geometrically convex function. Then*

$$F = \log \circ f \circ \exp : \log(I) \to \mathbb{R}$$

is a convex function. Conversely, if J is an interval (for which $\exp(J)$ is a subinterval of $\mathbb{R}^+$ and $F : J \to \mathbb{R}$ is a convex function, then

$$f = \exp \circ F \circ \log : \exp(J) \to \mathbb{R}^+$$

is geometrically convex function.

Theorem 1.5. *Let f be a geometrically convex function defined on $[a,b]$ such that $0<a<b$. Then, we have*

$$\begin{aligned} f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\ &\leq \exp\left(\frac{1}{\log b-\log a}\int_a^b \frac{\log f(t)}{t}dt\right) \\ &\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)} \end{aligned}$$

for $a,b\in I$.

Proof. Let $f:[a,b]\to\mathbb{R}$ be a geometrically convex function. So, by Lemma 1.4 we have

$$F(x)=\log\circ f\circ\exp(x):[\log a,\log b]\to\mathbb{R}$$

is convex.
Then, by [13, Remark 1.9.3]

$$\begin{aligned} F\left(\frac{\log a+\log b}{2}\right) &\leq \frac{1}{2}\left(F\left(\frac{3\log a+\log b}{4}\right)+F\left(\frac{\log a+3\log b}{4}\right)\right) \\ &\leq \frac{1}{\log b-\log a}\int_{\log a}^{\log b} F(x)dx \\ &\leq \frac{1}{2}\left(F\left(\frac{\log a+\log b}{2}\right)+\frac{F(\log a)+F(\log b)}{2}\right) \\ &\leq \frac{F(\log a)+F(\log b)}{2}. \end{aligned}$$

By definition of F, we obtain

$$\begin{aligned} \log\circ f\circ\exp(\log\sqrt{ab}) &\leq \frac{1}{2}\left(\log\circ f\circ\exp\left(\log a^{\frac{3}{4}}b^{\frac{1}{4}}\right)+\log\circ f\circ\exp\left(\log a^{\frac{3}{4}}b^{\frac{1}{4}}\right)\right) \\ &\leq \frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x)dx \\ &\leq \frac{1}{2}\left(\log\circ f\circ\exp\left(\log a^{\frac{1}{2}}b^{\frac{1}{2}}\right)+\frac{\log\circ f\circ\exp(\log a)+\log\circ f\circ\exp(\log b)}{2}\right) \\ &\leq \frac{\log\circ f\circ\exp(\log a)+\log\circ f\circ\exp(\log b)}{2}. \end{aligned}$$

It follows that

$$\begin{aligned}
\log f(\sqrt{ab}) &\leq \frac{1}{2}\left(\log f(a^{\frac{3}{4}}b^{\frac{1}{4}}) + \log f(a^{\frac{3}{4}}b^{\frac{1}{4}})\right) \\
&\leq \frac{1}{\log b - \log a}\int_{\log a}^{\log b} \log\circ f\circ \exp(x)dx \\
&\leq \frac{1}{2}\left(\log f\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right) + \frac{\log f(a) + \log f(b)}{2}\right) \\
&\leq \frac{\log f(a) + \log f(b)}{2}.
\end{aligned}$$

Since $\exp(x)$ is increasing, we have

$$\begin{aligned}
f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\
&\leq \exp\left(\frac{1}{\log b - \log a}\int_{\log a}^{\log b} \log f(\exp(x))dx\right) \\
&\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\
&\leq \sqrt{f(a)f(b)}.
\end{aligned}$$

Using change of variable $t = \exp(x)$ to obtain the desired result. the desired result. □

The author of [12, p. 158] showed that every polynomial $P(x)$ with non-negative coefficients is a geometrically convex function on $[0, \infty)$. More generally, every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is geometrically convex function on $(0, R)$ where R denotes the radius of convergence. This gives some different examples of geometrically convex function. It is easy to show that $\exp(x)$ is geometrically convex function.

In this paper, we introduce the concept of operator geometrically convex functions and prove the Hermite-Hadamard type inequalities for these class of functions. These results lead us to obtain some inequalities for trace functional of operators.

2. **Inequalities for operator geometrically convex functions**

In this section, we prove Hermite-Hadamard type inequality for operator geometrically convex function.

In [5] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I, the following inequalities holds

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dt \qquad (2.1)\\ &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \qquad (2.2)\\ &\leq \int_0^1 f\left((1-t)A+tB\right)dt\\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right] \qquad (2.3)\\ &\leq \frac{f(A)+f(B)}{2}, \qquad (2.4)\end{aligned}$$

for the first inequality in above, see [16].

To give operator geometrically convex function definition, we need following lemmas.

Lemma 2.1. [11, Lemma 3] *Let A and B be two operators in $\mathcal{A}^+$, and f a continuous function on* $\mathrm{Sp}(A)$. *Then, $AB = BA$ implies that $f(A)B = Bf(A)$.*

Since $f(t) = t^\lambda$ is continuous function for $\lambda \in [0, 1]$ and $\mathcal{A}$ is a commutative C^*-algebra, we have $A^\lambda B = BA^\lambda$. Moreover, by applying above lemma for $f(t) = t^{1-\lambda}$ again, we have $A^\lambda B^{1-\lambda} = B^{1-\lambda}A^\lambda$, for operators A and B in $\mathcal{A}^+$. It means A^λ and $B^{1-\lambda}$ commute together whenever A and B commute.

Lemma 2.2. *Let A and B be two operators in $\mathcal{A}^+$. Then*

$$\{A^\lambda B^{1-\lambda} : 0 \leq \lambda \leq 1\}$$

is convex.

Proof. We know that $\{\lambda A + (1-\lambda)B : 0 \leq \lambda \leq 1\}$ is convex for arbitrary operator A and B. So, $\{\lambda \log A + (1-\lambda)\log B : 0 \leq \lambda \leq 1\}$ is convex. Since A and B are commutative and knowing that e^f is convex when f is convex, we have

$$\begin{aligned} e^{(\lambda \log A+(1-\lambda)\log B)} &= e^{\lambda \log A}e^{(1-\lambda)\log B}\\ &= A^\lambda B^{1-\lambda}.\end{aligned}$$

So, $A^\lambda B^{1-\lambda}$ is convex for $0 \leq \lambda \leq 1$. □

Lemma 2.3. [18, Theorem 5.3] *Let A and B be in a Banach algebra such that $AB = BA$. Then*

$$\mathrm{Sp}(AB) \subset \mathrm{Sp}(A)\,\mathrm{Sp}(B).$$

Let A and B be two positive operators in $\mathcal{A}$ with spectra in I. Now, Lemma 2.1 and functional calculus [18, Theorem 10.3 (c)] imply that

$$\mathrm{Sp}(A^\lambda B^{1-\lambda}) \subset \mathrm{Sp}(A^\lambda)\,\mathrm{Sp}(B^{1-\lambda}) = \mathrm{Sp}(A)^\lambda\,\mathrm{Sp}(B)^{1-\lambda} \subseteq I$$

for $0 \le \lambda \le 1$.

Definition 2.4. A continuous function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is said to be operator geometrically convex if

$$f(A^\lambda B^{1-\lambda}) \le f(A)^\lambda f(B)^{1-\lambda}$$

for $A, B \in \mathcal{A}^+$ such that $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subseteq I$.

Now, we are ready to prove Hermite-Hadamard type inequality for operator geometrically convex functions.

Theorem 2.5. *Let f be an operator geometrically convex function. Then, we have*

$$\log f(\sqrt{AB}) \le \int_0^1 \log f(A^t B^{1-t})dt \le \log \sqrt{f(A)f(B)} \tag{2.5}$$

for $0 \le t \le 1$ and $A, B \in \mathcal{A}^+$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Proof. Since f is operator geometrically convex function, we have $f(\sqrt{AB}) \le \sqrt{f(A)f(B)}$. Let replace A and B by A^tB^{1-t} and $A^{1-t}B^t$ respectively, we obtain

$$f(\sqrt{AB}) \le \sqrt{f(A^tB^{1-t})f(A^{1-t}B^t)}. \tag{2.6}$$

It is well-known that $\log t$ is operator monotone function on $(0, \infty)$ (see [17]), i.e., $\log t$ is operator monotone function if $\log A \le \log B$ when $A \le B$. So, by above inequality, we have

$$\begin{aligned} \log f(\sqrt{AB}) &\le \log \sqrt{f(A^tB^{1-t})f(A^{1-t}B^t)} \\ &= \frac{1}{2}\log\left(f(A^tB^{1-t})f(A^{1-t}B^t)\right) \\ &= \frac{1}{2}\left(\log f(A^tB^{1-t}) + \log f(A^{1-t}B^t)\right). \end{aligned}$$

Therefore,

$$\log f(\sqrt{AB}) \le \frac{1}{2}\left(\log f(A^tB^{1-t}) + \log f(A^{1-t}B^t)\right).$$

Integrate above inequality over $[0, 1]$, we can write the following

$$\begin{aligned} \int_0^1 \log f(\sqrt{AB})dt &\le \frac{1}{2}\left(\int_0^1 \log f(A^tB^{1-t})dt + \int_0^1 \log f(A^{1-t}B^t)dt\right) \\ &= \int_0^1 \log f(A^tB^{1-t})dt. \end{aligned} \tag{2.7}$$

The last equality above follows by knowing that

$$\int_0^1 \log f(A^t B^{1-t})dt = \int_0^1 \log f(A^{1-t}B^t)dt.$$

Hence, from (2.7), we have

$$\log f(\sqrt{AB}) \leq \int_0^1 \log f(A^t B^{1-t})dt.$$

This proved left inequality of (2.5).
On the other hand, we have $f(A^tB^{1-t}) \leq f(A)^t f(B)^{1-t}$. It follows that

$$\begin{aligned} \log f(A^tB^{1-t}) &\leq \log f(A)^t f(B)^{1-t} \\ &= \log f(A)^t + \log f(B)^{1-t} \\ &= t\log f(A) + (1-t)\log f(B). \end{aligned}$$

So,

$$\log f(A^tB^{1-t}) \leq t\log f(A) + (1-t)\log f(B). \tag{2.8}$$

Now, integrate of (2.8) on $[0,1]$, we have

$$\begin{aligned} \int_0^1 \log f(A^tB^{1-t})dt &\leq \int_0^1 t\log f(A)dt + \int_0^1 (1-t)\log f(B)dt \\ &= \log f(A)\int_0^1 tdt + \log f(B)\int_0^1 (1-t)dt \\ &= \frac{1}{2}(\log f(A) + \log f(B)) \\ &= \log\sqrt{f(A)f(B)}. \end{aligned}$$

This completes the proof. □

We should mention, when f is operator geometrically convex function, then we have

$$\begin{aligned} f(\sqrt{AB}) &= f(\sqrt{A^tB^{1-t}A^{1-t}B^t}) \\ &\leq \sqrt{f(A^tB^{1-t})f(A^{1-t}B^t)} \\ &\leq \sqrt{f(A)^t f(B)^{1-t} f(A)^{1-t} f(B)^t} \\ &= \sqrt{f(A)f(B)}. \end{aligned}$$

So, we have

$$f(\sqrt{AB}) \leq \sqrt{f(A^tB^{1-t})f(A^{1-t}B^t)} \leq \sqrt{f(A)f(B)}.$$

Integrate above inequality over $[0,1]$, we obtain

$$f(\sqrt{AB}) \leq \int_0^1 \sqrt{f(A^tB^{1-t})f(A^{1-t}B^t)}dt \leq \sqrt{f(A)f(B)},$$

for $0 \leq t \leq 1$ and $A, B \in \mathcal{A}^+$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Let $A, B \in \mathcal{A}$ and $A \leq B$, by continuous functional calculus [18, Theorem 10.3 (b)], we can easily obtain $\exp(A) \leq \exp(B)$. This means $\exp(t)$ is operator monotone on $[0, \infty)$ for $A, B \in \mathcal{A}$.

On the other hand, like the classical case, the arithmetic-geometric mean inequality holds for operators as following

$$A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\nu} A^{\frac{1}{2}} \leq (1-\nu)A + \nu B, \quad \nu \in [0,1] \tag{2.9}$$

with respect to operator order for positive non-commutative operator in $B(H)$. Whenever, A and B commute together, then inequality (2.9) reduces to

$$A^{1-\nu} B^{\nu} \leq (1-\nu)A + \nu B, \quad \nu \in [0,1]. \tag{2.10}$$

Since $\exp(t)$ is an operator monotone function, by above inequality we have

$$\begin{aligned} \exp\left(A^{1-\nu} B^{\nu}\right) &\leq \exp\left((1-\nu)A + \nu B\right) \\ &= \exp((1-\nu)A)\exp(\nu B) \\ &= \exp(A)^{1-\nu} \exp(B)^{\nu}, \end{aligned}$$

for $A, B \in \mathcal{A}^+$ and $\nu \in [0,1]$. So, in this case $\exp(t)$ is an operator geometrically convex function on $[0, \infty)$.

Let replace f in Theorem 2.5 by $\exp(t)$ as an operator geometrically convex function, we have

$$\begin{aligned} \log\exp(\sqrt{AB}) \leq \int_0^1 \log\exp(A^t B^{1-t}) dt &\leq \log\sqrt{\exp(A)\exp(B)} \\ &= \frac{1}{2}\log\left(\exp(A)\exp(B)\right) \\ &= \frac{1}{2}\left(\log\exp(A) + \log\exp(B)\right). \end{aligned}$$

So,

$$\sqrt{AB} \leq \int_0^1 A^t B^{1-t} dt \leq \frac{A+B}{2}, \tag{2.11}$$

for $A, B \in \mathcal{A}^+$.

Here, we mention some remarks for operator geometrically convex functions.

Remark 2. $f(x) = \|x\|$ is geometrically convex function for usual operator norms since the following hold

$$f(A^{\alpha} B^{1-\alpha}) = \|A^{\alpha} B^{1-\alpha}\| \leq \|A\|^{\alpha} \|B\|^{1-\alpha} = f(A)^{\alpha} f(B)^{1-\alpha}.$$

Above inequality is a special case of McIntosh inequality.

Remark 3. If $f(t)$ is an operator geometrically convex function, then so is $g(t) = tf(t)$

$$\begin{aligned} g(A^\alpha B^{1-\alpha}) &= A^\alpha B^{1-\alpha} f(A^\alpha B^{1-\alpha}) \\ &\leq A^\alpha B^{1-\alpha} f(A)^\alpha f(B)^{1-\alpha} \\ &\leq A^\alpha f(A)^\alpha B^{1-\alpha} f(B)^{1-\alpha} \\ &= g(A)^\alpha g(B)^{1-\alpha} \end{aligned}$$

for $\alpha \in [0,1]$ and $A, B \in \mathcal{A}^+$

Remark 4. Operator geometrically convex functions is an algebra with some complication of operators spectra. To see this we make use of the following inequality

$$A^\alpha B^{1-\alpha} + C^\alpha D^{1-\alpha} \leq (A+C)^\alpha + (B+D)^{1-\alpha} \tag{2.12}$$

for $A, B, C, D \in \mathcal{A}^+$.

Let f and g be operator geometrically convex functions.
First, we prove that $f + g$ is an operator geometrically convex function

$$\begin{aligned} (f+g)(A^\alpha B^{1-\alpha}) &= f(A^\alpha B^{1-\alpha}) + g(A^\alpha B^{1-\alpha}) \\ &\leq f(A)^\alpha f(B)^{1-\alpha} + g(A)^\alpha g(B)^{1-\alpha} \\ &\leq (f(A) + g(A))^\alpha + (f(B) + g(B))^{1-\alpha} \\ &= ((f+g)(A))^\alpha + ((f+g)(B))^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$. In the last inequality above we applied (2.12).

Second, we show that mf is an operator geometrically convex function for a scalar m

$$\begin{aligned} (mf)(A^\alpha B^{1-\alpha}) &\leq mf(A)^\alpha f(B)^{1-\alpha} \\ &= (mf(A))^\alpha (mf(B))^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$.

Third, $h = fg$ is an operator geometrically convex function

$$\begin{aligned} h(A^\alpha B^{1-\alpha}) &= f(A^\alpha B^{1-\alpha}) g(A^\alpha B^{1-\alpha}) \\ &\leq f(A)^\alpha f(B)^{1-\alpha} g(A)^\alpha g(B)^{1-\alpha} \\ &= f(A)^\alpha g(A)^\alpha f(B)^{1-\alpha} g(B)^{1-\alpha} \\ &= h(A)^\alpha h(B)^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of H, we say that $A \in B(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{2.13}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $B_1(H)$ the set of trace class operators in $B(H)$.

We define the *trace* of a trace class operator $A \in B_1(H)$ to be

$$\operatorname{Tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{2.14}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.14) converges absolutely.

The following result collects some properties of the trace:

Theorem 2.6. *We have*

(i) If $A \in B_1(H)$ then $A^* \in B_1(H)$ and

$$\operatorname{Tr}(A^*) = \overline{\operatorname{Tr}(A)}; \tag{2.15}$$

(ii) If $A \in B_1(H)$ and $T \in B(H)$, then $AT, TA \in B_1(H)$ and

$$\operatorname{Tr}(AT) = \operatorname{Tr}(TA) \quad \text{and} \quad |\operatorname{Tr}(AT)| \leq \|A\|_1 \|T\|; \tag{2.16}$$

(iii) $\operatorname{Tr}(\cdot)$ is a bounded linear functional on $B_1(H)$ with $\|\operatorname{Tr}\| = 1$;

(iv) If $A, B \in B_1(H)$ then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

For the theory of trace functionals and their applications the reader is referred to [15].

For $A, B \geq 0$ we have $\operatorname{Tr}(AB) \leq \operatorname{Tr}(A)\operatorname{Tr}(B)$. Also, since $f(t) = t^{\frac{1}{2}}$ is monotone we have

$$\sqrt{\operatorname{Tr}(AB)} \leq \sqrt{\operatorname{Tr}(A)\operatorname{Tr}(B)} \tag{2.17}$$

for positive operator A and B in $B(H)$.

We know that $f(t) = \operatorname{Tr}(t)$ is operator geometrically convex function [7, p.513], i.e.

$$\operatorname{Tr}(A^t B^{1-t}) \leq \operatorname{Tr}(A)^t \operatorname{Tr}(B)^{1-t}$$

for $0 \leq t \leq 1$ and positive operators $A, B \in B_1(H)$.
For commutative case, we have

$$\sqrt{\operatorname{Tr}(AB)} \leq \operatorname{Tr}(\sqrt{AB}) \leq \sqrt{\operatorname{Tr}(A)\operatorname{Tr}(B)},$$

since $(\operatorname{Tr}(AB))^{\frac{1}{2}} \leq \operatorname{Tr}(AB)^{\frac{1}{2}}$.

Moreover, by Theorem 2.5 we can write

$$\begin{aligned}\log \operatorname{Tr}(\sqrt{AB}) &\leq \int_0^1 \log \operatorname{Tr}(A^t B^{1-t})dt \\ &\leq \log \sqrt{\operatorname{Tr}(A)\operatorname{Tr}(B)} \\ &= \frac{1}{2}(\log \operatorname{Tr}(A) + \log \operatorname{Tr}(B)).\end{aligned}$$

Let replace A and B by A^2 and B^2 in above inequality, respectively. By applying commutativity of algebra and knowing that $\operatorname{Tr}(A)^2 \leq (\operatorname{Tr} A)^2$ for positive operator A, we have

$$\log \operatorname{Tr}(AB) \leq \int_0^1 \log \operatorname{Tr}(A^{2t} B^{2(1-t)})dt \leq \log\left(\operatorname{Tr}(A)\operatorname{Tr}(B)\right).$$

3. **More results on trace functional class for product of operators**

In this section we prove some trace functional class inequalities for operators which are not necessarily commutative.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$. For $1 \leq p < \infty$, the Schatten p-norm of an operator $A \in B_1(H)$ defined by $\|A\|_p = (\operatorname{Tr}|A|^p)^{1/p}$. These Schatten p-norms are unitarily invariant.

In [2], Bhatia and Davis proved the following inequality

$$|||\,|A^*XB|^r|||^2 \leq |||\,|AA^*X|^r|||.|||\,|XBB^*|^r||| \tag{3.1}$$

for all operators A, B, X and $r \geq 0$.
As we know, $\|A\|_1 = \operatorname{Tr}|A|$. From (3.1) for $p = 1$, we have

$$\||A^*XB|^r\|_1^2 \leq \||AA^*X|^r\|_1.\||XBB^*|^r\|_1. \tag{3.2}$$

So, by inequality (3.2), we can write

$$(\operatorname{Tr}|A^*XB|^r)^2 \leq \operatorname{Tr}(|AA^*X|^r)\operatorname{Tr}(|XBB^*|^r), \tag{3.3}$$

for all operators $A, B \in B_1(H)$, $X \in B(H)$ and $r \geq 0$.
Let, $X = I$ in above inequality, we have

$$(\operatorname{Tr}|A^*B|^r)^2 \leq \operatorname{Tr}(|AA^*|^r)\operatorname{Tr}(|BB^*|^r).$$

Moreover, let $r = 1$ in inequality (3.3), we have

$$|\operatorname{Tr}(A^*XB)|^2 \leq (\operatorname{Tr}|A^*XB|)^2 \leq \operatorname{Tr}(|AA^*X|)\operatorname{Tr}(|XBB^*|).$$

Put X^* instead of X and applying the property of trace we have

$$|\operatorname{Tr}(AB^*X)|^2 \le \operatorname{Tr}(|AA^*X^*|)\operatorname{Tr}(|X^*BB^*|), \tag{3.4}$$

for all $A, B \in B_1(H)$ and $X \in B(H)$.

Let $X = I$ in (3.4).

Corollary 3.1. *Let $A, B \in B_1(H)$. Then*

$$|\operatorname{Tr}(AB^*)|^2 \le \operatorname{Tr}(AA^*)\operatorname{Tr}(BB^*). \tag{3.5}$$

In [6, Theorem 5], Dragomir proved the following inequality for $X \in B(H)$, $A, B \in B_1(H)$ and $\alpha \in [0,1]$

$$|\operatorname{Tr}(AB^*X)|^2 \le \operatorname{Tr}\left(|A^*|^2|X|^{2\alpha}\right)\operatorname{Tr}\left(|B^*|^2|X^*|^{2(1-\alpha)}\right).$$

Here, we give a generalization for above inequality when $\alpha \in \mathbb{R}$.

Theorem 3.2. *Let $X \in B_1(H)$, $A, B \in B(H)$ and $\alpha \in \mathbb{R}$. Then*

$$|\operatorname{Tr}(AB^*|X|)|^2 \le \operatorname{Tr}\left(|A^*|^2|X|^{2\alpha}\right)\operatorname{Tr}\left(|B^*|^2|X|^{2(1-\alpha)}\right). \tag{3.6}$$

Proof. Let replace A and B in Corollary 3.1 with $|X|^\alpha A$ and $|X|^{(1-\alpha)}B$, where $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned}
|\operatorname{Tr}(AB^*|X|)|^2 &\le \operatorname{Tr}\left(|X|^\alpha AA^*|X|^\alpha\right)\operatorname{Tr}\left(|X|^{(1-\alpha)}BB^*|X|^{(1-\alpha)}\right)\\
&= \operatorname{Tr}\left(AA^*|X|^\alpha|X|^\alpha\right)\operatorname{Tr}\left(BB^*|X|^{(1-\alpha)}|X|^{(1-\alpha)}\right)\\
&= \operatorname{Tr}\left(|A^*|^2|X|^{2\alpha}\right)\operatorname{Tr}\left(|B^*|^2|X|^{2(1-\alpha)}\right).
\end{aligned}$$

So, we have

$$|\operatorname{Tr}(AB^*|X|)|^2 \le \operatorname{Tr}\left(|A^*|^2|X|^{2\alpha}\right)\operatorname{Tr}\left(|B^*|^2|X|^{2(1-\alpha)}\right).$$

□

Let $A = B = I$ in Theorem 3.2, we have

$$|\operatorname{Tr}(X)|^2 \le \operatorname{Tr}\left(|X|^{2\alpha}\right)\operatorname{Tr}\left(|X|^{2(1-\alpha)}\right),$$

for $X \in B_1(H)$ and $\alpha \in \mathbb{R}$. Above inequality is a refinement for [6, Inequality (3.1)].

Also, let $X \in B_1(H)$ and normal operators $A, B \in B(H)$. For $\alpha \in \mathbb{R}$, we have

$$|\operatorname{Tr}(AB^*|X|)|^2 \le \operatorname{Tr}\left(|A|^2|X|^{2\alpha}\right)\operatorname{Tr}\left(|B|^2|X|^{2(1-\alpha)}\right).$$

In [4, Theorem 2.3], F. M. Dannan proved that if S_i and T_i $(i = 1, 2, \dots, n)$ are positive definite matrices, then we have

$$\left(\mathrm{Tr}\sum_{i=1}^{n} S_iT_i\right)^2 \leq \mathrm{Tr}\left(\sum_{i=1}^{n} S_i^2\right)\mathrm{Tr}\left(\sum_{i=1}^{n} T_i^2\right). \tag{3.7}$$

Moreover, if $S_iT_i \geq 0$, $(i = 1, 2, \dots, n)$. Then

$$\begin{aligned}\mathrm{Tr}\left(\sum_{i=1}^{n} S_iT_i\right)^2 &\leq \left(\mathrm{Tr}\sum_{i=1}^{n} S_iT_i\right)^2 \\ &\leq \mathrm{Tr}\left(\sum_{i=1}^{n} S_i^2\right)\mathrm{Tr}\left(\sum_{i=1}^{n} T_i^2\right).\end{aligned}$$

So,

$$\left(\mathrm{Tr}\sum_{i=1}^{n} S_iT_i\right)^2 \leq \mathrm{Tr}\left(\sum_{i=1}^{n} S_i^2\right)\mathrm{Tr}\left(\sum_{i=1}^{n} T_i^2\right).$$

Here, we prove inequality (3.7) for arbitrary operators.

Theorem 3.3. *Let S_i and T_i $(i = 1, 2, \dots, n)$ be arbitrary operators in $B_1(H)$. Then,*

$$\left|\mathrm{Tr}\left(\sum_{i=1}^{n} S_iT_i^*\right)\right|^2 \leq \mathrm{Tr}\left(\sum_{i=1}^{n} S_iS_i^*\right)\mathrm{Tr}\left(\sum_{i=1}^{n} T_iT_i^*\right). \tag{3.8}$$

Proof. Let $A = \begin{pmatrix} S_1 & S_2 & \dots & S_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} T_1 & T_2 & \dots & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So, we have

$$AB^* = \begin{pmatrix} \sum_{i=1}^{n} S_iT_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$AA^* = \begin{pmatrix} \sum_{i=1}^{n} S_iS_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$BB^* = \begin{pmatrix} \sum_{i=1}^n T_i T_i^* & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put A and B in inequality (3.5), by property of trace, we obtain the desired result.

□

Corollary 3.4. *Let S_i and T_i $(i = 1, 2, \ldots, n)$ be positive operators in $B_1(H)$. Then, we have*

$$\left(\operatorname{Tr}\sum_{i=1}^n S_i T_i\right)^2 \leq \operatorname{Tr}\left(\sum_{i=1}^n S_i^2\right)\operatorname{Tr}\left(\sum_{i=1}^n T_i^2\right).$$

Proof. By Theorem 3.3 for positive operators S_i and T_i, we obtain

$$\left|\operatorname{Tr}\left(\sum_{i=1}^n S_i T_i\right)\right|^2 \leq \operatorname{Tr}\left(\sum_{i=1}^n S_i^2\right)\operatorname{Tr}\left(\sum_{i=1}^n T_i^2\right).$$

Since S_i and T_i are positive operators, we have $\operatorname{Tr}(S_i T_i) \geq 0$. It follows that $\operatorname{Tr}(\sum_{i=1}^n S_i T_i) \geq 0$ because $\operatorname{Tr}(\sum_{i=1}^n S_i T_i) = \sum_{i=1}^n \operatorname{Tr}(S_i T_i)$. So,

$$\left(\operatorname{Tr}\sum_{i=1}^n S_i T_i\right)^2 \leq \operatorname{Tr}\left(\sum_{i=1}^n S_i^2\right)\operatorname{Tr}\left(\sum_{i=1}^n T_i^2\right).$$

□

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