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INEQUALITIES OF JENSEN TYPE FOR GA-CONVEX FUNCTIONS

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Abstract. Some integral inequalities of Jensen type for GA-convex functions defined on intervals of real line are given. Applications in relation to Hermite-Hadamard inequalities and Jensen discrete inequalities are provided. Inequalities for GA-convex functions of selfadjoint operators on complex Hilbert spaces are established as well.

1. Introduction

We recall some facts on the lateral derivatives and subdifferential of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{\mathrm{I}}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $I \subset (0, \infty)$ be an interval; a real-valued function $h : I \to \mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$h\left(x^{1-\lambda}y^{\lambda}\right) \leq (\geq)(1-\lambda)h(x) + \lambda h(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Since the condition (1.1) can be written as

$$h \circ \exp((1-\lambda)\ln x + \lambda \ln y) \leq (\geq)(1-\lambda) h \circ \exp(\ln x) + \lambda h \circ \exp(\ln y), \tag{1.2}$$

then we observe that $h : I \to \mathbb{R}$ is *GA-convex* (concave) on I if and only if $h \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a,b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $h(x) = \ln(1+x)$ is GA-convex on $(0,\infty)$ [3].

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For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [27] that the function $h : (0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0, \infty)$ while the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0, \infty)$.

If $[a, b] \subset (0, \infty)$ and the function $g : [\ln a, \ln b] \to \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $h : [a, b] \to \mathbb{R}$, $h(t) = g(\ln t)$ is GA-convex (concave) on $[a, b]$.

Indeed, if $x, y \in [a, b]$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} h\left(x^{1-\lambda} y^{\lambda}\right) &= g\left(\ln\left(x^{1-\lambda} y^{\lambda}\right)\right) = g\left[(1-\lambda)\ln x + \lambda \ln y\right] \\ &\leq (\geq)(1-\lambda) g(\ln x) + \lambda g(\ln y) = (1-\lambda) h(x) + \lambda h(y) \end{aligned}$$

showing that h is GA-convex (concave) on $[a, b]$.

The *identric mean* $I(a, b)$ is defined by

$$I(a, b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}.$$

Now, since $h \circ \exp$ is convex on $[\ln a, \ln b]$ it follows that h has finite lateral derivatives on $(\ln a, \ln b)$ and by gradient inequality for convex functions we have

$$h \circ \exp(x) - h \circ \exp(y) \geq (x - y)\varphi(\exp y)\exp y \tag{1.3}$$

where $\varphi(\exp y) \in \left[h'_-(\exp y), h'_+(\exp y)\right]$ for any $x, y \in (\ln a, \ln b)$.

If $s, t \in (a, b)$ and we take in (1.3) $x = \ln t, y = \ln s$, then we get

$$h(t) - h(s) \geq (\ln t - \ln s)\varphi(s) s \tag{1.4}$$

where $\varphi(s) \in \left[h'_-(s), h'_+(s)\right]$.

2. Some Jensen's Type Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma -$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ –a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)| d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) d\mu(x)$.

If $f,\ g:\Omega\rightarrow\mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg\in L_{w}\left(\Omega,\mu\right)$, then we may consider the *Čebyšev functional*

$$T_{w}\left(f,g\right):=\int_{\Omega}wfgd\mu-\int_{\Omega}wfd\mu\int_{\Omega}wgd\mu. \tag{2.1}$$

The following result is known in the literature as the *Grüss inequality*

$$\left|T_{w}\left(f,g\right)\right|\leq\frac{1}{4}\left(\Gamma-\gamma\right)\left(\Delta-\delta\right), \tag{2.2}$$

provided

$$-\infty<\gamma\leq f\left(x\right)\leq\Gamma<\infty,\ \ -\infty<\delta\leq g\left(x\right)\leq\Delta<\infty \tag{2.3}$$

for μ –a.e. (almost every) $x\in\Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If we assume that $-\infty<\gamma\leq f\left(x\right)\leq\Gamma<\infty$ for μ –a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ and by the Schwarz's integral inequality, we have

$$\int_{\Omega}w\left|f-\int_{\Omega}wfd\mu\right|d\mu\leq\left[\int_{\Omega}wf^{2}d\mu-\left(\int_{\Omega}wfd\mu\right)^{2}\right]^{\frac{1}{2}}\leq\frac{1}{2}\left(\Gamma-\gamma\right). \tag{2.4}$$

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [7] the following result:

Theorem 1. *Let $\Phi:[m,M]\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\rightarrow[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ \left(\Phi'\circ f\right)f\in L_{w}\left(\Omega,\mu\right)$, where $w\geq0$ μ-a.e. (almost everywhere) on Ω with $\int_{\Omega}wd\mu=1$. Then we have the inequality:*

$$\begin{aligned}0&\leq\int_{\Omega}\left(\Phi\circ f\right)wd\mu-\Phi\left(\int_{\Omega}fwd\mu\right)\\&\leq\int_{\Omega}\left(\Phi'\circ f\right)fwd\mu-\int_{\Omega}\left(\Phi'\circ f\right)wd\mu\int_{\Omega}wfd\mu\\&\leq\frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\int_{\Omega}w\left|f-\int_{\Omega}fwd\mu\right|d\mu.\end{aligned} \tag{2.5}$$

If $\mu\left(\Omega\right)<\infty$ and $\Phi\circ f,\ f,\ \Phi'\circ f,\ \left(\Phi'\circ f\right)f\in L\left(\Omega,\mu\right)$, then we have the inequality:

$$\begin{aligned}0&\leq\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\left(\Phi\circ f\right)d\mu-\Phi\left(\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}fd\mu\right)\\&\leq\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\left(\Phi'\circ f\right)fd\mu-\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\left(\Phi'\circ f\right)d\mu\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}fd\mu\\&\leq\frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}\left|f-\frac{1}{\mu\left(\Omega\right)}\int_{\Omega}fd\mu\right|d\mu.\end{aligned} \tag{2.6}$$

The following discrete inequality is of interest as well.

Corollary 1. *Let $\Phi:[m,M]\rightarrow\mathbb{R}$ be a differentiable convex function on (m,M). If $x_{i}\in[m,M]$ and $w_{i}\geq0$ $(i=1,\ldots,n)$ with $W_{n}:=\sum_{i=1}^{n}w_{i}=1$, then one has* *the counterpart of Jensen's weighted discrete inequality:*

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|. \tag{2.7}$$

Remark 1. *We notice that the inequality between the first and the second term in (2.7) was proved in 1994 by Dragomir & Ionescu, see* [14].

On making use of the results (2.5) and (2.4), we can state the following string of reverse inequalities:

Lemma 1. *Let* $\Phi : I \to \mathbb{R}$ *be a differentiable convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$, $\mathring{I}$ *is the interior of* I. *If* $f : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfies the bounds*

$$-\infty < m \leq f(x) \leq M < \infty \quad \text{for} \quad \mu\text{-a.e. } x \in \Omega \tag{2.8}$$

and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$, *then*

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \\ &\leq \int_\Omega (\Phi' \circ f) f w d\mu - \int_\Omega (\Phi' \circ f) w d\mu \int_\Omega f w d\mu \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega w f d\mu \right| w d\mu \\ &\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_\Omega f^2 w d\mu - \left(\int_\Omega f w d\mu\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M - m). \end{aligned} \tag{2.9}$$

Remark 2. *We notice that the inequality between the first, second and last term from (2.9) was proved in the general case of positive linear functionals in 2001 by S.S. Dragomir in* [6].

If the differentiability condition is removed, then Φ' *can be replaced in the inequality (2.9) above with a section* φ *of the subdifferential* $\partial\Phi$. *We omit the details.*

The following reverse of the Jensen's inequality holds [10], [11]:

Lemma 2. *Let* $\Phi : I \to \mathbb{R}$ *be a convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$. *If* $f : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfies the bounds (2.8) and* $f, \Phi \circ f \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$, *then*

$$0 \leq \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \tag{2.10}$$
$$\leq \frac{\left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right)}{M - m} \sup_{t \in (m,M)} \Psi_{\Phi}(t; m, M)$$
$$\leq \left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right) \frac{\Phi'_{-}(M) - \Phi'_{+}(m)}{M - m}$$
$$\leq \frac{1}{4}(M - m)\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right],$$

where $\bar{f}_{\Omega,w} := \int_{\Omega} w(x) f(x) d\mu(x) \in [m, M]$ *and* $\Psi_{\Phi}(\cdot; m, M) : (m, M) \to \mathbb{R}$ *is defined by*

$$\Psi_{\Phi}(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We also have the inequality

$$0 \leq \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \leq \frac{1}{4}(M - m)\, \Psi_{\Phi}\left(\bar{f}_{\Omega,w}; m, M\right) \tag{2.11}$$
$$\leq \frac{1}{4}(M - m)\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right],$$

provided that $\bar{f}_{\Omega,w} \in (m, M)$.

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ –a.e. $x \in \Omega$, is a μ-measurable function with $\int_{\Omega} w d\mu = 1$.

We also have:

Lemma 3. *With the assumptions of Lemma 2, we have the inequalities*

$$0 \leq \int_{\Omega} w (\Phi \circ f)\, d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \tag{2.12}$$
$$\leq 2 \max\left\{\frac{M - \bar{f}_{\Omega,w}}{M - m}, \frac{\bar{f}_{\Omega,w} - m}{M - m}\right\}\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right)\right]$$
$$\leq \frac{1}{2} \max\left\{M - \bar{f}_{\Omega,w}, \bar{f}_{\Omega,w} - m\right\}\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right].$$

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points $\alpha, \beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

The following result holds [12]:

Lemma 4. *Let* $\Phi : I \to \mathbb{R}$ *be a convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$. *If* $f : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfying the bounds (2.8) and* f, $\Phi \circ f \in L_w(\Omega, \mu)$, *then by denoting*

$$\overline{f}_{\Omega,w} := \int_{\Omega} w f d\mu \in [m, M]$$

and assuming that $\overline{f}_{\Omega,w} \neq m, M,$ *we have*

$$\left|\int_\Omega \left|\Phi(f) - \Phi\left(\overline{f}_{\Omega,w}\right)\right| sgn\left[f - \overline{f}_{\Omega,w}\right] w d\mu\right| \tag{2.13}$$
$$\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\overline{f}_{\Omega,w}\right)$$
$$\leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_w(f)$$
$$\leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_{w,2}(f)$$
$$\leq \frac{1}{4}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right)(M - m),$$

where

$$D_w(f) := \int_\Omega w\left|f - \overline{f}_{\Omega,w}\right| d\mu$$

and

$$D_{w,2}(f) := \left[\int_\Omega w f^2 d\mu - \left(\overline{f}_{\Omega,w}\right)^2\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ *in the second inequality from (2.10) is best possible.*

For recent results related to Jensen's inequality, see [1]-[9], [15]-[26] and the references therein.

Motivated by the above results, in this paper we establish some Jensen type inequalities for the class of *GA-convex (concave)* functions. Some applications for power and logarithmic functions are provided as well. Some inequalities for functions of selfadjoint operators in Hilbert spaces are also established.

We have the following result concerning Jensen's type inequalities for *GA*-convex functions.

Theorem 2. *Let* $h : I \subset (0, \infty) \to \mathbb{R}$ *be a GA-convex function and* $[k, K] \subset \mathring{I}$ *Assume also that* $x : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfying the bounds*

$$0 < k \leq x(t) \leq K < \infty \quad \text{for } \mu\text{-a.e. } t \in \Omega \tag{2.14}$$

and $w \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$.

(i) If $\varphi \in \partial h$*, the subdifferential of* h *and* $h \circ x$, $\ln x$, $(\ln x)^2$, $(\varphi \circ x)\, x \ln x$ *and* $(\varphi \circ x) x \in L_w(\Omega, \mu)$, *then*

$$0 \leq \int_\Omega (h \circ x) w d\mu - h \circ \exp\left(\int_\Omega w \ln x d\mu\right) \tag{2.15}$$
$$\leq \int_\Omega (\varphi \circ x) w x \ln x d\mu - \int_\Omega (\varphi \circ x) x w d\mu \int_\Omega w \ln x d\mu$$
$$\leq \frac{1}{2}\left[h'_-(K) K - h'_+(k) k\right] \int_\Omega \left|\ln x - \int_\Omega w \ln x d\mu\right| w d\mu$$
$$\leq \frac{1}{2}\left[h'_-(K) K - h'_+(k) k\right] \left[\int_\Omega w (\ln x)^2 d\mu - \left(\int_\Omega w \ln x d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[h'_-(K) K - h'_+(k) k\right](\ln K - \ln k).$$

(ii) Consider $\Psi_h(\cdot;k,K):(k,K)\to\mathbb{R}$ *defined by*

$$\Psi_h(t;k,K)=\frac{h(K)-h(t)}{\ln K-\ln t}-\frac{h(t)-h(k)}{\ln t-\ln k}.$$

If $h\circ x,\ \ln x\in L_w(\Omega,\mu)$, *then*

$$\begin{aligned}0&\le\int_\Omega(h\circ x)\,wd\mu-h\circ\exp\left(\int_\Omega w\ln xd\mu\right)\\&\le\frac{\left(\ln K-\int_\Omega w\ln xd\mu\right)\left(\int_\Omega w\ln xd\mu-\ln k\right)}{\ln K-\ln k}\sup_{t\in(k,K)}\Psi_h(t;k,K)\\&\le\left(\ln K-\int_\Omega w\ln xd\mu\right)\left(\int_\Omega w\ln xd\mu-\ln k\right)\frac{h'_-(K)K-h'_+(k)k}{\ln K-\ln k}\\&\le\frac{1}{4}(\ln K-\ln k)\left[h'_-(K)K-h'_+(k)k\right],\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}0&\le\int_\Omega(h\circ x)\,wd\mu-h\circ\exp\left(\int_\Omega w\ln xd\mu\right)\\&\le\frac{1}{4}(\ln K-\ln k)\,\Psi_h\left(\int_\Omega w\ln xd\mu;k,K\right)\\&\le\frac{1}{4}(\ln K-\ln k)\left[h'_-(K)K-h'_+(k)k\right].\end{aligned}\tag{2.17}$$

(iii) If $h\circ x,\ \ln x\in L_w(\Omega,\mu)$, *then*

$$\begin{aligned}0&\le\int_\Omega(h\circ x)\,wd\mu-h\circ\exp\left(\int_\Omega w\ln xd\mu\right)\\&\le2\max\left\{\frac{\ln K-\int_\Omega w\ln xd\mu}{\ln K-\ln k},\frac{\int_\Omega w\ln xd\mu-\ln k}{\ln K-\ln k}\right\}\\&\times\left[\frac{h(k)+h(K)}{2}-h\left(\sqrt{kK}\right)\right]\\&\le\frac{1}{2}\max\left\{\frac{\ln K-\int_\Omega w\ln xd\mu}{\ln K-\ln k},\frac{\int_\Omega w\ln xd\mu-\ln k}{\ln K-\ln k}\right\}\\&\times\left[h'_-(K)K-h'_+(k)k\right].\end{aligned}\tag{2.18}$$

In particular, we have

$$\begin{aligned}0&\le\int_\Omega(h\circ x)\,wd\mu-h\circ\exp\left(\int_\Omega w\ln xd\mu\right)\\&\le2\left[\frac{h(k)+h(K)}{2}-h\left(\sqrt{kK}\right)\right].\end{aligned}\tag{2.19}$$

(iv) If $h\circ x,\ \ln x\in L_w(\Omega,\mu)$, *then*

$$\begin{aligned}&\left|\int_\Omega\left|h\circ x-h\left(\int_\Omega w\ln xd\mu\right)\right|sgn\left[\ln x-\int_\Omega w\ln xd\mu\right]wd\mu\right|\\&\le\int_\Omega(h\circ x)\,wd\mu-h\circ\exp\left(\int_\Omega w\ln xd\mu\right)\end{aligned}\tag{2.20}$$

$$\leq \frac{1}{2}\left(\frac{h(K)-h\circ\exp\left(\int_\Omega w\ln x d\mu\right)}{\ln K-\int_\Omega w\ln x d\mu}-\frac{h\circ\exp\left(\int_\Omega w\ln x d\mu\right)-h(k)}{\int_\Omega w\ln x d\mu-\ln k}\right)$$
$$\times\int_\Omega\left|\ln x-\int_\Omega w\ln x d\mu\right| w d\mu$$
$$\leq \frac{1}{2}\left(\frac{h(K)-h\circ\exp\left(\int_\Omega w\ln x d\mu\right)}{\ln K-\int_\Omega w\ln x d\mu}-\frac{h\circ\exp\left(\int_\Omega w\ln x d\mu\right)-h(k)}{\int_\Omega w\ln x d\mu-\ln k}\right)$$
$$\times\left[\int_\Omega w(\ln x)^2 d\mu-\left(\int_\Omega w\ln x d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\frac{h(K)-h\circ\exp\left(\int_\Omega w\ln x d\mu\right)}{\ln K-\int_\Omega w\ln x d\mu}-\frac{h\circ\exp\left(\int_\Omega w\ln x d\mu\right)-h(k)}{\int_\Omega w\ln x d\mu-\ln k}\right)$$
$$\times(\ln K-\ln k).$$

Proof. (i). Since h is a GA-convex function on I, then the function $\Phi : I \to \mathbb{R}$, $\Phi(s) = h\circ\exp$ is convex on $[\ln k, \ln K]$. If we take the function $f : \Omega \to \mathbb{R}$, $f(t) = \ln x(t)$, $t \in \Omega$, then f is μ-measurable and satisfying the bounds

$$-\infty < \ln k \leq f(t) \leq \ln K < \infty \text{ for } \mu-\text{a.e. } t\in\Omega.$$

We also have

$$\partial\Phi(s) = \{\varphi(e^s)e^s \text{ with } \varphi\in\partial(h)\} \text{ for } s\in\mathring{I}.$$

Now, using the inequality (2.9) from Lemma 1 we have

$$0 \leq \int_\Omega w(h\circ\exp)(\ln x)d\mu-(h\circ\exp)\left(\int_\Omega w\ln x d\mu\right)$$
$$\leq \int_\Omega w\varphi\left(e^{\ln x}\right)e^{\ln x}\ln x d\mu-\int_\Omega\varphi\left(e^{\ln x}\right)e^{\ln x}w d\mu\int_\Omega w\ln x d\mu$$
$$\leq \frac{1}{2}\left[h'_-\left(e^{\ln K}\right)e^{\ln K}-h'_+\left(e^{\ln k}\right)e^{\ln k}\right]\int_\Omega\left|\ln x-\int_\Omega w\ln x d\mu\right| w d\mu$$
$$\leq \frac{1}{2}\left[h'_-\left(e^{\ln K}\right)e^{\ln K}-h'_+\left(e^{\ln k}\right)e^{\ln k}\right]$$
$$\times\left[\int_\Omega w(\ln x)^2 d\mu-\left(\int_\Omega w\ln x d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[h'_-\left(e^{\ln K}\right)e^{\ln K}-h'_+\left(e^{\ln k}\right)e^{\ln k}\right](\ln K-\ln k)$$

and the inequality (2.15) is obtained.

(ii). Using Lemma 2 we have

$$\begin{aligned}
0 &\le \int_\Omega w\left(h\circ\exp\right)\left(\ln x\right)d\mu-\left(h\circ\exp\right)\left(\int_\Omega w\ln x d\mu\right)\\
&\le \frac{\left(\ln K-\int_\Omega w\ln xd\mu\right)\left(\int_\Omega w\ln xd\mu-\ln k\right)}{\ln K-\ln k}\sup_{t\in(k,K)}\Psi_h\left(t;k,K\right)\\
&\le \left(\ln K-\int_\Omega w\ln xd\mu\right)\left(\int_\Omega w\ln xd\mu-\ln k\right)\frac{h'_-\left(K\right)K-h'_+\left(k\right)k}{\ln K-\ln k}\\
&\le \frac{1}{4}\left(\ln K-\ln k\right)\left[h'_-\left(K\right)K-h'_+\left(k\right)k\right],
\end{aligned}$$

which proves the inequality (2.16).

The inequality (2.17) follows by (2.11).

(iii). Follows by Lemma 3.

(iv). Follows by Lemma 4. □

The following result also holds:

Theorem 3. *Let $h:I\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function and $[k,K]\subset\mathring{I}$. Assume also that $x:\Omega\to\mathbb{R}$ is μ-measurable, satisfying the condition (2.14) and $w\ge 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$. Then*

$$\int_\Omega\left(h\circ x\right)wd\mu\le\frac{\ln K-\int_\Omega w\ln xd\mu}{\ln K-\ln k}h\left(k\right)+\frac{\int_\Omega w\ln xd\mu-\ln k}{\ln K-\ln k}h\left(K\right).\tag{2.21}$$

Proof. Observe that for $s\in[k,K]$ we have

$$\ln s=\frac{\left(\ln K-\ln s\right)\ln k+\left(\ln s-\ln k\right)\ln K}{\ln K-\ln k}.$$

By the convexity of $h\circ\exp$ on $[\ln,\ln K]$ we have

$$\begin{aligned}
h\left(s\right)&=\left(h\circ\exp\right)\left(\ln s\right)=\left(h\circ\exp\right)\left(\frac{\left(\ln K-\ln s\right)\ln k+\left(\ln s-\ln k\right)\ln K}{\ln K-\ln k}\right)\\
&\le\frac{\ln K-\ln s}{\ln K-\ln k}\left(h\circ\exp\right)\left(\ln k\right)+\frac{\ln s-\ln k}{\ln K-\ln k}\left(h\circ\exp\right)\left(\ln K\right)\\
&=\frac{\ln K-\ln s}{\ln K-\ln k}h\left(k\right)+\frac{\ln s-\ln k}{\ln K-\ln k}h\left(K\right)
\end{aligned}\tag{2.22}$$

for any $s\in[k,K]$.

Using (2.22) we have

$$h\left(x\left(t\right)\right)\le\frac{\ln K-\ln x\left(t\right)}{\ln K-\ln k}h\left(k\right)+\frac{\ln x\left(t\right)-\ln k}{\ln K-\ln k}h\left(K\right)\tag{2.23}$$

for any $t\in\Omega$.

If we multiply (2.23) by $w\left(t\right)\ge0$ for almost every $t\in\Omega$ and then integrate on Ω to get

$$\begin{aligned}
&\int_\Omega\left(h\circ x\right)wd\mu\\
&\le\frac{\int_\Omega wd\mu\ln K-\int_\Omega w\ln xd\mu}{\ln K-\ln k}h\left(k\right)+\frac{\int_\Omega w\ln xd\mu-\int_\Omega wd\mu\ln k}{\ln K-\ln k}h\left(K\right)
\end{aligned}$$

and since $\int_\Omega wd\mu=1$, the inequality (2.21) is proved. □

3. Some Weighted Hermite-Hadamard Type Inequalities

Let $h : I \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function and $[a,b] \subset \mathring{I}$. Assume also that $w(t) \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt > 0$. Using the results from the previous section, we can state the following weighted inequalities for functions of a single real variable .

If $\varphi \in \partial h$ then by (2.15) we have

$$
\begin{aligned}
(3.1)\qquad 0 &\leq \frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(s)\,ds} - h\circ\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(s)\,ds}\right)\\
&\leq \frac{\int_a^b \varphi(t)\,w(t)\,t\ln t dt}{\int_a^b w(s)\,ds} - \frac{\int_a^b \varphi(t)\,tw(t)\,dt}{\int_a^b w(s)\,ds}\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(s)\,ds}\\
&\leq \frac{1}{2}\frac{h'_-(b)\,b - h'_+(a)\,a}{\int_a^b w(s)\,ds}\int_a^b\left|\ln t - \frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)\,ds}\right| w(t)\,dt\\
&\leq \frac{1}{2}\left[h'_-(b)\,b - h'_+(a)\,a\right]\left[\frac{\int_a^b w(t)(\ln t)^2\,dt}{\int_a^b w(s)\,ds} - \left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(s)\,ds}\right)^2\right]^{\frac{1}{2}}\\
&\leq \frac{1}{4}\left[h'_-(b)\,b - h'_+(a)\,a\right](\ln b - \ln a)\,.
\end{aligned}
$$

If h is differentiable on (a,b), then φ from the inequality (3.1) can be replaced by h'.

If $h : I \subset (0,\infty) \to \mathbb{R}$ is a differentiable GA-convex function and $[a,b] \subset \mathring{I}$, then by taking $w(s) = 1$, $s \in [a,b]$ in (3.1) we get

$$
\begin{aligned}
(3.2)\qquad 0 &\leq \frac{1}{b-a}\int_a^b h(t)\,dt - h(I(a,b))\\
&\leq \frac{1}{b-a}\int_a^b h'(t)\,t\ln t dt - \frac{1}{b-a}\ln I(a,b)\int_a^b h'(t)\,t dt\\
&\leq \frac{1}{2}\frac{h'_-(b)\,b - h'_+(a)\,a}{b-a}\int_a^b |\ln t - \ln I(a,b)|\,dt\\
&\leq \frac{1}{2}\left[h'_-(b)\,b - h'_+(a)\,a\right]\left[\frac{1}{b-a}\int_a^b (\ln t)^2\,dt - (\ln I(a,b))^2\right]^{\frac{1}{2}}\\
&\leq \frac{1}{4}\left[h'_-(b)\,b - h'_+(a)\,a\right](\ln b - \ln a)\,.
\end{aligned}
$$

If we take in (3.1) $w(s) = \frac{1}{s}$, $s \in [a,b]\,$, then we get

$$
\begin{aligned}
(3.3)\qquad 0 &\leq \frac{1}{\ln b - \ln a}\int_a^b \frac{h(t)}{t}dt - h\left(\sqrt{ab}\right)\\
&\leq \frac{1}{\ln b - \ln a}\int_a^b h'(t)\ln t dt - \frac{h(b) - h(a)}{\ln b - \ln a}\ln\left(\sqrt{ab}\right)
\end{aligned}
$$

$$\le \frac{1}{2}\frac{h'_{-}(b)\,b-h'_{+}(a)\,a}{\ln b-\ln a}\int_a^b\left|\ln t-\ln\left(\sqrt{ab}\right)\right|\frac{1}{t}dt$$
$$\le \frac{1}{4\sqrt{3}}\left[h'_{-}(b)\,b-h'_{+}(a)\,a\right](\ln b-\ln a)\,.$$

Some of the integrals involved in (3.2) and (3.3) maybe calculated even further, however we do not present the details here.

Consider $\Psi_h(\cdot;a,b):(a,b)\to\mathbb{R}$ defined by

$$\Psi_h(t;a,b)=\frac{h(b)-h(t)}{\ln b-\ln t}-\frac{h(t)-h(a)}{\ln t-\ln a}.$$

Then by (2.16) we get

$$0\le\frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}-h\circ\exp\left(\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)\,ds}\right) \tag{3.4}$$
$$\le\frac{\left(\ln b-\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)ds}\right)\left(\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)ds}-\ln a\right)}{\ln b-\ln a}\sup_{t\in(a,b)}\Psi_h(t;a,b)$$
$$\le\left(\ln b-\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)\,ds}\right)\left(\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)\,ds}-\ln a\right)$$
$$\times\frac{h'_{-}(b)\,b-h'_{+}(a)\,a}{\ln b-\ln a}$$
$$\le\frac{1}{4}(\ln b-\ln a)\left[h'_{-}(b)\,b-h'_{+}(a)\,a\right],$$

and

$$0\le\frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}-h\circ\exp\left(\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)\,ds}\right) \tag{3.5}$$
$$\le\frac{1}{4}(\ln b-\ln a)\,\Psi_h\left(\frac{\int_a^b w(t)\ln tdt}{\int_a^b w(s)\,ds};a,b\right)$$
$$\le\frac{1}{4}(\ln b-\ln a)\left[h'_{-}(b)\,b-h'_{+}(a)\,a\right].$$

If we take $w(s)=1,\,s\in[a,b]$ in (3.4) and (3.5), then we get

$$0\le\frac{1}{b-a}\int_a^b h(t)\,dt-h(I(a,b)) \tag{3.6}$$
$$\le\frac{(\ln b-\ln I(a,b))(\ln I(a,b)-\ln a)}{\ln b-\ln a}\sup_{t\in(a,b)}\Psi_h(t;a,b)$$
$$\le(\ln b-\ln I(a,b))(\ln I(a,b)-\ln a)\frac{h'_{-}(b)\,b-h'_{+}(a)\,a}{\ln b-\ln a}$$
$$\le\frac{1}{4}(\ln b-\ln a)\left[h'_{-}(b)\,b-h'_{+}(a)\,a\right],$$

and

$$0\le\frac{1}{b-a}\int_a^b h(t)\,dt-h(I(a,b))\le\frac{1}{4}(\ln b-\ln a)\Psi_h(\ln I(a,b);a,b)\le\frac{1}{4}(\ln b-\ln a)\left[h'_-(b)b-h'_+(a)a\right]. \tag{3.7}$$

If we take $w(s)=\frac{1}{s}$, $s\in[a,b]$ in (3.4) and (3.5), then we get

$$0\le\frac{1}{\ln b-\ln a}\int_a^b\frac{h(t)}{t}dt-h\left(\sqrt{ab}\right)\le\frac{1}{4}(\ln b-\ln a)\sup_{t\in(a,b)}\Psi_h(t;a,b)\le\frac{1}{4}(\ln b-\ln a)\left[h'_-(b)b-h'_+(a)a\right] \tag{3.8}$$

and

$$0\le\frac{1}{\ln b-\ln a}\int_a^b\frac{h(t)}{t}dt-h\left(\sqrt{ab}\right)\le\frac{1}{4}(\ln b-\ln a)\Psi_h\left(\ln\sqrt{ab};a,b\right)\le\frac{1}{4}(\ln b-\ln a)\left[h'_-(b)b-h'_+(a)a\right]. \tag{3.9}$$

We have from (2.18) that

$$\begin{aligned}0&\le\frac{\int_a^b h(t)w(t)\,dt}{\int_a^b w(s)\,ds}-h\circ\exp\left(\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)\,ds}\right)\\&\le 2\max\left\{\frac{\ln b-\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)ds}}{\ln b-\ln a},\frac{\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)ds}-\ln a}{\ln b-\ln a}\right\}\\&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\sqrt{ab}\right)\right]\\&\le\frac{1}{2}\max\left\{\frac{\ln b-\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)ds}}{\ln b-\ln a},\frac{\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)ds}-\ln a}{\ln b-\ln a}\right\}\\&\quad\times\left[h'_-(b)b-h'_+(a)a\right].\end{aligned} \tag{3.10}$$

In particular, we have

$$0\le\frac{\int_a^b h(t)w(t)\,dt}{\int_a^b w(s)\,ds}-h\circ\exp\left(\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(s)\,ds}\right)\le 2\left[\frac{h(a)+h(b)}{2}-h\left(\sqrt{ab}\right)\right]. \tag{3.11}$$

If we take in (3.10) and (3.11) $w(s)=1, s\in[a,b]$, then we have

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,dt-h(I(a,b)) \tag{3.12}$$
$$\leq 2\max\left\{\frac{\ln b-\ln I(a,b)}{\ln b-\ln a},\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}\right\}$$
$$\times\left[\frac{h(a)+h(b)}{2}-h\left(\sqrt{ab}\right)\right]$$
$$\leq\frac{1}{2}\max\left\{\frac{\ln b-\ln I(a,b)}{\ln b-\ln a},\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}\right\}$$
$$\times\left[h'_-(b)\,b-h'_+(a)\,a\right].$$

and

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,dt-h(\ln I(a,b))\leq 2\left[\frac{h(a)+h(b)}{2}-h\left(\sqrt{ab}\right)\right]. \tag{3.13}$$

We also have from (2.20)

$$\left|\frac{\int_a^b\left|h(t)-h\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)\right| sgn\left[\ln t-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right]w(t)\,dt}{\int_a^b w(s)\,ds}\right| \tag{3.14}$$
$$\leq\frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}-h\circ\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq\frac{1}{2}\left(\frac{h(b)-h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)}{\ln b-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}}-\frac{h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)-h(a)}{\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}-\ln a}\right)$$
$$\times\frac{\int_a^b\left|\ln t-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right|w(t)\,dt}{\int_a^b w(s)\,ds}$$

$$\leq\frac{1}{2}\left(\frac{h(b)-h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)}{\ln b-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}}-\frac{h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)-h(a)}{\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}-\ln a}\right)$$
$$\times\left[\frac{\int_a^b w(t)(\ln t)^2\,dt}{\int_a^b w(s)\,ds}-\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(s)\,ds}\right)^2\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{4}\left(\frac{h(b)-h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)}{\ln b-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}}-\frac{h\circ\exp\left(\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}\right)-h(a)}{\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}-\ln a}\right)$$
$$\times(\ln b-\ln a).$$

If we take in (3.14) $w(s)=1$, then we get

$$\left|\frac{\int_a^b |h(t)-h(\ln I(a,b))|\, sgn[\ln t-\ln I(a,b)]\,dt}{b-a}\right| \tag{3.15}$$
$$\le \frac{1}{b-a}\int_a^b h(t)\,dt-h(I(a,b))$$
$$\le \frac{1}{2}\left(\frac{h(b)-h(I(a,b))}{\ln b-\ln I(a,b)}-\frac{h(I(a,b))-h(a)}{\ln I(a,b)-\ln a}\right)$$
$$\times \frac{\int_a^b |\ln t-\ln I(a,b)|\,dt}{b-a}$$
$$\le \frac{1}{2}\left(\frac{h(b)-h(I(a,b))}{\ln b-\ln I(a,b)}-\frac{h(I(a,b))-h(a)}{\ln I(a,b)-\ln a}\right)$$
$$\times \left[\frac{1}{b-a}\int_a^b (\ln t)^2\,dt-(\ln I(a,b))^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left(\frac{h(b)-h(I(a,b))}{\ln b-\ln I(a,b)}-\frac{h(I(a,b))-h(a)}{\ln I(a,b)-\ln a}\right)(\ln b-\ln a).$$

If we take in (3.14) $w(s)=\frac{1}{s}$, then we get

$$\left|\frac{\int_a^b \left|h(t)-h\left(\ln\sqrt{ab}\right)\right| sgn\left(\ln t-\ln\sqrt{ab}\right)\frac{1}{t}dt}{\ln b-\ln a}\right| \tag{3.16}$$
$$\le \frac{1}{\ln b-\ln a}\int_a^b \frac{h(t)}{t}dt-h\left(\sqrt{ab}\right)$$
$$\le \frac{h(b)-h(a)}{\ln b-\ln a}\frac{1}{\ln b-\ln a}\int_a^b \left|\ln t-\ln\sqrt{ab}\right|\frac{1}{t}dt$$
$$\le \frac{1}{2\sqrt{3}}[h(b)-h(a)](\ln b-\ln a).$$

We also have from (2.21) that

$$\frac{\int_a^b h(t)w(t)\,dt}{\int_a^b w(s)\,ds}\le \frac{\ln b-\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}}{\ln b-\ln a}h(a)+\frac{\frac{\int_a^b w(s)\ln s ds}{\int_a^b w(s)ds}-\ln a}{\ln b-\ln a}h(b). \tag{3.17}$$

If we take in (3.17) $w(s)=1, s\in[a,b]$ then we get

$$\frac{1}{b-a}\int_a^b h(t)\,dt\le \frac{\ln b-\ln I(a,b)}{\ln b-\ln a}h(a)+\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}h(b). \tag{3.18}$$

Now, we observe that

$$\frac{\ln b-\ln I(a,b)}{\ln b-\ln a}=\frac{\ln b-\frac{b\ln b-a\ln a}{b-a}+1}{\ln b-\ln a}$$
$$=\frac{(b-a)\ln b-b\ln b+a\ln a+b-a}{(b-a)(\ln b-\ln a)}$$
$$=\frac{b-a-a(\ln b-\ln a)}{(b-a)(\ln b-\ln a)}=\frac{L(a,b)-a}{b-a}$$

and, similarly

$$\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}=\frac{b-L(a,b)}{b-a}.$$

Therefore (3.18) becomes

$$\frac{1}{b-a}\int_a^b h(t)\,dt \le \frac{b-L(a,b)}{b-a}h(b)+\frac{L(a,b)-a}{b-a}h(a) \tag{3.19}$$

that has been obtained in a different way in ([27]).

If we take in (3.17) $w(s)=\frac{1}{s}$, $s\in[a,b]$ then we get (see also [21])

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{h(t)}{t}dt \le \frac{h(a)+h(b)}{2}. \tag{3.20}$$

We observe that, for $r\neq 0$, the function $g_r:\mathbb{R}\to\mathbb{R}$ defined by $g_r(x)=\exp(rx)=(\exp x)^r$ is convex on $\mathbb{R}$. Then the function $h_r(t)=t^r$, $t>0$ is GA-convex on $(0,\infty)$.

We observe that one can apply the above Hermite-Hadamard type inequalities (3.2), (3.3), (3.6), (3.7), (3.9) etc. to obtain various inequalities for special means as in [13].

If we use, for instance, the inequality (3.13) for the GA-convex function $h_r(t)=t^r$, $t>0$, $r\neq 0$ then we get

$$0\le L_r^r(a,b)-(\ln I(a,b))^r \le 2\left[A(a^r,b^r)-G^r(a,b)\right], \tag{3.21}$$

where

$$L_r^r(a,b):=\frac{1}{r+1}\frac{b^{r+1}-a^{r+1}}{b-a},\ r\neq -1,\ L(a,b):=\frac{b-a}{\ln b-\ln a},\ r=-1$$

and $G(a,b):=\sqrt{ab}$.

If we use the inequality between the first and last term in (3.2) then we have for $r\neq 0$ that

$$0\le L_r^r(a,b)-(\ln I(a,b))^r \le \frac{1}{4}r^2\frac{L_{r-1}^{r-1}(a,b)}{L(a,b)}(b-a)^2. \tag{3.22}$$

4. Discrete Inequalities

Let $p=(p_1,...,p_n)$ be a probability distribution, i.e. $p_i\ge 0$, $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$. If we write the inequalities from Theorem 2 for the discrete measure we can get the following discrete inequalities.

Let $x_i\in[k,K]\subset(0,\infty)$ for $i\in\{1,...,n\}$ and $p=(p_1,...,p_n)$ be a probability distribution. If $h:[k,K]\to\mathbb{R}$ is GA-convex and differentiable, then by (2.15) we get

$$\begin{aligned} 0 &\le \sum_{i=1}^n p_i h(x_i)-h\left(\prod_{i=1}^n x_i^{p_i}\right) \\ &\le \sum_{i=1}^n p_i h'(x_i)x_i\ln x_i-\ln\left(\prod_{i=1}^n x_i^{p_i}\right)\sum_{i=1}^n p_i h'(x_i)x_i \end{aligned} \tag{4.1}$$

$$\leq \frac{1}{2}\left[h'_{-}(K)K - h'_{+}(k)k\right]\sum_{i=1}^{n}\left|\ln x_i - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right| p_i$$

$$\leq \frac{1}{2}\left[h'_{-}(K)K - h'_{+}(k)k\right]\left[\sum_{i=1}^{n} p_i(\ln x_i)^2 - \left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{4}\left[h'_{-}(K)K - h'_{+}(k)k\right](\ln K - \ln k).$$

Consider $\Psi_h(\cdot;k,K):(k,K)\to\mathbb{R}$ defined by

$$\Psi_h(t;k,K) = \frac{h(K)-h(t)}{\ln K - \ln t} - \frac{h(t)-h(k)}{\ln t - \ln k}.$$

If we use the inequalities (2.16) and (2.17), then we have

$$0 \leq \sum_{i=1}^{n} p_i h(x_i) - h\left(\prod_{i=1}^{n} x_i^{p_i}\right) \tag{4.2}$$

$$\leq \frac{\left(\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k\right)}{\ln K - \ln k}\sup_{t\in(k,K)}\Psi_h(t;k,K)$$

$$\leq \left(\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k\right)\frac{h'_{-}(K)K - h'_{+}(k)k}{\ln K - \ln k}$$

$$\leq \frac{1}{4}(\ln K - \ln k)\left[h'_{-}(K)K - h'_{+}(k)k\right],$$

and

$$0 \leq \sum_{i=1}^{n} p_i h(x_i) - h\left(\prod_{i=1}^{n} x_i^{p_i}\right) \leq \frac{1}{4}(\ln K - \ln k)\Psi_h\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right);k,K\right) \tag{4.3}$$

$$\leq \frac{1}{4}(\ln K - \ln k)\left[h'_{-}(K)K - h'_{+}(k)k\right].$$

By the inequalities (2.18) and (2.19) we get

$$0 \leq \sum_{i=1}^{n} p_i h(x_i) - h\left(\prod_{i=1}^{n} x_i^{p_i}\right) \tag{4.4}$$

$$\leq 2\max\left\{\frac{\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)}{\ln K - \ln k}, \frac{\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k}{\ln K - \ln k}\right\}$$

$$\times\left[\frac{h(k)+h(K)}{2} - h\left(\sqrt{kK}\right)\right]$$

$$\leq \frac{1}{2}\max\left\{\frac{\ln K-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)}{\ln K-\ln k},\frac{\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)-\ln k}{\ln K-\ln k}\right\}$$
$$\times\left[h'_-(K)K-h'_+(k)k\right].$$

and

$$0\leq\sum_{i=1}^{n}p_ih(x_i)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)\leq 2\left[\frac{h(k)+h(K)}{2}-h\left(\sqrt{kK}\right)\right]. \tag{4.5}$$

Also, by the use of the inequality (2.20) we have

$$\left|\sum_{i=1}^{n}p_i\left|h(x_i)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)\right| sgn\left[\ln x_i-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)\right]\right| \tag{4.6}$$
$$\leq\sum_{i=1}^{n}p_ih(x_i)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)$$
$$\leq\frac{1}{2}\left(\frac{h(K)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)}{\ln K-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)}-\frac{h\left(\prod_{i=1}^{n}x_i^{p_i}\right)-h(k)}{\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)-\ln k}\right)$$
$$\times\sum_{i=1}^{n}p_i\left|\ln x_i-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)\right|$$

$$\leq\frac{1}{2}\left(\frac{h(K)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)}{\ln K-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)}-\frac{h\left(\prod_{i=1}^{n}x_i^{p_i}\right)-h(k)}{\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)-\ln k}\right)$$
$$\times\left[\sum_{i=1}^{n}p_i(\ln x_i)^2-\left(\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)\right)^2\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{4}\left(\frac{h(K)-h\left(\prod_{i=1}^{n}x_i^{p_i}\right)}{\ln K-\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)}-\frac{h\left(\prod_{i=1}^{n}x_i^{p_i}\right)-h(k)}{\ln\left(\prod_{j=1}^{n}x_j^{p_j}\right)-\ln k}\right)$$
$$\times(\ln K-\ln k).$$

Finally, by the use of the inequality (2.21) we have

$$\sum_{i=1}^{n} p_i h(x_i) \leq \frac{\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)}{\ln K - \ln k} h(k) + \frac{\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k}{\ln K - \ln k} h(K). \tag{4.7}$$

We consider the GA-convex function $h_r(t) = t^r$, $t > 0$, $r \neq 0$. Then by (4.1) we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i x_i^r - \left(\prod_{i=1}^{n} x_i^{p_i}\right)^r \\ &\leq r\left[\sum_{i=1}^{n} p_i x_i^r \ln x_i - \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right) \sum_{i=1}^{n} p_i x_i^r\right] \\ &\leq \frac{1}{2} r (K^r - k^r) \sum_{i=1}^{n} \left|\ln x_i - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right| p_i \\ &\leq \frac{1}{2} r (K^r - k^r) \left[\sum_{i=1}^{n} p_i (\ln x_i)^2 - \left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4} r (K^r - k^r)(\ln K - \ln k), \end{aligned} \tag{4.8}$$

where $x_i \in [k, K] \subset (0, \infty)$ for $i \in \{1, ..., n\}$.

Consider $\Psi_r(\cdot; k, K) : (k, K) \to \mathbb{R}$ defined by

$$\Psi_r(t; k, K) = \frac{K^r - t^r}{\ln K - \ln t} - \frac{t^r - k^r}{\ln t - \ln k}.$$

By the inequalities (4.2) and (4.3) we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i x_i^r - \left(\prod_{i=1}^{n} x_i^{p_i}\right)^r \\ &\leq \frac{\left(\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k\right)}{\ln K - \ln k} \sup_{t \in (k,K)} \Psi_r(t; k, K) \\ &\leq r\left(\ln K - \ln\left(\prod_{j=1}^{n} x_j^{p_j}\right)\right)\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right) - \ln k\right) \frac{K^r - k^r}{\ln K - \ln k} \\ &\leq \frac{1}{4} r (\ln K - \ln k)(K^r - k^r), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i x_i^r - \left(\prod_{i=1}^{n} x_i^{p_i}\right)^r \leq \frac{1}{4} (\ln K - \ln k) \Psi_r\left(\ln\left(\prod_{j=1}^{n} x_j^{p_j}\right); k, K\right) \\ &\leq \frac{1}{4} r (\ln K - \ln k)(K^r - k^r), \end{aligned} \tag{4.10}$$

where $x_i \in [k, K] \subset (0, \infty)$ for $i \in \{1, ..., n\}$.

Finally, if we use the inequalities (4.4) and (4.5) we get

$$
\begin{aligned}
(4.11)\qquad 0 &\leq \sum_{i=1}^{n} p_i x_i^r - \left(\prod_{i=1}^{n} x_i^{p_i} \right)^r \\
&\leq \max \left\{ \frac{\ln K - \ln \left(\prod_{j=1}^{n} x_j^{p_j} \right)}{\ln K - \ln k}, \frac{\ln \left(\prod_{j=1}^{n} x_j^{p_j} \right) - \ln k}{\ln K - \ln k} \right\} \left(K^{r/2} - k^{r/2} \right)^2 \\
&\leq \frac{1}{2} r \max \left\{ \frac{\ln K - \ln \left(\prod_{j=1}^{n} x_j^{p_j} \right)}{\ln K - \ln k}, \frac{\ln \left(\prod_{j=1}^{n} x_j^{p_j} \right) - \ln k}{\ln K - \ln k} \right\} (K^r - k^r).
\end{aligned}
$$

and

$$
(4.12)\qquad 0 \leq \sum_{i=1}^{n} p_i x_i^r - \left(\prod_{i=1}^{n} x_i^{p_i} \right)^r \leq \left(K^{r/2} - k^{r/2} \right)^2
$$

where $x_i \in [k, K] \subset (0, \infty)$ for $i \in \{1, ..., n\}$.

The interested reader may obtain other similar results by employing the inequalities (4.6) and (4.7). The details are omitted.

5. Applications for Functions of Selfadjoint Operators

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $\varphi : [m, M] \to [a, b]$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [15, p. 257]):

$$
(5.1)\qquad \langle \varphi(A) x, y \rangle = \int_{m-0}^{M} \varphi(\lambda) \, d(\langle E_\lambda x, y \rangle),
$$

and

$$
(5.2)\qquad \|\varphi(A) x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 \, d \|E_\lambda x\|^2,
$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$
g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y \rangle
$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

Now, assume that $\Phi : [k, K] \subset I \to (0, \infty)$ is continuous *GA*-convex function on the interval of real numbers I, $f : [m, M] \to [k, K]$, $p : [m, M] \to (0, \infty)$ are continuous functions on $[m,M]$ and $g:[m,M]\to\mathbb{R}$ is monotonic nondecreasing on $[m,M]$.

Using the first inequality in (2.15) and the inequality (2.21) written for the Riemann-Stieltjes integral of monotonic nondecreasing integrators we can state that

$$\begin{aligned}(5.3)\qquad &\Phi\left(\exp\left(\frac{\int_m^M p(t)\ln f(t)\,dg(t)}{\int_m^M p(t)\,dg(t)}\right)\right)\\ &\leq \frac{\int_m^M p(t)\,\Phi(f(t))\,dg(t)}{\int_m^M p(t)\,dg(t)}\\ &\leq \frac{\ln K-\frac{\int_m^M p(t)\ln(f(t))dg(t)}{\int_m^M p(t)dg(t)}}{\ln K-\ln k}\Phi(k)+\frac{\frac{\int_m^M p(t)\ln(f(t))dg(t)}{\int_m^M p(t)dg(t)}-\ln k}{\ln K-\ln k}\Phi(K).\end{aligned}$$

Assume that $Sp(A)$ is included in the interval $[m,M]\subset(0,\infty)$. Now, if we apply the inequalities (5.3) for the monotonic nondecreasing function $g_x(\lambda):=\langle E_\lambda x,x\rangle$, $x\in H$, where $\{E_\lambda\}_\lambda$ is the spectral family of A, then we get

$$\begin{aligned}(5.4)\qquad &\Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)\,x,x\rangle}{\langle p(A)\,x,x\rangle}\right)\right)\\ &\leq \frac{\langle p(A)\,\Phi(f(A))\,x,x\rangle}{\langle p(A)\,x,x\rangle}\\ &\leq \frac{\ln K-\frac{\langle p(A)\ln(f(A))x,x\rangle}{\langle p(A)x,x\rangle}}{\ln K-\ln k}\Phi(k)+\frac{\frac{\langle p(A)\ln(f(A))x,x\rangle}{\langle p(A)x,x\rangle}-\ln k}{\ln K-\ln k}\Phi(K)\end{aligned}$$

for any $x\in H$, $x\neq 0$.

In particular, if p is taken to be the constant 1, then for any $x\in H$, $\|x\|=1$, we have

$$\begin{aligned}(5.5)\qquad &\Phi(\exp(\langle\ln f(A)\,x,x\rangle))\\ &\leq\langle\Phi(f(A))\,x,x\rangle\\ &\leq\frac{\ln K-\langle\ln(f(A))\,x,x\rangle}{\ln K-\ln k}\Phi(k)+\frac{\langle\ln(f(A))\,x,x\rangle-\ln k}{\ln K-\ln k}\Phi(K).\end{aligned}$$

Moreover, if in (5.5) we take $[m,M]=[k,K]$ and $f(t)=t$, then we have

$$\begin{aligned}(5.6)\qquad \Phi(\exp\langle\ln Ax,x\rangle)&\leq\langle\Phi(A)\,x,x\rangle\\ &\leq\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}\Phi(m)+\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}\Phi(M),\end{aligned}$$

for any $x\in H$, $\|x\|=1$, provided $\Phi:[m,M]\subset J\to(0,\infty)$ is continuous GA-convex function on the interval of real numbers J.

Making use of the inequalities from Theorem 2 written for Riemann-Stieltjes integral of monotonic nondecreasing integrators, we have the following inequalities.

Assume that $Sp(A)$ is included in the interval $[m,M]\subset(0,\infty)$ and $\Phi:[k,K]\subset I\to(0,\infty)$ is continuous differentiable GA-convex function on the interval of real numbers I. Then for any $x \in H$ we have the inequalities

$$
\begin{aligned}
(5.7)\qquad 0 &\leq \frac{\langle p(A)\Phi(f(A))x,x\rangle}{\langle p(A)x,x\rangle} - \Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\right) \\
&\leq \frac{\langle p(A)\Phi'(f(A))f(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle} \\
&\quad - \frac{\langle p(A)\Phi'(f(A))f(A)x,x\rangle}{\langle p(A)x,x\rangle}\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle} \\
&\leq \frac{1}{2}\left[\Phi'_-(K)K - \Phi'_+(k)k\right] \\
&\quad \times \frac{\left\langle p(A)\left|\ln f(A) - \frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}1_H\right| x,x\right\rangle}{\langle p(A)x,x\rangle} \\
&\leq \frac{1}{2}\left[\Phi'_-(K)K - \Phi'_+(k)k\right] \\
&\quad \times \left[\frac{\left\langle p(A)[\ln f(A)]^2 x,x\right\rangle}{\langle p(A)x,x\rangle} - \left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'_-(K)K - \Phi'_+(k)k\right](\ln K - \ln k).
\end{aligned}
$$

In particular, we have the simpler inequality

$$
\begin{aligned}
(5.8)\qquad 0 &\leq \langle \Phi(A)x,x\rangle - \Phi(\exp\langle \ln Ax,x\rangle) \\
&\leq \langle \Phi'(A)A\ln Ax,x\rangle - \langle \Phi'(A)Ax,x\rangle\langle \ln Ax,x\rangle \\
&\leq \frac{1}{2}\left[\Phi'_-(M)M - \Phi'_+(m)m\right]\langle |\ln A - \langle \ln Ax,x\rangle 1_H| x,x\rangle \\
&\leq \frac{1}{2}\left[\Phi'_-(M)M - \Phi'_+(m)m\right]\left[\left\langle (\ln A)^2 x,x\right\rangle - \langle \ln Ax,x\rangle^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'_-(M)M - \Phi'_+(m)m\right](\ln M - \ln m),
\end{aligned}
$$

for any $x \in H$, $\|x\| = 1$ and provided that $\Phi : [m,M] \subset J \to (0,\infty)$ is continuous differentiable GA-convex function on the interval of real numbers J.

For $\Phi : [k,K] \subset I \to (0,\infty)$ a continuous GA-convex function on the interval of real numbers I, consider $\Psi_\Phi(\cdot;k,K) : (k,K) \to \mathbb{R}$ defined by

$$\Psi_\Phi(t;k,K) = \frac{\Phi(K) - \Phi(t)}{\ln K - \ln t} - \frac{\Phi(t) - \Phi(k)}{\ln t - \ln k}.$$

If $Sp(A)$ is included in the interval $[m,M]\subset(0,\infty)$, then

$$
\begin{aligned}
(5.9)\qquad 0 &\le \frac{\langle p(A)\Phi(f(A))x,x\rangle}{\langle p(A)x,x\rangle}-\Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\right)\\
&\le \frac{\left(\ln K-\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}-\ln k\right)}{\ln K-\ln k}\sup_{t\in(k,K)}\Psi_{\Phi}(t;k,K)\\
&\le \left(\ln K-\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}-\ln k\right)\\
&\times\frac{\Phi'_{-}(K)K-\Phi'_{+}(k)k}{\ln K-\ln k}\\
&\le \frac{1}{4}(\ln K-\ln k)\left[\Phi'_{-}(K)K-\Phi'_{+}(k)k\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(5.10)\qquad 0 &\le \frac{\langle p(A)\Phi(f(A))x,x\rangle}{\langle p(A)x,x\rangle}-\Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\right)\\
&\le \frac{1}{4}(\ln K-\ln k)\Psi_{\Phi}\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle};k,K\right)\\
&\le \frac{1}{4}(\ln K-\ln k)\left[\Phi'_{-}(K)K-\Phi'_{+}(k)k\right]
\end{aligned}
$$

for any $x\in H$.

In particular, if $\Phi:[m,M]\subset J\to(0,\infty)$ is continuous GA-convex function on the interval of real numbers J, then we have

$$
\begin{aligned}
(5.11)\qquad 0 &\le \langle\Phi(A)x,x\rangle-\Phi(\exp\langle\ln Ax,x\rangle)\\
&\le \frac{(\ln M-\langle\ln Ax,x\rangle)(\langle\ln Ax,x\rangle-\ln m)}{\ln M-\ln m}\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M)\\
&\le (\ln M-\langle\ln Ax,x\rangle)(\langle\ln Ax,x\rangle-\ln m)\frac{\Phi'_{-}(M)M-\Phi'_{+}(m)m}{\ln M-\ln m}\\
&\le \frac{1}{4}(\ln M-\ln m)\left[\Phi'_{-}(M)M-\Phi'_{+}(m)m\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(5.12)\qquad 0 &\le \langle\Phi(A)x,x\rangle-\Phi(\exp\langle\ln Ax,x\rangle)\\
&\le \frac{1}{4}(\ln M-\ln m)\Psi_{\Phi}(\langle\ln Ax,x\rangle;m,M)\\
&\le \frac{1}{4}(\ln M-\ln m)\left[\Phi'_{-}(M)M-\Phi'_{+}(m)m\right]
\end{aligned}
$$

for any $x\in H$, $\|x\|=1$.

With the above assumptions for p, f and Φ, we also have

$$(5.13)\quad \begin{aligned} 0 &\leq \frac{\langle p(A)\Phi(f(A))x,x\rangle}{\langle p(A)x,x\rangle} - \Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\right) \\ &\leq 2\max\left\{\frac{\ln K - \frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}}{\ln K - \ln k}, \frac{\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle} - \ln k}{\ln K - \ln k}\right\} \\ &\times \left[\frac{\Phi(k)+\Phi(K)}{2} - \Phi\left(\sqrt{kK}\right)\right] \\ &\leq \frac{1}{2}\max\left\{\frac{\ln K - \frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}}{\ln K - \ln k}, \frac{\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle} - \ln k}{\ln K - \ln k}\right\} \\ &\times \left[\Phi'_-(K)K - \Phi'_+(k)k\right] \end{aligned}$$

for any $x \in H$.

In particular, we have

$$(5.14)\quad \begin{aligned} 0 &\leq \frac{\langle p(A)\Phi(f(A))x,x\rangle}{\langle p(A)x,x\rangle} - \Phi\left(\exp\left(\frac{\langle p(A)\ln f(A)x,x\rangle}{\langle p(A)x,x\rangle}\right)\right) \\ &\leq 2\left[\frac{\Phi(k)+\Phi(K)}{2} - \Phi\left(\sqrt{kK}\right)\right] \end{aligned}$$

for any $x \in H$.

If $\Phi : [m, M] \subset J \to (0,\infty)$ is continuous GA-convex function on the interval of real numbers J, then we have

$$(5.15)\quad \begin{aligned} 0 &\leq \langle \Phi(A)x,x\rangle - \Phi(\exp\langle \ln Ax,x\rangle) \\ &\leq 2\max\left\{\frac{\ln M - \langle \ln Ax,x\rangle}{\ln M - \ln m}, \frac{\langle \ln Ax,x\rangle - \ln m}{\ln M - \ln m}\right\} \\ &\times \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\sqrt{mM}\right)\right] \\ &\leq \frac{1}{2}\max\left\{\frac{\ln M - \langle \ln Ax,x\rangle}{\ln M - \ln m}, \frac{\langle \ln Ax,x\rangle - \ln m}{\ln M - \ln m}\right\} \\ &\times \left[\Phi'_-(M)M - \Phi'_+(m)m\right] \end{aligned}$$

and

$$(5.16)\quad 0 \leq \langle \Phi(A)x,x\rangle - \Phi(\exp\langle \ln Ax,x\rangle) \leq 2\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\sqrt{mM}\right)\right]$$

for any $x \in H$, $\|x\| = 1$.

If we consider the GA-convex function $h_r(t) = t^r$, $t > 0$, $r \neq 0$, then for any selfadjoint operator A such that $Sp(A)$ is included in the interval $[m, M] \subset (0,\infty)$, we have

$$(5.17)\quad \begin{aligned} \exp\langle \ln A^r x,x\rangle &\leq \langle A^r x,x\rangle \\ &\leq \frac{\ln M - \langle \ln Ax,x\rangle}{\ln M - \ln m}m^r + \frac{\langle \ln Ax,x\rangle - \ln m}{\ln M - \ln m}M^r, \end{aligned}$$

$$
\begin{aligned}
0 &\le \langle A^r x, x\rangle - \exp\langle \ln A^r x, x\rangle \\
&\le r\left[\langle A^r \ln Ax, x\rangle - \langle A^r x, x\rangle \langle \ln Ax, x\rangle\right] \\
&\le \frac{1}{2} r\left(M^r - m^r\right)\langle \left|\ln A - \langle \ln Ax, x\rangle 1_H\right| x, x\rangle \\
&\le \frac{1}{2} r\left(M^r - m^r\right)\left[\left\langle (\ln A)^2 x, x\right\rangle - \langle \ln Ax, x\rangle^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4} r\left(M^r - m^r\right)\left(\ln M - \ln m\right),
\end{aligned} \tag{5.18}
$$

and

$$
0 \le \langle A^r x, x\rangle - \exp\langle \ln A^r x, x\rangle \le \left(M^{r/2} - m^{r/2}\right)^2 \tag{5.19}
$$

for any $x \in H$, $\|x\| = 1$.

The interested reader may apply the above inequalities for $\Phi(x) = \ln(1 + x)$ which is GA-convex on $(0, \infty)$. We omit the details.

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