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HERMITE-HADAMARD INEQUALITY FOR GEOMETRICALLY QUASICONVEX FUNCTIONS ON CO-ORDINATES

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ABSTRACT. In the paper, we introduce a new concept 'geometrically quasiconvex function on co-ordinates' and establish some Hermite-Hadamard type integral inequalities for functions defined on a rectangle in plane.

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1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if for every $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This remarkable result is well known in the literature as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Since then some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by number of authors, see for example [1-3,5,7,8,10,14-16]). In

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[4], S.S. Dragomir defined convex functions on the co-ordinates (or co-ordinated convex functions) on the set $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mappings,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are convex. This means that for every $(x,y),(z,w) \in \Delta$ and $t,s \in [0,1]$,

$$\begin{aligned} &f(tx+(1-t)z, sy+(1-s)w) \\ &\leq tsf(x,y) + s(1-t)f(z,y) \\ &\quad + t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermit-Hadamard type inequality for co-ordinated convex functions was also proved in [4].

Theorem 1.1. *suppose that $f : \Delta \to \mathbb{R}$ is convex on co-ordinates Δ. Then,*

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(a,y)dy\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

Since then several important generalizations introduced on this category, see [11, 18-20] and references therein. Recall that a function

$f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be quasiconvex if for every $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

In [13], M.E. Özdemir et al. introduced the notion of co-ordinated quasiconvex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1.2. A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to quasiconvex on the co-ordinates on Δ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mapping,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v)$$

are quasiconvex. This means that for every $(x,y),(z,w) \in \Delta$ and $s, t \in [0,1]$,

$$\begin{aligned} & f(tx + (1-t)z, sy + (1-s)w) \\ \leq \ & \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}. \end{aligned}$$

Since then several important generalizations on this category proved by by M.E. Özdemir et al. in [9, 12, 13].

On the other hand F. Qi and B.A. Xi in [18] introduced the notion of geometrically quasiconvex functions and established some integral inequalities of Hermite-Hadamard type.

Definition 1.3. A function $f : I \subseteq \mathbb{R}_0 := [0,\infty) \to \mathbb{R}_0$, is said to be geometrically quasiconvex on I if for every $x, y \in I$ and $\lambda \in [0,1]$,

$$f(x^{\lambda} y^{1-\lambda}) \leq \max\{f(x), f(y)\}.$$

Note that if f decreasing and geometrically quasiconvex then, it is quasiconvex. If f increasing and quasiconvex then, it is geometrically quasiconvex. We recall some results introduced [18].

Lemma 1.1. *Let $f : I \subseteq \mathbb{R}_+ := (0,\infty) \to \mathbb{R}$, be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ then,*
(i)

$$\begin{aligned} & \frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \\ = & \int_0^1 a^{1-t}b^t \ln(a^{1-t}b^t) f'(a^{1-t}b^t)dt. \end{aligned} \tag{1}$$

(ii)

$$
\begin{aligned}
M(a,b) &:= \int_0^1 |\ln(a^{1-t}b^t)|dt \\
&= \begin{cases} \dfrac{\ln a+\ln b}{2}, & a\geq 1, \\ \dfrac{(\ln a)^2+(\ln b)^2}{\ln b-\ln a}, & a<1<b, \\ -\dfrac{\ln a+\ln b}{2}, & b\leq 1. \end{cases}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
N(a,b) &:= \int_0^1 a^{1-t}b^t|\ln(a^{1-t}b^t)|dt \\
&= \begin{cases} \dfrac{b\ln b-a\ln a-(b-a)}{\ln b-\ln a}, & a\geq 1, \\ \dfrac{b\ln b+a\ln a+2-b-a}{\ln b-\ln a}, & a<1<b, \\ \dfrac{b-a-(b\ln b-a\ln a)}{\ln b-\ln a}, & b\leq 1. \end{cases}
\end{aligned} \tag{3}
$$

Theorem 1.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$ for $a,b \in I^\circ$ with $a<b$. If $|f'|$ is geometrically quasiconvex on $[a,b]$ then,*

$$
\begin{aligned}
&\left|\frac{(\ln b)f(b)-(\ln a)f(a)}{\ln b-\ln a}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq N(a,b)\sup\{|f'(a)|,|f'(b)|\}.
\end{aligned} \tag{4}
$$

Theorem 1.3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$ for $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is geometrically quasiconvex on $[a,b]$ for $q>1$ then,*

$$
\begin{aligned}
&\left|\frac{(\ln b)f(b)-(\ln a)f(a)}{\ln b-\ln a}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq [M(a,b)]^{\frac{1}{q}}\left[\frac{q-1}{q}N(a^{q/q-1},b^{q/q-1})\right]^{1-1/q} \\
&\quad\times\left[\sup\{|f'(a)|^q,|f'(b)|^q\}\right]^{\frac{1}{q}}.
\end{aligned} \tag{5}
$$

Theorem 1.4. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$ for $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is geometrically*

quasiconvex on $[a,b]$ *for* $q>1$ *and* $q>r>0$ *then,*

$$\left|\frac{(\ln b)f(b)-(\ln a)f(a)}{\ln b-\ln a}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$
$$\leq\left(\frac{q-1}{q-r}\right)^{1-1/q}\left(\frac{1}{r}\right)^{1/q}\left[N(a^r,b^r)\right]^{\frac{1}{q}}$$
$$\times\left[N(a^{(q-r)/q-1},b^{(q-r)/q-1})\right]^{1-1/q}\times\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}}. \tag{6}$$

Theorem 1.5. *Let* $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_0$ *be a differentiable function on* I° *and* $f\in L([a,b])$ *for* $a,b\in I^\circ$ *with* $a<b$. *If* $|f'|$ *is geometrically quasiconvex on* $[a,b]$ *then,*

$$f((ab)^{1/2})\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\leq\sup\left\{f(a),f(b)\right\}. \tag{7}$$

In [14], M. E. Özdemir defined geometrically convex functions on the co-ordinates as following:

Definition 1.4. Let $\Delta_+:=[a,b]\times[c,d]$ be a subset of ${\mathbb{R}_+}^2$ with $a<b$ and $c<d$. A function $f:\Delta_+\to\mathbb{R}$ is said to be geometrically convex on the co-ordinates if for every $y\in[c,d]$ and $x\in[a,b]$ the partial mappings,

$$f_y:[a,b]\to\mathbb{R},\qquad f_y(u)=f(u,y),$$

and

$$f_x:[c,d]\to\mathbb{R},\qquad f_x(v)=f(x,v),$$

are geometrically convex function. This means that for every $(x,y),(z,w)\in\Delta_+$ and $t,s\in[0,1]$,

$$\begin{aligned}&f(x^tz^{1-t},y^sw^{1-s})\\ &\leq tsf(x,y)+s(1-t)f(z,y)\\ &\quad+t(1-s)f(x,w)+(1-t)(1-s)f(z,w).\end{aligned}$$

The main purpose of this paper is to establish new Hadamard-type inequalities for geometrically quasiconvex functions on the co-ordinates.

2. MAIN RESULTS

In this section we introduce the notion; "geometrically quasiconvex functions on the co-ordinates" for a functions defined on a rectangle in $\mathbb{R}_+^2$, which is a generalization of the notion " geometrically convex functions on the co-ordinates" given in [14]. Then, we establish some Hermite-Hadamard type inequalities for this class of functions.

Definition 2.1. Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of ${\mathbb{R}_+}^2$ with $a < b$ and $c < d$. A function $f : \Delta_+ \to \mathbb{R}$ issaid to be geometrically quasiconvex on the co-ordinates on $\Delta_+ \subseteq \mathbb{R}^2_+$ if for every $y \in [c,d]$ and $x \in [a,b]$ the partial mappings

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v)$$

are geometrically quasiconvex. This means that for every $(x,y),(z,w) \in \Delta_+$ and $s,t \in [0,1]$,

$$f(x^t z^{1-t}, y^s w^{1-s}) \le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}.$$

Note that every geometrically convex function on co-ordinates is geometrically quasiconvex on co-ordinates, but the converse is not holds. In the following we give an example of a geometrically quasiconvex on co-ordinates which is not geometrically convex function on the co-ordinates.

Example 2.1. Let $\Delta_+ := [1,4]\times[4,9]$ and consider the function $f : \Delta_+ \to \mathbb{R}$ defined by

$$f(x,y) := x^2 - y^2.$$

It is easy to see that the functions

$$f_y(x) = x^2 - y^2, \ x \in [1,4],$$

and

$$f_x(y) = x^2 - y^2, \ y \in [4,9],$$

are geometrically quasiconvex. Hence, f is geometrically quasiconvex on co-ordinates on Δ_+. This function is not geometrically convex function on co-ordinates on Δ_+. Indeed, if we take two points, $(x,y) = (1,4)$, $(z,w) = (4,9)$ and $s = t = \frac{1}{2}$, then

$$f(x^t z^{1-t}, y^s w^{1-s}) = f(2,6) = -32,$$

and

$$\begin{aligned} & tsf(x,y) + s(1-t)f(z,y) + t(1-s)f(x,w) \\ & + (1-t)(1-s)f(z,w) \\ & = \frac{1}{4}\{f(x,y), f(x,w), f(z,w), f(z,y)\} = -40 \\ & < f(x^t z^{1-t}, y^s w^{1-s}). \end{aligned}$$

Lemma 2.1. *Let* $\Delta_+ := [a,b]\times[c,d]$ *be a subset of* ${\mathbb{R}_+}^2$ *with* $a<b$ *and* $c<d$. *Suppose that* $f:\Delta_+\to\mathbb{R}$ *is a partial differentiable function on* $\operatorname{int}(\Delta_+)$. *If* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$, *then*

$$\begin{aligned}
&\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\\
&\times\Bigg(C+D+\int_a^b\Big[(\ln c)\frac{f(x,c)}{x}-(\ln d)\frac{f(x,d)}{x}\Big]dx\\
&+\int_c^d\Big[(\ln a)\frac{f(a,y)}{y}-(\ln b)\frac{f(b,y)}{y}\Big]dy+\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx\Bigg) \quad (8)\\
=&\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dtds,
\end{aligned}$$

where

$$C := (\ln d)[(\ln b)f(b,d)-(\ln a)f(a,d)],$$

and

$$D := (\ln c)[(\ln a)f(a,c)-(\ln b)f(b,c)].$$

Proof. If we denote the right hand side of (8) by I and integrating by parts on Δ_+, we have

$$\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)I\\
&=(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s\\
&\quad\times\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dtds\\
&=(\ln b-\ln a)(\ln d-\ln c)\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\\
&\quad\times\Big[\int_0^1 a^{1-t}b^t\ln(a^{1-t}b^t)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dt\Big]ds \quad (9)\\
&=(\ln b-\ln a)(\ln d-\ln c)\\
&\quad\times\Big(\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\Big[\frac{\ln(a^{1-t}b^t)}{(\ln b)-(\ln a)}\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)\Big|_0^1\\
&\qquad-\int_0^1\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)dt\Big]ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&= (\ln b - \ln a)(\ln d - \ln c) \\
&\quad \times \Big(\int_0^1 c^{1-s} d^s \ln(c^{1-s}d^s) \Big[\frac{\ln b}{\ln b - \ln a} \frac{\partial f}{\partial s}(b, c^{1-s}d^s) \\
&\quad - \frac{\ln a}{\ln b - \ln a} \frac{\partial f}{\partial s}(a, c^{1-s}d^s) - \int_0^1 \frac{\partial f}{\partial s}(a^{1-t}b^t, c^{1-s}d^s) dt \Big] ds \Big) \\
&= (\ln d - \ln c)(\ln b) \int_0^1 c^{1-s} d^s \ln(c^{1-s}d^s) \frac{\partial f}{\partial s}(b, c^{1-s}d^s) ds \\
&\quad - (\ln d - \ln c)(\ln a) \int_0^1 c^{1-s} d^s \ln(c^{1-s}d^s) \frac{\partial f}{\partial s}(a, c^{1-s}d^s) ds \\
&\quad - (\ln b - \ln a)(\ln d - \ln c) \\
&\quad \times \Big(\int_0^1 \Big[\int_0^1 c^{1-s} d^s \ln(c^{1-s}d^s) \frac{\partial f}{\partial s}(a^{1-t}b^t, c^{1-s}d^s) ds \Big] dt \Big).
\end{aligned}
$$

Similarly integration by parts in the right side of (9) deduce that

$$
\begin{aligned}
&(\ln b - \ln a)(\ln d - \ln c) I \\
&= (\ln b) \left(\ln(c^{1-s}d^s) f(b, c^{1-s}d^s) \Big|_0^1 - (\ln d - \ln c) \int_0^1 f(b, c^{1-s}d^s) ds \right) \\
&\quad - (\ln a) \left(\ln(c^{1-s}d^s) f(a, c^{1-s}d^s) \Big|_0^1 - (\ln d - \ln c) \int_0^1 f(a, c^{1-s}d^s) ds \right) \\
&\quad - (\ln b - \ln a) \int_0^1 \left(\ln(c^{1-s}d^s) f(a^{1-t}b^t, c^{1-s}d^s) \Big|_0^1 \right) dt \\
&\quad + (\ln b - \ln a)(\ln d - \ln c) \int_0^1 \int_0^1 f(a^{1-t}b^t, c^{1-s}d^s) dt ds \\
&= (\ln b) \Big([(\ln d) f(b, d) - (\ln c) f(b, c)] \\
&\qquad\qquad - (\ln d - \ln c) \int_0^1 f(b, c^{1-s}d^s) ds \Big) \\
&\quad - (\ln a) \Big([(\ln d) f(a, d) - (\ln c) f(a, c)] \\
&\qquad\qquad - (\ln d - \ln c) \int_0^1 f(a, c^{1-s}d^s) ds \Big) \\
&\quad - (\ln b - \ln a) \left((\ln d) \int_0^1 f(a^{1-t}b^t, d) dt - (\ln c) \int_0^1 f(a^{1-t}b^t, c) dt \right) \\
&\quad + (\ln b - \ln a)(\ln d - \ln c) \int_0^1 \int_0^1 f(a^{1-t}b^t, c^{1-s}d^s) dt ds.
\end{aligned}
$$

If we using the change of variables $x = a^{1-t}b^t$ and $y = c^{1-s}d^s$ for $t, s \in [0,1]$, we obtain

$$\begin{aligned}
&(\ln b - \ln a)(\ln d - \ln c)I \\
&= (\ln b)\left([(\ln d)f(b,d) - (\ln c)f(b,c)] - \int_c^d \frac{f(b,y)}{y}dy\right) \\
&\quad - (\ln a)\left([(\ln d)f(a,d) - (\ln c)f(a,c)] - \int_c^d \frac{f(a,y)}{y}dy\right) \\
&\quad - (\ln d)\int_a^b \frac{f(x,d)}{x} + (\ln c)\int_a^b \frac{f(x,c)}{x}dx + \int_a^b\int_c^d \frac{f(x,y)}{yx}dydx.
\end{aligned} \tag{10}$$

Dividing both sides of (10) by $(\ln b - \ln a)(\ln d - \ln c)$ implies that the equation (8) holds and proof is completed. □

Theorem 2.1. *Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. Suppose that $f : \Delta_+ \to \mathbb{R}$ is a partial differentiable function on $\text{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b - \ln a)(\ln d - \ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b - \ln a)(\ln d - \ln c)} - B\right| \\
&\le N(a,b)\ N(c,d) \\
&\quad \times \max\left\{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|, \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|, \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|, \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|\right\},
\end{aligned} \tag{11}$$

where, C, D and $N(a,b)$ are defined, respectively, in Lemma 2.1 and Lemma 1.1, and

$$\begin{aligned}
B = &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \times \left(\int_a^b \left[(\ln d)\frac{f(x,d)}{x} - (\ln c)\frac{f(x,c)}{x}\right]dx\right. \\
&\left. + \int_c^d \left[(\ln b)\frac{f(b,y)}{y} - (\ln a)\frac{f(a,y)}{y}\right]dy\right).
\end{aligned}$$

Proof. From Lemma 2.1, it follows that

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b - \ln a)(\ln d - \ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b - \ln a)(\ln d - \ln c)} - B\right| \\
&\le \int_0^1\int_0^1 a^{1-t}b^t c^{1-s}d^s |\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)| \\
&\quad \times \left|\frac{\partial^2 f}{\partial t \partial s}(a^{1-t}b^t, c^{1-s}d^s)\right| dtds.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is geometrically quasiconvex on the co-ordinates we have

$$\begin{aligned} &\Big|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t, c^{1-s}d^s)\Big| \\ \leq\ & \max\left\{\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|\right\}, \end{aligned}$$

where $t, s \in [0,1]$. From this inequality and relationship (3) in Lemma 1.1, it follows that

$$\begin{aligned} &\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\Big|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t, c^{1-s}d^s)\Big|dtds \\ \leq\ & \max\left\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial s}\Big|, \Big|\frac{\partial^2 f(a,d)}{\partial t\partial s}\Big|, \Big|\frac{\partial^2 f(b,c)}{\partial t\partial s}\Big|, \Big|\frac{\partial^2 f(b,d)}{\partial t\partial s}\Big|\right\} \\ &\times \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds \\ =\ & N(a,b)\ N(c,d) \\ &\times \max\left\{\Big|\frac{\partial^2 f}{\partial t\partial s}(a,c)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(a,d)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(b,c)\Big|, \Big|\frac{\partial^2 f}{\partial t\partial s}(b,d)\Big|\right\}, \end{aligned}$$

which is the required inequality (11), since

$$\begin{aligned} &\int_0^1\int_0^1 a^{(1-t)}b^tc^{(1-s)}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds \\ =\ & \left(\int_0^1 a^{(1-t)}b^t|\ln(a^{1-t}b^t)|dt\right)\left(\int_0^1 c^{(1-s)}d^s|\ln(c^{1-s}d^s)|ds\right) \\ =\ & N(a,b)\ N(c,d). \end{aligned}$$

The proof of theorem is completed. □

The following corollary is an immediate consequence of theorem 2.1.

Corollary 2.1. *Suppose the conditions of the Theorem 2.1 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is increasing on the co-ordinates on* Δ_+*, then*

$$\begin{aligned} &\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)} - B\right| \\ \leq\ & N(a,b)\ N(c,d)\ \Big|\frac{\partial^2}{\partial t\partial s}f(b,d)\Big|. \end{aligned} \tag{12}$$

(2) $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is decreasing on the co-ordinates on* Δ_+*, then*

$$\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \leq N(a,b)\ N(c,d)\ \left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|, \tag{13}$$

where, C*,* D*,* B *and* $N(a,b)$ *are defined, respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.*

Proof. Follows directly from Theorem 2.1. □

Theorem 2.2. *Let* $\Delta_+ := [a,b]\times[c,d]$ *be a subset of* $\mathbb{R}_+^2$ *with* $a<b$ *and* $c<d$*. Suppose that* $f:\Delta_+\to\mathbb{R}$ *is a partial differentiable function on* $\operatorname{int}(\Delta_+)$ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is a geometrically quasiconvex function on the co-ordinates on* Δ_+ *and* $p,q>1$*,* $\frac{1}{p}+\frac{1}{q}=1$*, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\ &\leq \left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}}\times\Bigg[\max\Bigg\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\\ &\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\Bigg\}\Bigg]^{1/q},\end{aligned} \tag{14}$$

where, C*,* D*,* B *and* $N(a,b)$ *are defined, respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.*

Proof. suppose $p>1$. From Lemma 2.1 and well-known Hölder inequality for double integrals, we obtain

$$\begin{aligned}&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\ &\leq \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds\\ &\leq \left(\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|^pdtds\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^qdtds\right)^{\frac{1}{q}}.\end{aligned} \tag{15}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is geometrically quasiconvex on the co-ordinates on Δ_+, we obtain

$$\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t, c^{1-s}d^s)\right|^q dtds$$
$$\leq \max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}. \quad (16)$$

We also notice that

$$\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|^p dtds$$
$$= \left(\int_0^1 a^{p(1-t)}b^{pt}|\ln(a^{1-t}b^t)|^p dt\right)\left(\int_0^1 c^{p(1-s)}d^{ps}|\ln(c^{1-s}d^s)|^p ds\right) \quad (17)$$
$$= N(a^p, b^p)\ N(c^p, d^p).$$

A combination of (15), (16) and (17), gives the desired inequality (14). Hence the proof of the theorem is completed. □

Corollary 2.2. *Suppose the conditions of the Theorem 2.2 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is increasing on the co-ordinates on* Δ_+*, then*

$$\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)} - B\right|$$
$$\leq \left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}} \left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|. \quad (18)$$

(2) $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is decreasing on the co-ordinates on* Δ_+*, then*

$$\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)} - B\right|$$
$$\leq \left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}} \left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|, \quad (19)$$

where, C*,* D*,* B *and* $N(a,b)$ *are defined, respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.*

Proof. It is direct consequence of Theorem 2.2. □

Theorem 2.3. *Let* $\Delta_+ := [a,b]\times[c,d]$ *be a subset of* $\mathbb{R}^2_+$ *with* $a<b$ *and* $c<d$*. Suppose that* $f:\Delta_+\to\mathbb{R}$ *is a partial differentiable function on* $\operatorname{int}(\Delta_+)$ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is a geometrically*

quasiconvex function on the co-ordinates on Δ_+ for $q>1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\
&\le [M(a,b)\ M(c,d)]^{1/q} \\
&\quad\times\left[\left(\frac{q-1}{q}\right)^2 N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\ N\left(c^{q/(q-1)},d^{q/(q-1)}\right)\right]^{1-1/q} \qquad (20)\\
&\quad\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}$$

where, C, D, B and $M(a,b), N(a,b)$ are defined, respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.

Proof. By Lemma 2.1, Hölder's inequality, and the geometric quasicanvexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\
&\le \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds \\
&\le \left[\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\right.\\
&\quad\left.\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^q dtds\right]^{1/q}\\
&\le \left[\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\right.\\
&\quad\left.\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1/q}\\
&\quad\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

Note that relationship (3) in Lemma 1.1 shows,

$$\begin{aligned}
&\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\\
&\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\\
=&\left(\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}|\ln(a^{1-t}b^t)|dt\right)\\
&\times\left(\int_0^1 c^{q(1-s)/(q-1)}d^{qs/(q-1)}|\ln(c^{1-s}d^s)|ds\right)\\
=&\frac{(q-1)^2}{q^2}N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\ N\left(c^{q/(q-1)},d^{q/(q-1)}\right),
\end{aligned}$$

and

$$\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds=M(a,b)\ M(c,d).$$

The proof of theorem is completed. □

Theorem 2.4. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a<b$ and $c<d$. Suppose that $f:\Delta_+\to\mathbb{R}$ is a partial differentiable function on $\operatorname{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ and $q>\ell>0$, then*

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
\le&\left(\frac{q-1}{q-\ell}\right)^{2(1-1/q)}\left(\frac{1}{\ell}\right)^{2/q}\left[N\left(a^\ell,b^\ell\right)\ N\left(c^\ell,d^\ell\right)\right]^{1/q}\\
&\times\left[N\left(a^{(q-\ell)/(q-1)},b^{(q-\ell)/(q-1)}\right)\ N\left(c^{(q-\ell)/(q-1)},d^{(q-\ell)/(q-1)}\right)\right]^{(1-1/q)}\\
&\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}\tag{21}$$

where, C, D, B and $N(a,b)$ are defined, respectively, in Lemma 2.1, Theorem 2.1 and Lemma 1.1.

Proof. From Lemma 2.1, Hölder's inequality, and the geometric quasi-convexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on Δ_+ and by (3) it follows that,

$$
\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\
\leq\ & \int_0^1\int_0^1 a^{1-t}b^t c^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)| \\
&\times\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds \\
\leq\ & \Bigg[\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)} \\
&\times d^{(q-\ell)s/(q-1)}\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\Bigg]^{1-1/q} \\
&\times\Bigg[\int_0^1\int_0^1|\ln(a^{\ell(1-t)}b^{\ell t})\ln(c^{\ell(1-s)}d^{\ell s})| \\
&\times\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^q dtds\Bigg]^{1/q} \\
\leq\ & \Bigg[\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)} \\
&\times d^{(q-\ell)s/(q-1)}\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\Bigg]^{1-1/q} \\
&\times\Bigg[\int_0^1\int_0^1|a^{\ell(1-t)}b^{\ell t}c^{\ell(1-s)}d^{\ell s}\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\Bigg]^{1/q} \\
&\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}} \\
=\ & \left(\frac{q-1}{q-\ell}\right)^{2(1-1/q)}\Big[N\big(a^{(q-\ell)/(q-1)},b^{(q-\ell)/(q-1)}\big) \\
&\times N\big(c^{(q-\ell)/(q-1)},d^{(q-\ell)/(q-1)}\big)\Big]^{1-1/q} \\
&\times\left(\frac{1}{\ell}\right)^{2/q}\Big[N\big(a^\ell,b^\ell\big)\ N\big(c^\ell,d^\ell\big)\Big]^{1/q} \\
&\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof of theorem is completed. □

Theorem 2.5. *Let* $\Delta_+ := [a,b]\times[c,d]$ *be a subset of* $\mathbb{R}^2_+$ *with* $a<b$ *and* $c<d$*. Suppose that* $f:\Delta_+\to\mathbb{R}$ *is a geometrically quasiconvex function on the co-ordinates on* Δ_+*. If* $f\in L(\Delta_+)$*, then*

$$\begin{aligned} f\big((ab)^{1/2},(cd)^{1/2}\big) &\le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx \\ &\le \max\{f(a,c),f(a,d),f(b,c),f(b,d)\}. \qquad (22)\end{aligned}$$

Proof. By geometric quasiconvexity of f on co-ordinates on Δ_+, for $t\in[0,1]$, we have

$$\begin{aligned} & f\big((ab)^{1/2},(cd)^{1/2}\big) \\ \le\ & \max\{f(a^{1-t}b^t,c^{1-s}d^s),f(a^tb^{1-t},c^sd^{1-s})\} \qquad (23)\\ \le\ & \max\{f(a,c),f(a,d),f(b,c),f(b,d)\}.\end{aligned}$$

Since

$$\begin{aligned}\int_0^1\int_0^1 f(a^{1-t}b^t,c^{1-s}d^s)dtds &= \int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s})dtds \\ &= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx,\end{aligned}$$

by integrating in (23) we get

$$\begin{aligned} & f\big((ab)^{1/2},(cd)^{1/2}\big) \\ &\le \max\left\{\int_0^1\int_0^1 f(a^{1-t}b^t,c^{1-s}d^s)dtds, \int_0^1\int_0^1 f(a^tb^{1-t},c^sd^{1-s})dtds\right\} \\ &= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx \\ &\le \max\{f(a,c),f(a,d),f(b,c),f(b,d)\}.\end{aligned}$$

and proof is completed. □

Theorem 2.6. *Let* $\Delta_+ := [a,b]\times[c,d]$ *be a subset of* $\mathbb{R}^2_+$ *with* $a<b$ *and* $c<d$*. Suppose that* $f,g:\Delta_+\to\mathbb{R}$ *are geometrically quasiconvex functions on the co-ordinates on* Δ_+*. If* $fg\in L(\Delta_+)$*. Then,*

$$\begin{aligned} &\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}g(x,y)dydx \\ \le\ & \max\big\{f(u,v)\ g(w,z)\ \big|\ u,w\in\{a,b\},\ v,z\in\{c,d\}\big\}.\end{aligned}$$

Proof. Let $x = a^{1-t}b^t,\ y = a^{1-s}b^s, s, t \in [0, 1]$ and using the geometric quasiconvexity of f, g on Δ_+ yields

$$\begin{aligned} &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_a^b \int_c^d \frac{f(x,y)}{yx} g(x,y) dydx \\ = &\int_0^1 \int_0^1 f(a^{1-t}b^t, c^{1-s}d^s)\ g(a^{1-t}b^t, c^{1-s}d^s) dtds \\ \leq &\max\{f(a,c), f(a,d), f(b,c), f(b,d)\} \\ &\times \max\{g(a,c), g(a,d), g(b,c), g(b,d)\}, \end{aligned}$$

and proof is completed. □

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