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EQUIVALENT NORM INEQUALITIES FOR NORM IDEALS IN C^*-ALGEBRA

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Abstract. In this paper, we shall investigate several inequalities in norm ideals in C^*-algebra, in particular to the noncommutative L^p-spaces of semi-finite von Neumann algebra. It is well known that the Corach-Porta-Recht's inequality holds for invertible self-adjoint operators $A, B \in B(H)$ for the usual operator norm. Here we will show that this inequality holds for invertible positive operators $A, B \in B(H)$ for norm ideals in C^*-algebra and it is equivalent to the famous Heinz inequality, to the arithmetic-geometric-mean inequality and to several other interesting inequalities.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators on a separable complex Hilbert space H. Let $K(H)$ denote the ideal of compact operators on H. For any compact operator A, let $s_1(A), s_2(A), ...$ be the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ in decreasing order and repeated according to multiplicity. A compact operator A is said to be in the Schatten p-class $C_p\,(1 \le p < \infty)$, if $\sum_i s_i(A)^p < \infty$. The Schatten p-norm of A is defined by

$$||A||_p = \left(\sum_i s_i(A)^p\right)^{\frac{1}{p}}.$$

This norm makes C_p into a Banach space. Hence C_1 is the trace class and C_2 is the Hilbert-Schmidt class. It is reasonable to let C_∞ denote the ideal of compact operators $K(H)$, and $||.||_\infty$ stand for the usual operator norm. If $A \in C_p\,(1 \le p < \infty)$ and $\{e_i\}$ is any orthonormal set in H, then

$$||A||_p^p \ge \sum_i |(Ae_i, e_i)|^p.$$

We refer to [5] for further properties of the Schatten p-classes. Each ideal of $B(H)$ is contained in the ideal of compact operators. Each symmetric gauge function Φ on sequences gives rise to a symmetric norm or a unitarily invariant norm on operators defined by $||A||_\Phi =$

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$\Phi(\{s_j(A)\})$. we will denote by the symbol $|||.|||$ any such norm. Each such norm satisfies the invariance property

$$|||UAV||| = |||A|||$$

for all A and unitary U, V. With each such norm is associated a norm ideal of $B(H)$ on which it is bounded, and this ideal is closed in the topology generated by this norm. Two special families of unitarily invariant norms are the Schatten p-norms and the Ky Fan norms defined as

$$||A||_k = \sum_{j=1}^{k} s_j(A),\ k = 1, 2, ...$$

Let $\mathcal{M}$ be a C^*-algebra and let $\mathcal{I}_0$ be a normed ideal in $\mathcal{M}$ with norm $||.||_{\mathcal{I}}$ such that

$$||XYZ||_{\mathcal{I}} \leq ||X||||Y||_{\mathcal{I}}||Z||$$

for $X, Z \in \mathcal{M}$ and $Y \in \mathcal{I}_0$. Let $\mathcal{I}$ stand for the completion of the linear space $\mathcal{I}_0$ relative to the norm $||.||_{\mathcal{I}}$. We say that $\mathcal{I}$ is a normed ideal in $\mathcal{M}$ with an unitarily invariant norm. The obvious examples being the noncommutative L^p spaces $L^p(M, \tau)$ of semi finite von Neumann algebra M with trace τ, which are the completion of the ideals

$$\mathcal{I}^{\tau,p} = \{x \in M :\ \tau(|x|^p) < \infty\}$$

relative to the unitary invariant norm given by $||x||_p = \tau(|x|^p)^{\frac{1}{p}}$, where as usual $p \in [1, \infty]$ and $L^\infty(M, T) = M$ (see [14]).

Let $A \in B(H)$ be an invertible self-adjoint operator. In one of their papers Corach-Porta-Recht [3] proved that

$$||A^{-1}XA + AXA^{-1}|| \geq 2||X|| \tag{1}$$

for every $X \in B(H)$ which is a key factor in their study of differential geometry of self adjoint operators. They proved this inequality by using the integral representation of a self-adjoint operator with respect to a spectral measure. McIntosh [12] proved that for all $A, B, X \in B(H)$,

$$||A^*AX + XBB^*|| \geq 2||AXB|| \tag{2}$$

the so called the arithmetic-geometric-mean inequality. Also in [11] G.Larotonda proved for $A, B, X \in B(H)$,

$$|||A^*AX + XB^*B||| \geq 2|||AXB^*||| \tag{3}$$

for any unitarily invariant norms. In 1951 Heinz [7] proved a series of very useful norm inequalities that are closely related to the Condes inequality [4] and the Furuta inequality [10]. Heinz inequality states that for positive operators $S_1, S_2 \in B(H)$ and $Q \in B(H)$,

$$||S_1Q + QS_2|| \geq ||S_1^{\alpha}QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^{\alpha}||, \tag{4}$$

for $0 \leq \alpha \leq 1$. Which is one of the most essential inequalities in operator theory. Recently Larotonda [11] has extended all the above inequalities to norm ideals in C^*-algebra. He proved that for a normed ideal $\mathcal{I}$ with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$ the following inequalities hold:

$$||S_1Q + QS_2||_{\mathcal{I}} \geq \left|\left|S_1^{\alpha}QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^{\alpha}\right|\right|_{\mathcal{I}},$$

where $S_1, S_2 \in \mathcal{M}$ are positive, $Q \in \mathcal{I}$ and $0 \leq \alpha \leq 1$;

$$||A^*AX + XB^*B||_{\mathcal{I}} \geq 2||AXB||_{\mathcal{I}},$$

where $A, B \in \mathcal{M}$ are invertible positive and $X \in \mathcal{I}$;

$$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right|_{\mathcal{I}} \geq 2\,||X||_{\mathcal{I}},$$

where $A, B \in B(H)$ are invertible operators and $X \in \mathcal{I}$;

$$\left|\left|AXB^{-1} + A^{-1}XB\right|\right|_{\mathcal{I}} \geq 2\,||X||_{\mathcal{I}},$$

where $A, B \in \mathcal{M}$ are invertible self-adjoint and $X \in \mathcal{I}$. This is not the place to give a thorough discussion on the subject of norm inequalities in spaces of operators, but let us just mention that it has deep connections with the theory of majorization and Schur products for matrices [1], and that recently Hiai and Kosaki [8] developed a striking technique that involves Fouries transforms to prove norm inequalities for bounded operators. In this paper, we shall investigate several inequalities in norm ideals in C^*-algebra, in particular to the noncommutative L^p-spaces of semi-finite von Neumann algebra. It is well known that the Corach-Porta-Recht's inequality holds for invertible self-adjoint operators $A, B \in B(H)$ for the usual operator norm. Here we will show that this inequality holds for invertible positive operators $A, B \in B(H)$ for norm ideals in C^*-algebra and it is equivalent to the famous Heinz inequality, to the arithmetic-geometric-mean inequality and to several other interesting inequalities.

2. Main Results

First we will show that the arithmetic-geometric-mean inequality is equivalent to the Corach-Porta-Recht's inequality.

Theorem 2.1. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. The following inequalities are equivalent:*

$$||STR^{-1} + S^{-1}TR||_{\mathcal{I}} \geq 2||T||_{\mathcal{I}}, \tag{5}$$

where $A, B \in \mathcal{M}$ are invertible positive and $X \in \mathcal{I}$.

$$||A^*AX + XB^*B||_{\mathcal{I}} \geq 2||A^*XB||_{\mathcal{I}}, \tag{6}$$

where $A, B \in \mathcal{M}$ and $X \in \mathcal{I}$.

Proof. (5)$\Rightarrow$ (6). We may assume that A^*A and B^*B are both invertible. Then we have

$$||A^*AX + XB^*B||_{\mathcal{I}} = \left|\left|\mid A \mid\mid A \mid X \mid B^* \mid\mid B^* \mid^{-1} + \mid A \mid^{-1}\mid A \mid X \mid B^* \mid B^* \mid\right|\right|_{\mathcal{I}}$$

$$\geq 2 \left|\left|\left|\left| A \mid X \mid B^* \right|\right|\right|\right| = 2 \left|\left|AXB\right|\right|_{\mathcal{I}}.$$

(6)$\Rightarrow$ (5). We have

$$\begin{aligned}\left|\left|AXB^{-1} + A^{-1}XB\right|\right|_{\mathcal{I}} &= \left|\left|AA(A^{-1}XB^{-1}) + (A^{-1}XB^{-1})BB\right|\right|_{\mathcal{I}} \\ &\geq 2 \left|\left|\left|A(A^{-1}XB^{-1})B\right|\right|\right| = 2 \left|\left|X\right|\right|_{\mathcal{I}}.\end{aligned}$$

□

Now we will extend the Corach-Porta-Recht's inequality to positive invertible operators $A, B \in B(H)$ for every unitarily invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$.

Theorem 2.2. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$ and let $A, B \in \mathcal{M}$ be invertible positive. Then*

$$||AXB^{-1} + A^{-1}XB||_{\mathcal{I}} \geq 2||X||_{\mathcal{I}}$$

for all $X \in \mathcal{I}$.

Proof. On $H \oplus H$, let

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then

$$T^{-1}YT + TYT^{-1} = \begin{pmatrix} 0 & AXB^{-1} + A^{-1}XB \\ 0 & 0 \end{pmatrix}.$$

Since $||Y||_{\mathcal{I}} = ||X||_{\mathcal{I}}$, hence (5) applied to T and Y yields

$$\left|\left|T^{-1}YT + TYT^{-1}\right|\right|_{\mathcal{I}} = \left|\left|AXB^{-1} + A^{-1}XB\right|\right|_{\mathcal{I}} \geq 2 \left|\left|X\right|\right|_{\mathcal{I}}.$$

□

Now we shall extend the Corach-Porta-Recht's inequality to invertible $A, B \in \mathcal{M}$ (not necessary positive)

Theorem 2.3. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. If*

$$\left|\left|AXB^{-1} + A^{-1}XB\right|\right|_{\mathcal{I}} \geq 2 \left|\left|X\right|\right|_{\mathcal{I}}, \tag{7}$$

where $A, B \in \mathcal{M}$ are invertible positive and $X \in \mathcal{I}$, then

$$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right|_{\mathcal{I}} \geq 2 \left|\left|X\right|\right|_{\mathcal{I}}, \tag{8}$$

where $A, B \in \mathcal{M}$ are invertible and $X \in \mathcal{I}$.

Proof. Let $A, B \in \mathcal{M}$) be invertible. Put $A = U|A|$, $B = V|B|$ be polar decompositions of A, B. Then

$$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right|_{\mathcal{I}} = \left|\left|\left| A \mid UXV^*|B|^{-1} + |A|^{-1}UXV^* \mid B \right|\right|\right|_{\mathcal{I}}.$$

Applying (8) to get

$$\left|\left|\left| A \mid UXV^*|B|^{-1} + |A|^{-1}UXV^* \mid B \right|\right|\right|_{\mathcal{I}}$$

$$\geq 2\,||UXV^*||_{\mathcal{I}} = 2\,||X||_{\mathcal{I}}\,.$$

□

We also have the following interesting equivalent inequalities for invertible $A, B \in \mathcal{M}$ (not necessary positive)

Theorem 2.4. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. The following inequalities are equivalent.*

$$\left|\left|STR^{-1} + (S^*)^{-1}TR^*\right|\right|_{\mathcal{I}} \geq 2\,||T||_{\mathcal{I}}\,, \tag{9}$$

where $A, B \in \mathcal{M}$ are invertible and $T \in \mathcal{I}$.

$$||AA^*X + XBB^*||_{\mathcal{I}} \geq 2\,||A^*XB||_{\mathcal{I}}\,, \tag{10}$$

where $A, B \in \mathcal{M}$ are invertible and $X \in \mathcal{I}$.

Proof. (9)⇒ (10). Arguing as in the proof of [11, Proposition 3.3]. Letting $A = S, R = B$, $T = A^*XB$ in (9) proves (10) for invertible $A, B \in \mathcal{M}$. Suppose that A, B are positive (not necessary invertible) and let $A_\epsilon = A + \epsilon$ and $B_\epsilon = B + \epsilon$. Since A_ϵ, B_ϵ are invertible for all $\epsilon > 0$, we have

$$||A_\epsilon A_\epsilon^* X + XB_\epsilon B_\epsilon^*||_{\mathcal{I}} \geq 2\,||A_\epsilon^* XB_\epsilon||_{\mathcal{I}}\,.$$

Letting $\epsilon \to 0$. Then the inequality holds for positive $A, B \in \mathcal{M}$. Assume now that $A = U|A|$ and $B = V|B|$ (polar decomposition). Then

$$||AA^*X + XBB^*||_{\mathcal{I}} = |||A|^2X + X|B|^2||_{\mathcal{I}} \geq |||A|X||B|||_{\mathcal{I}}$$
$$= 2||U^*|A|X|B|V||_{\mathcal{I}} = 2||A^*XB||_{\mathcal{I}}.$$

(10) ⇒ (9). We may assume that A^*A and B^*B are both invertible. By choosing $A = S, R = B$, $X = (A^*)^{-1}TB$ in (10), we obtain (9) by (10). □

Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Let $A, B \in \mathcal{M}$ be invertible positive and $T \in \mathcal{I}$. Then the following inequality can be considered as a generalization of inequality (8).

$$||A^{2m+n}TB^{-n} + A^{-n}TB^{2m+n}||_{\mathcal{I}} \geq ||A^{2m}T + TB^{2m}||_{\mathcal{I}}, \tag{11}$$

where m, n are both nonnegative integers.

Now we will show that Heinz inequality for every norm ideals in C^*-algebra implies the previous mentioned inequality. Recall that Kittanah and Davies [2] generalized Heinz inequality to every unitarily invariant norm $|||.|||$.

Theorem 2.5. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Let $S_1, S_2 \in \mathcal{M}$ be positive, $0 \leq \alpha \leq 1$. If*

$$||S_1Q + QS_2||_{\mathcal{I}} \geq \left|\left|S_1^\alpha QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^\alpha\right|\right|_{\mathcal{I}}\,. \tag{12}$$

Then

$$||A^{2m+n}TB^{-n} + A^{-n}TB^{2m+n}||_{\mathcal{I}} \geq ||A^{2m}T + TB^{2m}||_{\mathcal{I}}, \tag{13}$$

where $A, B \in \mathcal{M}$ are invertible positive, $T \in \mathcal{I}$ and m, n are both nonnegative integers.

Proof. It suffices to take $S_1 = A^{2m+2n}$, $S_2 = B^{2m+2n}$, $Q = A^{-n}TB^{-n}$ and $\alpha = (2m+n)(2m+2n)^{-1}$ in (12) to get (13). □

Using Theorem 2.5 with $m = 0$ and n nonnegative integer, we have the following interesting implication:

Corollary 2.1. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Let S_1, S_2 be positive and $0 \leq \alpha \leq 1$. If*

$$||S_1Q + QS_2||_{\mathcal{I}} \geq \left|\left|S_1^{\alpha}QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^{\alpha}\right|\right|_{\mathcal{I}}, \tag{14}$$

$S_1, S_2 \in \mathcal{M}$ are positive. Then

$$||A^nTB^{-n} + A^{-n}TB^n||_{\mathcal{I}} \geq 2|||T||_{\mathcal{I}}, \tag{15}$$

where $A, B \in \mathcal{M}$ are invertible positive, $T \in \mathcal{I}$ and n is a nonnegative integer.

Now we are ready to give a more general version of Corach-Porta-Recht's inequality for every unitarily invariant norm $||.||_{\mathcal{I}}$.

Theorem 2.6. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Let $A, B \in \mathcal{M}$ be an invertible positive and $X \in \mathcal{I}$). Then*

$$||A^nXB^{-n} + A^{-n}XB^n||_{\mathcal{I}} \geq 2||X||_{\mathcal{I}}, \tag{16}$$

where n is a nonnegative integer.

Proof. On $H \oplus H$, let

$$T = \begin{pmatrix} A^n & 0 \\ 0 & B^n \end{pmatrix}, Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then T is invertible positive and we have

$$T^{-1}YT + TYT^{-1} = \begin{pmatrix} 0 & A^nXB^{-n} + A^{-n}XB^n \\ 0 & 0 \end{pmatrix}.$$

Since $||Y||_{\mathcal{I}} = ||X||_{\mathcal{I}}$, hence (5) applied to $T = S = R$ and Y yields

$$\left|\left|T^{-1}YT + TYT^{-1}\right|\right|_{\mathcal{I}} = \left|\left|AXB^{-n} + A^{-n}XB\right|\right|_{\mathcal{I}} \geq 2\,||X||_{\mathcal{I}}.$$

□

The following theorem must be of interest.

Theorem 2.7. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Let $A, B \in \mathcal{M}$) be invertible positive. Then*
(17)
$$\max(||A^nXB^{-n} + A^{-n}XB^n||_{\mathcal{I}}, ||A^nX^*B^{-n} + A^{-n}X^*B^n||_{\mathcal{I}}) \geq 2||X||_{\mathcal{I}}$$
for every $X \in \mathcal{I}$.

Proof. On $H \oplus H$, let
$$S = \begin{pmatrix} A^n & 0 \\ 0 & A^n \end{pmatrix}, T = \begin{pmatrix} B^n & 0 \\ 0 & B^n \end{pmatrix}, Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$
Then S, T are invertible and positive. We have
$$S^{-1}YT+SYT^{-1} = \begin{pmatrix} 0 & A^nXB^{-n} + A^{-n}XB^n \\ A^nX^*B^{-n} + A^{-n}X^*B^n & 0 \end{pmatrix}.$$
Since $||Y||_{\mathcal{I}} = ||X||_{\mathcal{I}}$, hence (8) applied to S, T and Y yields

$$\begin{aligned}\left|\left|S^{-1}YT + SYT^{-1}\right|\right|_{\mathcal{I}} &= \max(\left|\left|AXB^{-n} + A^{-n}XB\right|\right|_{\mathcal{I}}, \left|\left|AX^*B^{-n} + A^{-n}X^*B\right|\right|_{\mathcal{I}}) \\ &\geq 2\left|\left|X\right|\right|_{\mathcal{I}}.\end{aligned}$$
□

Now we are ready to prove the equivalence between Heinz inequality and several other norm inequalities.

Theorem 2.8. *Let $\mathcal{I}$ be a normed ideal with an unitary invariant norm $||.||_{\mathcal{I}}$ in a C^*-algebra $\mathcal{M}$. Then following inequalities are mutually equivalent:*

(18) $$||S_1Q + QS_2||_{\mathcal{I}} \geq \left|\left|S_1^{\alpha}QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^{\alpha}\right|\right|_{\mathcal{I}},$$
where $S_1, S_2 \in \mathcal{M}$ are positive, $Q \in \mathcal{I}$ and $0 \leq \alpha \leq 1$;

(19) $$||A^*AX + XB^*B||_{\mathcal{I}} \geq 2||AXB||_{\mathcal{I}},$$
where $A, B \in \mathcal{M}$ are invertible positive and $X \in \mathcal{I}$;

(20) $$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right|_{\mathcal{I}} \geq 2\left|\left|X\right|\right|_{\mathcal{I}},$$
where $A, B \in B(H)$ are invertible operators and $X \in \mathcal{I}$;

(21) $$\left|\left|AXB^{-1} + A^{-1}XB\right|\right|_{\mathcal{I}} \geq 2\left|\left|X\right|\right|_{\mathcal{I}},$$
where $A, B \in \mathcal{M}$ are invertible positive and $X \in \mathcal{I}$;

(22) $$\left|\left|AXA^{-1} + A^{-1}XA\right|\right|_{\mathcal{I}} \geq 2\left|\left|X\right|\right|_{\mathcal{I}},$$
where $A \in \mathcal{M}$ is invertible positive and $X \in \mathcal{I}$;

(23) $$||A^nXB^{-n} + A^{-n}XB^n||_{\mathcal{I}} \geq 2||X||_{\mathcal{I}},$$
where $A, B \in \mathcal{M}$ are invertible positive, $X \in \mathcal{I}$ and n is a nonnegative integer.

(24) $$|||A^{2m+n}XB^{-n}+A^{-n}XB^{2m+n}||_{\mathcal{I}}\geq 2||A^{2m}X+XB^{2m}||_{\mathcal{I}},$$

where $A,B\in\mathcal{M}$ are invertible positive operators, $X\in\mathcal{I}$ and m,n are both nonnegative integers.

Proof. To prove the theorem it suffices to show that the following implications (18) $\Rightarrow$ (24) $\Rightarrow$ (23) $\Rightarrow$ (22) $\Rightarrow$ (21) $\Rightarrow$ (20) $\Rightarrow$ (19) $\Rightarrow$ (18) hold. We have only to prove that (19) $\Rightarrow$ (18). The other implications are already proved.

(19) $\Rightarrow$ (18). The proof can be achieve by a slight modification in the proof of this implication in the usual norm of operator given by McIntosh [12], which is quite similar to a proof presented by Pedersen [13]. Indeed, note that (1) remains hold for $\alpha=0,1$. Assume that $0\leq\mu=\alpha-p<\alpha<\alpha+p=\lambda\leq 1$. We will show that (1) holds true for μ and λ. Now applying (2) to get,

$$f(\alpha)=\left|\left|S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}\right|\right|_{\mathcal{I}}=\left|\left|S_1^{p}(S_1^{\mu}QS_2^{1-\lambda}+S_1^{1-\lambda}QS_2^{\mu})S_2^{p}\right|\right|_{\mathcal{I}}$$

$$\leq\frac{\left|\left|S_1^{2p}(S_1^{\mu}QS_2^{1-\lambda}+S_1^{1-\lambda}QS_2^{\mu})+(S_1^{\mu}QS_2^{1-\lambda}+S_1^{1-\lambda}QS_2^{\mu})S_2^{2p}\right|\right|_{\mathcal{I}}}{2}$$

$$\leq\frac{\left|\left|S_1^{\lambda}QS_2^{1-\lambda}+S_1^{1-\lambda}QS_2^{\lambda}\right|\right|_{\mathcal{I}}}{2}+\frac{\left|\left|S_1^{\mu}QS_2^{1-\mu}+S_1^{1-\mu}QS_2^{\mu}\right|\right|_{\mathcal{I}}}{2}$$

This implies that

$$f(\alpha)\leq\frac{(f(\alpha+p)+f(\alpha-p))}{2}.$$

it is clear that S_1^{α} and S_2^{α} depend continuously on $\alpha\in(0,1]$ in the topology generated by this norm. Since

$$f(\alpha)\leq\frac{(f(\alpha+p)+f(\alpha-p))}{2},$$

it follows that $f(\alpha)$ is convex on [0,1]. Since

$$f(\alpha)=\left|\left|S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}\right|\right|_{\mathcal{I}}$$

is convex on $[0,1]$, by using the inequality (2.8) from [6] we get for $0\leq\alpha,\beta\leq 1$

$$\frac{f\,(1)+f\,(0)}{2}-f\left(\frac{1}{2}\right)\geq\frac{f(\alpha)+f\,(\beta)}{2}-f\left(\frac{\alpha+\beta}{2}\right)\geq 0$$

giving that

$$\begin{aligned}
&\|S_1Q+QS_2\|_{\mathcal{I}}-2\left\|S_1^{1/2}QS_2^{1/2}\right\|_{\mathcal{I}}\\
&\geq\frac{1}{2}\left[\left\|S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}\right\|_{\mathcal{I}}+\left\|S_1^{\beta}QS_2^{1-\beta}+S_1^{1-\beta}QS_2^{\beta}\right\|_{\mathcal{I}}\right]\\
&-\left\|S_1^{\frac{\alpha+\beta}{2}}QS_2^{1-\frac{\alpha+\beta}{2}}+S_1^{1-\frac{\alpha+\beta}{2}}QS_2^{\frac{\alpha+\beta}{2}}\right\|_{\mathcal{I}}\\
&\geq 0
\end{aligned}$$

namely

$$\begin{aligned}
&\left\|\frac{S_1Q+QS_2}{2}\right\|_{\mathcal{I}}-\left\|S_1^{1/2}QS_2^{1/2}\right\|_{\mathcal{I}}\\
&\geq\frac{1}{2}\left[\left\|\frac{S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}}{2}\right\|_{\mathcal{I}}+\left\|\frac{S_1^{\beta}QS_2^{1-\beta}+S_1^{1-\beta}QS_2^{\beta}}{2}\right\|_{\mathcal{I}}\right]\\
&-\left\|\frac{S_1^{\frac{\alpha+\beta}{2}}QS_2^{1-\frac{\alpha+\beta}{2}}+S_1^{1-\frac{\alpha+\beta}{2}}QS_2^{\frac{\alpha+\beta}{2}}}{2}\right\|_{\mathcal{I}}\\
&\geq 0.
\end{aligned}$$

If in the above inequality we take $\beta=1-\alpha$, then we otain

$$\|S_1Q+QS_2\|_{\mathcal{I}}\geq\left\|S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}\right\|_{\mathcal{I}}$$

for any α. This implies the desired inequality (18).

By using (2.15) from [6] we also have for $0\leq\alpha<\beta\leq 1$

$$\frac{f(1)+f(0)}{2}-\int_0^1 f(t)\,dt\geq\frac{f(\alpha)+f(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)\,dt\geq 0,$$

which gives

$$\begin{aligned}
&\|S_1Q+QS_2\|_{\mathcal{I}}-\int_0^1\left\|S_1^{t}QS_2^{1-t}+S_1^{1-t}QS_2^{t}\right\|_{\mathcal{I}}dt\\
&\geq\left\|\frac{S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}}{2}\right\|_{\mathcal{I}}+\left\|\frac{S_1^{\beta}QS_2^{1-\beta}+S_1^{1-\beta}QS_2^{\beta}}{2}\right\|_{\mathcal{I}}\\
&-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left\|S_1^{t}QS_2^{1-t}+S_1^{1-t}QS_2^{t}\right\|_{\mathcal{I}}dt\\
&\geq 0.
\end{aligned}$$

If we take in this inequality $\beta = 1-\alpha$ and assume that $\alpha < 1/2$, then we get

$$||S_1Q + QS_2||_{\mathcal{I}} - \int_0^1 \left|\left|S_1^tQS_2^{1-t} + S_1^{1-t}QS_2^t\right|\right|_{\mathcal{I}} dt$$
$$\geq \left|\left|S_1^\alpha QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^\alpha\right|\right|_{\mathcal{I}} - \frac{1}{1-2\alpha}\int_\alpha^{1-\alpha} \left|\left|S_1^tQS_2^{1-t} + S_1^{1-t}QS_2^t\right|\right|_{\mathcal{I}} dt$$
$$\geq 0.$$

□

Specializing the previous theorem to the usual norm of operators on a Hilbert space yields

Corollary 2.2.

$$|||S_1Q + QS_2||| \geq \left|\left|\left|S_1^\alpha QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^\alpha\right|\right|\right|, \tag{25}$$

where S_1, S_2 are positive operators and $0 \leq \alpha \leq 1$;

$$||A^*AX + XB^*B|| \geq 2||AXB||, \tag{26}$$

where $A, B \in B(H)$ are invertible positive operators and $X \in B(H)$;

$$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right| \geq 2\,||X||\,, \tag{27}$$

where $A, B \in B(H)$ are invertible operators and $X \in B(H)$;

$$\left|\left|AXB^{-1} + A^{-1}XB\right|\right| \geq 2\,||X||\,, \tag{28}$$

where $A, B \in B(H)$ are invertible positive operators and $X \in B(H)$;

$$\left|\left|AXA^{-1} + A^{-1}XA\right|\right| \geq 2\,||X||\,, \tag{29}$$

where $A \in B(H)$ is invertible positive operator and $X \in B(H)$;

$$||A^nXB^{-n} + A^{-n}XB^n|| \geq 2||X||, \tag{30}$$

where $A, B \in B(H)$ are invertible positive operators, $X \in B(H)$ and n is a nonnegative integer.

$$||A^{2m+n}XB^{-n} + A^{-n}XB^{2m+n}||| \geq 2|||A^{2m}X + XB^{2m}||, \tag{31}$$

where $A, B \in B(H)$ are invertible positive operators, $X \in B(H)$ and m, n are both nonnegative integers.

Note that the previous corollary is proved by J. Fujii *etal* [9, Theorem 1] for self-adjoint operators in $B(H)$. Specializing the previous corollary to the Schatten p-norms, we obtain the following corollary.

Corollary 2.3.

$$||S_1Q + QS_2||_p^p \geq \left|\left|S_1^\alpha QS_2^{1-\alpha} + S_1^{1-\alpha}QS_2^\alpha\right|\right|_p^p, \tag{32}$$

where S_1, S_2 are positive operators, $Q \in C_p$ and $0 \leq \alpha \leq 1$;

$$||A^*AX + XB^*B||_p^p \geq 2^p||AXB||_p^p, \tag{33}$$

where $A, B \in B(H)$ *are invertible positive operators and* $X \in C_p$*;*

$$\left|\left|AXB^{-1} + (A^*)^{-1}XB^*\right|\right|_p^p \geq 2^p \left|\left|X\right|\right|_p^p, \tag{34}$$

where $A, B \in B(H)$ *are invertible operators and* $X \in B(H)$*;*

$$\left|\left|AXB^{-1} + A^{-1}XB\right|\right|_p^p \geq 2^p \left|\left|X\right|\right|_p^p, \tag{35}$$

where $A, B \in B(H)$ *are invertible positive operators and* $X \in C_p$*;*

$$\left|\left|AXA^{-1} + A^{-1}XA\right|\right|_p^p \geq 2^p \left|\left|X\right|\right|_p^p, \tag{36}$$

where $A \in B(H)$ *is invertible positive operator and* $X \in C_p$*;*

$$||A^n XB^{-n} + A^{-n}XB^n||_p^p \geq 2^p ||X||_p^p, \tag{37}$$

where $A, B \in B(H)$ *are invertible positive operators,* $X \in C_p$ *and* n *is a nonnegative integer.*

$$||A^{2m+n}XB^{-n} + A^{-n}XB^{2m+n}||_p^p \geq 2^p ||A^{2m}X + XB^{2m}||_p^p, \tag{38}$$

where $A, B \in B(H)$ *are invertible operators,* $X \in\in C_p$ *and* m, n *are both nonnegative integers.*

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