Received 05/04/15

Some inequalities related to η-convex functions

[1]M. Eshaghi Gordji, [1]M. Rostamian Delavar and [2]S. S. Dragomir
[1]Department of Mathematics, Faculty of Basic Sciences,
Semnan University, P.O.Box. 35195-363, Semnan, Iran.
[2]Mathematics, School of Engineering and Science, Victoria University,
PO Box 14428, Melbourne City, MC 8001, Australia
{madjid.eshaghi,rostamian333}@gmail.com and Sever.Dragomir@vu.edu.au

Abstract

Keywords: η-convex function, Hermite-Hadamard inequality, Fejer inequality, Jensen inequality.
2010 MSC: 26A51, 26D15, 52A01.

1 Introduction and Preliminaries

The elegance in shape and property of convex functions makes it attractive to study this branch of mathematical analysis. It should be noticed that in new problems related to convexity, generalized notions about convex functions are required to obtain applicable results. During recently years many efforts have gone on generalization of notion of convex functions. Most important generalizations can be found in works that change the definition of convex functions to a generalized form such as quasi-convex [1], pseudo-convex [7], strongly convex [10], logarithmically convex [9], approximately convex [4], midconvex [5] functions etc.

On the other hand Hermite-Hadamard-Fejer inequality an interesting result related to convex functions has been proved in [3] as the following:

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1}$$

where $g : [a, b] \to \mathbb{R}^+ = [0, +\infty)$ is integrable and symmetric about $x = \frac{a+b}{2}$.

If in (1) we consider $g \equiv 1$ then we obtain Hermite-Hadamard inequality as the following:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

An interesting question in (2) was estimating the difference between left and middle terms and between right and middle terms. In [2] and [8] we can find some results about difference between right and middle terms in (2). Also in

[6], the difference between the middle and left terms in (2) has been estimated as the following:

Theorem 2 *Consider I^* as the interior of interval $I \subset \mathbb{R}$. Let $f : I^* \to \mathbb{R}$ be a differentiable mapping on I^*, $a, b \in I^*$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\Big|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\Big| \leq \frac{b-a}{8}\big(|f'(a)| + |f'(b)|\big) \tag{3}$$

Motivated by these works we introduce the notion of η-convex functions as generalization of convex functions and estimate the difference between the middle and left terms in (1) when $|f'|$ is an η-convex function. Also as an application we give an error estimate for midpoint formula.

Definition 3 *Let I* be an interval in real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called *convex with respect to bifunction* $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (briefly η-convex), if

$$f(tx + (1-t)y) \leq f(y) + t\eta\big(f(x), f(y)\big) \tag{4}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In fact above definition geometrically says that if a function is η-convex on I, then it's graph between any $x, y \in I$ is on or under the path starting from $\big(y, f(y)\big)$ and ending at $\big(x, f(y) + \eta(f(x), f(y))\big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. Note that by taking $x = y$ in (4) we get $t\eta(f(x), f(x)) \geq 0$ for any $x \in I$ and $t \in [0, 1]$ which implies that

$$\eta\big(f(x), f(x)\big) \geq 0$$

for any $x \in I$. Also if we take t $= 1$ in (4) we get

$$f(x) - f(y) \leq \eta\big(f(x), f(y)\big)$$

for any $x, y \in I$. If $f : I \to \mathbb{R}$ is a convex function and $\eta : I \times I \to \mathbb{R}$ is an arbitrary bifunction that satisfies

$$\eta(x, y) \geq x - y$$

for any $x, y \in I$, then

$$f(tx + (1-t)y) \leq f(y) + t[f(x) - f(y)] \leq f(y) + t\eta\big(f(x), f(y)\big)$$

showing that f is η-convex.

There are simple examples about η-convexity of a function.

Example 4 (1) For a convex function f, we may find another function η other than the function $\eta(x,y) = x - y$ such that f is η-convex. Consider $f(x) = x^2$ and $\eta(x,y) = 2x + y$. Then we have

$$f\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big)^2 \leq$$
$$y^2 + \lambda x^2 + \lambda(1-\lambda)2xy \leq y^2 + \lambda x^2 + \lambda(1-\lambda)(x^2+y^2) \leq$$
$$y^2 + \lambda(x^2 + x^2 + y^2) = y^2 + \lambda(2x^2 + y^2) = f(y) + \lambda\eta\big(f(x), f(y)\big)$$

for all $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$. Also the facts $x^2 \leq y^2 + (2x^2+y^2)$ and $y^2 \leq y^2$, for all $x, y \in \mathbb{R}$ show the correctness of inequality for $\lambda = 1$ and $\lambda = 0$ respectively which means that f is η-convex. Note that the function $f(x) = x^2$ is η-convex w.r.t all $\eta(x,y) = ax + by$ with $a \geq 1$, $b \geq -1$ and $x, y \in \mathbb{R}$.

(2) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

and define a bifunction η as $\eta(x,y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that f is a η-convex function but not a convex one.

(3) Define the function $f : \mathbb{R}^+ = [0, +\infty) \to \mathbb{R}^+$ as $f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$
and a bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $\eta(x,y) = \begin{cases} x + y, & x \leq y; \\ 2(x+y), & x > y. \end{cases}$

Then f is η-convex but is not convex.

As a basic result we investigate that when an η-convex function can be continuous. We need two definitions.

Definition 5 *[11] A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if corresponding to any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any collection $\{a_i, b_i\}_1^n$ of disjoint open intervals of $[a,b]$ with $\sum_1^n (b_i - a_i) < \delta$, $\sum_1^n |f(b_i) - f(a_i)| < \varepsilon$.*

Definition 6 *[11] A function $f : [a,b] \to \mathbb{R}$ is said to satisfy Lipschitz condition on $[a,b]$ if there is a constant K so that for any two points $x, y \in [a,b]$, $|f(x) - f(y)| \leq K|x - y|$.*

Lemma 7 *Suppose that $f : I \to \mathbb{R}$ is an η-convex function and η is bounded from above on $f(I) \times f(I)$. Then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Hence, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Proof. Let M_η be the upper bound of η on $f(I) \times f(I)$. Consider closed interval $[a,b]$ in I° and choose $\varepsilon > 0$ such that $[a-\varepsilon, b+\varepsilon]$ belongs to I. Suppose that x, y are distinct points of $[a,b]$. Set $z = y + \frac{\varepsilon}{|y-x|}(y-x)$ and $t = \frac{|y-x|}{\varepsilon + |y-x|}$. So it is not hard to see that $z \in [a-\varepsilon, b+\varepsilon]$ and $y = tz + (1-t)x$. Then

$$f(y) \leq f(x) + t\,\eta\big(f(z), f(x)\big) \leq f(x) + tM_\eta. \tag{5}$$

This implies that

$$f(y) - f(x) \leq tM_\eta = \frac{|y-x|}{\varepsilon + |y-x|} M_\eta \leq \frac{|y-x|}{\varepsilon} M_\eta = K\,|y-x|\,, \tag{6}$$

where $K = \frac{M_\eta}{\varepsilon}$.

Also if we change the place of x, y in above argument we have $f(x) - f(y) \leq K\,|y-x|$. Therefore $\mid f(y) - f(x) \mid \leq K\,|y-x|$.

It follows that if we choose $\delta < \varepsilon/K$, then f is absolutely continuous. Finally since $[a,b]$ is arbitrary on I°, then f is continuous on I°.

As a consequence of Lemma7, an η-convex function $f : [a,b] \to \mathbb{R}$ where η is bounded from above on $f\big([a,b]\big) \times f\big([a,b]\big)$ is integrable.

2 Main Result

The following lemma is generalization of Lemma 2.1 in [6].

Lemma 8 *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable mapping, $g : [a,b] \to \mathbb{R}^+$ is a continuous mapping and f' is integrable on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a} f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx =$$
$$(b-a)\Big[\int_0^{1/2} M(t) f'\big(ta + (1-t)b\big)dt + \int_{1/2}^1 N(t) f'\big(ta + (1-t)b\big)dt\Big],$$

where

$$M(t) = \int_0^t g\big(ua + (1-u)b\big)du \quad \text{and} \quad N(t) = -\int_t^1 g\big(ua + (1-u)b\big)du.$$

Proof. Using the change of variable $x = ta + (1-t)b$,

$$I_1 = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a} f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx =$$
$$\int_0^1 f\big(ta + (1-t)b\big) g\big(ta + (1-t)b\big)dt - f\Big(\frac{a+b}{2}\Big)\int_0^1 g\big(ta + (1-t)b\big)dt = I_2.$$

It follows that

$$I_2 = \int_0^{1/2} f\big(ta + (1-t)b\big) g\big(ta + (1-t)b\big)dt - f\Big(\frac{a+b}{2}\Big)\int_0^{1/2} g\big(ta + (1-t)b\big)dt +$$
$$\int_{1/2}^1 f\big(ta + (1-t)b\big) g\big(ta + (1-t)b\big)dt - f\Big(\frac{a+b}{2}\Big)\int_{1/2}^1 g\big(ta + (1-t)b\big)dt.$$

The Leibniz integral rule gives

$$g\big(ta+(1-t)b\big)=\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)'=\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)'.$$

So

$$\begin{aligned}I_2=&\int_0^{1/2} f\big(ta+(1-t)b\big)\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)'dt-f\Big(\frac{a+b}{2}\Big)\int_0^{1/2} g\big(ta+(1-t)b\big)dt\\&+\int_{1/2}^1 f\big(ta+(1-t)b\big)\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)'dt-f\Big(\frac{a+b}{2}\Big)\int_{1/2}^1 g\big(ta+(1-t)b\big)dt.\end{aligned}$$

Using integration by parts in last I_2 we have

$$\begin{aligned}I_2=&\Big(\textstyle\int_0^t g\big(ua+(1-u)b\big)du\Big).f\big(ta+(1-t)b\big)\Big|_0^{1/2}\\&-\textstyle\int_0^{1/2}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)(a-b)dt\\&-f\Big(\tfrac{a+b}{2}\Big)\textstyle\int_0^{1/2} g\big(ta+(1-t)b\big)dt+\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big).f\big(ta+(1-t)b\big)\Big|_{1/2}^1\\&+\textstyle\int_{1/2}^1\Big(\int_t^1 g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)(a-b)dt-f\Big(\tfrac{a+b}{2}\Big)\int_{1/2}^1 g\big(ta+(1-t)b\big)dt.\end{aligned}$$

If we apply the limits we have

$$\begin{aligned}I_2=(b-a)\Big[&\int_0^{1/2}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)dt\\&+\int_{1/2}^1\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)dt\Big].\end{aligned}$$

Since $I_1=I_2$, the result is obtained. ■

Remark 9 *In Lemma 8, if we use the change of variable $x=tb+(1-t)a$, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\\&=(b-a)\Big[\int_0^{1/2} M(t)f'\big(tb+(1-t)a\big)dt+\int_{1/2}^1 N(t)f'\big(tb+(1-t)a\big)dt\Big],\end{aligned}$$

where

$$M(t)=-\int_0^t g\big(ub+(1-u)a\big)du \quad \textit{and} \quad N(t)=\int_t^1 g\big(ub+(1-u)a\big)du.$$

Using Lemma 8, we can prove the following theorem to estimate the difference between the middle and left terms in (1).

Theorem 10 *Suppose that $f:[a,b]\to\mathbb{R}$ is an integrable mapping, $g:[a,b]\to\mathbb{R}^+$ is a continuous mapping symmetric with respect to $\frac{a+b}{2}$ and $|f'|$ is an η-convex mapping on $[a,b]$ with a bounded η from above. Then*

$$\left|\int_a^b f(x)g(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|$$
$$\leq \frac{1}{(b-a)}\int_{\frac{a+b}{2}}^b \Big[(x-a)^2-(b-x)^2\Big]g(x)K dx,$$

where

$$K=\min\Big\{\big|f'(b)\big|+\frac{\big|\eta\big(f'(a),f'(b)\big)\big|}{2},\big|f'(a)\big|+\frac{\big|\eta\big(f'(b),f'(a)\big)\big|}{2}\Big\}$$

Proof. From Lemma 7, Lemma 8 and η-convexity of $|f'|$ it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|\leq$$
$$(b-a)\Big\{\int_0^{1/2}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)\Big(\big|f'(b)\big|+t\big|\eta(f'(a),f'(b))\big|\Big)dt+$$
$$\int_{1/2}^1\Big(\int_t^1 g\big(ua+(1-u)b\big)du\Big)\Big(\big|f'(b)\big|+t\big|\eta(f'(a),f'(b))\big|\Big)dt\Big\}=J_1$$

Changing the order of integrals and calculation of internal integrals in J_1 imply that

$$J_1=(b-a)\Big\{\int_0^{1/2}\int_u^{1/2} g\big(ua+(1-u)b\big)\Big(\big|f'(b)\big|+t\big|\eta\big(f'(a),f'(b)\big)\big|\Big)dtdu+$$
$$\int_{1/2}^1\int_{1/2}^u g\big(ua+(1-u)b\big)\Big(\big|f'(b)\big|+t\big|\eta\big(f'(a),f'(b)\big)\big|\Big)dtdu\Big\}=$$
$$(b-a)\Big\{\int_0^{1/2}\Big(t\big|f'(b)\big|+\frac{1}{2}t^2\big|\eta\big(f'(a),f'(b)\big)\big|\Big|_u^{1/2}\Big)g\big(ua+(1-u)b\big)du+$$
$$\int_{1/2}^1\Big(t\big|f'(b)\big|+\frac{1}{2}t^2\big|\eta\big(f'(a),f'(b)\big)\big|\Big|_{1/2}^u\Big)g\big(ua+(1-u)b\big)du\Big\}=$$
$$(b-a)\Big\{\int_0^{1/2}(\frac{1}{2}-u)\big|f'(b)\big|+(\frac{1}{8}-\frac{1}{2}u^2)\big|\eta\big(f'(a),f'(b)\big)\big|g\big(ua+(1-u)b\big)du+$$
$$\int_{1/2}^1(u-\frac{1}{2})\big|f'(b)\big|+(\frac{1}{2}u^2-\frac{1}{8})\big|\eta\big(f'(a),f'(b)\big)\big|g\big(ua+(1-u)b\big)du\Big\}=J_2.$$

Changing the variable by $x=ua+(1-u)b$ in J_2 implies that

$$J_2=\int_{\frac{a+b}{2}}^b\Big(\frac{1}{2}-\frac{x-b}{a-b}\Big)\big|f'(b)\big|+\Big(\frac{1}{8}-\frac{1}{2}(\frac{x-b}{a-b})^2\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx+$$

$$\int_a^{\frac{a+b}{2}} \Big(\frac{x-b}{a-b}-\frac{1}{2}\Big)\big|f'(b)\big| + \Big(\frac{1}{2}(\frac{x-b}{a-b})^2-\frac{1}{8}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx =$$

$$\int_{\frac{a+b}{2}}^b \Big(\frac{a+b-2x}{2(a-b)}\Big)\big|f'(b)\big| + \Big(\frac{(a-b)^2-4(x-b)^2}{8(a-b)^2}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx +$$

$$\int_a^{\frac{a+b}{2}} \Big(\frac{2x-(a+b)}{2(a-b)}\Big)\big|f'(b)\big| + \Big(\frac{4(x-b)^2-(a-b)^2}{8(a-b)^2}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx = J_3.$$

Since for any $x \in [a,b]$ we have $g(x) = g(a+b-x)$, then

$$J_3 = \int_{\frac{a+b}{2}}^b \Big(\frac{a+b-2x}{2(a-b)}\Big)\big|f'(b)\big| + \Big(\frac{(a-b)^2-4(x-b)^2}{8(a-b)^2}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx +$$

$$\int_{\frac{a+b}{2}}^b \Big(\frac{a+b-2x}{2(a-b)}\Big)\big|f'(b)\big| + \Big(\frac{4(x-a)^2-(a-b)^2}{8(a-b)^2}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx =$$

$$\int_{\frac{a+b}{2}}^b \Big(\frac{a+b-2x}{(a-b)}\Big)\big|f'(b)\big| + \Big(\frac{4(x-a)^2-4(x-b)^2}{8(a-b)^2}\Big)\big|\eta\big(f'(a),f'(b)\big)\big|g(x)dx =$$

$$\frac{1}{(a-b)^2}\int_{\frac{a+b}{2}}^b \Big((a-b)(a+b-2x)\Big)\big|f'(b)\big| + \Big((x-a)^2-(x-b)^2\Big)\frac{\big|\eta\big(f'(a),f'(b)\big)\big|}{2}g(x)dx =$$

$$\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b \Big((x-a)^2-(b-x)^2\Big)\Big(\big|f'(b)\big| + \frac{\big|\eta\big(f'(a),f'(b)\big)\big|}{2}\Big)g(x)dx = J_4.$$

On the other hand according to Remark 9, if we use the change of variable $x = ub+(1-u)a$ then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right| \leq$$

$$\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b \Big((x-a)^2-(b-x)^2\Big)\Big(\big|f'(a)\big| + \frac{\big|\eta\big(f'(b),f'(a)\big)\big|}{2}\Big)g(x)dx = J_5.$$

We can deduce the result from

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right| \leq \min\{J_4, J_5\}.$$

■

Remark 11 *Theorem 10 reduces to Theorem 2, if we set* $\eta(x,y) = x-y$ *and* $g \equiv 1$.

Finally as an application of Theorem 10, we give an error estimate for midpoint formula that is generalization of Proposition 4.1 in [6].

Suppose that d is a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of interval $[a,b]$ and consider formula

$$\int_a^b f(x)g(x)dx = T(f,g,d) + E(f,g,d),$$

where

$$T(f,g,d) = \sum_{i=0}^{n-1} f\Big(\frac{x_i + x_{i+1}}{2}\Big) \int_{x_i}^{x_{i+1}} g(x)dx$$

and $E(f,g,d)$ is the approximation error.

Theorem 12 *Suppose that $f : [a,b] \to \mathbb{R}$ is an integrable mapping, $g : [a,b] \to \mathbb{R}^+$ is a continuous mapping symmetric with respect to $\frac{a+b}{2}$ and $|f'|$ is an η-convex mapping on $[a,b]$ with a bounded η from above. Then*

$$\Big|E(f,g,d)\Big| \leq \sum_{i=0}^{n-1} \frac{1}{(x_{i+1} - x_i)} \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} \Big[(x - x_i)^2 - (x_{i+1} - x)^2\Big] g(x) K_i dx,$$

where

$$K_i = \min\left\{ \Big[\big|f'(x_{i+1})\big| + \frac{\big|\eta\big(f'(x_i), f'(x_{i+1})\big)\big|}{2}\Big], \Big[\big|f'(x_i)\big| + \frac{\big|\eta\big(f'(x_{i+1}), f'(x_i)\big)\big|}{2}\Big] \right\},$$

for $i = 0, 1, \cdots, n-1$.

Proof. It is enough to apply Theorem 10 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ of the partition d for interval $[a,b]$, and to sum all achieved inequalities over i and then using triangle inequality. ∎

References

[1] B. Definetti, *Sulla stratificazioni convesse*, Ann. Math. Pura. Appl. **30** (1949) 173–183.

[2] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998) 91–95.

[3] L. Fejer, *Uberdie fourierreihen*, II, Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906) 369–390.

[4] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952) 821–828.

[5] J. L. W. V. Jensen, *On konvexe funktioner og uligheder mellem middlvaerdier*, Nyt. Tidsskr. Math. B. **16** (1905) 49-69.

[6] U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp. **147**(1) (2004) 137–146.

[7] O. L. Mangasarian, *Pseudo-Convex functions*, SIAM Journal on Control, **3** (1965) 281–290.

[8] C. E. M. Pearce and J. Pecaric, *Inequalities for differentiable mappings with application to special means and quadrature formula*, Appl. Math. Lett. **13** (2000) 51–55.

[9] J. E. Pecaric , F. Proschan and Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, 1992.

[10] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966) 72–75.

[11] A. W. Robert and D. E. Varbeg, *Convex Functions*, Academic Press, (1973).