Received 26/04/15

FEJÉR TYPE INTEGRAL INEQUALITIES RELATED WITH GEOMETRICALLY-ARITHMETICALLY-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, a new identity involving a geometrically symmetric function and a differentiable function is established. Some new Fejér type integral inequalities connected with the left part of Hemite-Hadamard type inequalities for geometrically-arithmetically convex functions are presented by using the Hölder integral inequality and the notion of geometrically-arithmetically convexity. Applications of our results to special means of positive real numbers are given.

1. Introduction

The classical convexity is defined as follows:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all x, $y \in I$ and $t \in [0,1]$.

A vast literature has been reported on inequalities concerning classical convexity during the past three decades but the most celebrated inequalities in mathematical analysis for convex functions is the Hermite-Hadamard inequality, which is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The inequalities in (1.1) hold in upturned direction if f is a concave function.

In the past few years the theory of convex sets and convex functions have been a subject of extensive research and these notions have been generalized and extended in diverse directions, and as a result many new proofs, noteworthy extensions, generalizations, refinements, new Hermite-Hadamard-type inequalities and numerous applications of the inequalities (1.1) have arisen in the literature of mathematical analysis and in various other branches of pure and applied mathematics, see for instance [2, 3], [8], [11], [17], [18], [21]-[23], [25], [27] and the references therein.

We begin with the following generalization of the usual or classical convexity which is so called the geometrically-arithmetical convexity or GA-convexity:

Date: November 15, 2012.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26A51, 26E60, 41A55.

Key words and phrases. Hermite-Hadamard's inequality, Fejér's inequality, convex function, GA-convex function, Hölder's inequality, power mean inequality.

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Definition 1. [15, 16] *A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

The above notion of GA-convexity is further generalized as GA-s-convexity in the second sense as follows.

Definition 2. [9] *A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be s-GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x,\ y \in I$, $\lambda \in [0,1]$ and for some $s \in (0,1]$.

For the properties of GA-convex functions and GA-s-convex functions, we refer the reader to [4, 5, 7, 15, 16] and the reference there in.

Most recently, İşcan [9] proved the following result for GA-s-convex functions.

Theorem 1. [9] *Suppose that $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is GA-s-convex in the second sense and $a,\ b \in I$ with $a < b$. If $f \in L\,[a,b]$, then one has the inequalities:*

$$2^{s-1} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(x\right)}{x}dx \leq \frac{f\left(a\right) + f\left(b\right)}{s+1}. \tag{1.2}$$

If f in Theorem 1 is GA-convex function, then we get the following inequalities.

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(x\right)}{x}dx \leq \frac{f\left(a\right) + f\left(b\right)}{2}. \tag{1.3}$$

In [13], the authors introduced the notion of geometrically symmetric functions as follows.

Definition 3. [13] *A function $g : [a,b] \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be geometrically symmetric with respect to $\sqrt{ab}$ if the inequality*

$$g\left(\frac{ab}{x}\right) = g\left(x\right)$$

holds for all $x \in [a,b]$.

The authors in [13] also proved the following Fejér type integral inequalities by using the notion of geometrically symmetric functions and geometrically-arithmetically convex functions, which provide a weighted generalization of the inequality given by (1.3).

Theorem 2. [13] *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a GA-convex function and $a,\ b \in I$ with $a < b$. Let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g\left(x\right)}{x}dx \leq \int_a^b \frac{f\left(x\right)g\left(x\right)}{x}dx \leq \frac{f\left(a\right) + f\left(b\right)}{2}\int_a^b \frac{g\left(x\right)}{x}dx. \tag{1.4}$$

Several new integral inequalities were also established in [13] for the right part of the inequality (1.4) which not only provide weighted generalization of the results from [9] connected with the right part of (1.3) but also give refinements of those results for particular choice of the geometrically symmetric weight functions.

For more results on Hermite-Hadamard type inequalities concerning GA-convex functions, s-GA-convex functions and their applications we refer the reader to [4], [5], [7], [9], [12], [13], [24], [26] and closely related articles mentioned in the references therein.

In Section 2, we will prove a weighted integral identity for the left part of the inequality (1.4) involving a differentiable mapping and a geometrically symmetric function. We will use this identity, the geometrical-arithmetical convexity and some auxiliary results to obtain some new Fejér type integral inequalities related with the left part of (1.4). The results of Section 2 provide a weighted version of the results given in [9] for the left part of the inequality (1.3) and also refine those results for particular choice of the geometrically symmetric function involved. Some applications of our results to special means of positive real numbers are given in Section 3.

2. Fejér Type Integral Inequalities For Geometrically-Arithmetically-Convex Functions

Throughout this section we will use the notations $U(t) = a^{(1-t)/2}b^{(1+t)/2}$ and $L(t) = a^{(1+t)/2}b^{(1-t)/2}$ for our convenience.

The Beta function and the integral from of the hypergeometric function are defined as follows to be used in the sequel of the paper

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \alpha>0, \beta>0$$

and

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt$$

for $|z|<1, \gamma>\beta>0$.

Now we prove a weighted integral identity which play a key role in establishing our main results.

Lemma 1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a<b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. If $f' \in L([a,b])$, then the following equality holds*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx$$
$$= \frac{\ln b - \ln a}{2}\int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x}dx\right)\left[L(t) f'(L(t)) - U(t) f'(U(t))\right] dt. \quad (2.1)$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x}dx\right) L(t) f'(L(t))\,dt$$

and

$$I_2=\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x}dx\right)U(t)\,f^{'}(U(t))\,dt.$$

Since $g:[a,b]\to[0,\infty)$ is geometrically symmetric to $\sqrt{ab}$, hence $g(U(t))=g(L(t))$ and

$$\int_a^{L(t)}\frac{g(x)}{x}dx=\int_{U(t)}^{b}\frac{g(x)}{x}dx$$

for all $t\in[0,1]$.

Now we observe that

$$\begin{aligned}
I_1&=\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x}dx\right)L(t)\,f^{'}(L(t))\,dt\\
&=-\frac{2}{\ln b-\ln a}\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x}dx\right)d\left[f(L(t))\right]\\
&=-\frac{2}{\ln b-\ln a}\left(\int_a^{L(t)}\frac{g(x)}{x}dx\right)f(L(t))\Bigg|_0^1-\int_0^1 g(L(t))\,f(L(t))\,dt\\
&=\frac{2f\left(\sqrt{ab}\right)}{\ln b-\ln a}\int_a^{\sqrt{ab}}\frac{g(x)}{x}dx-\int_0^1 g(L(t))\,f(L(t))\,dt\\
&=\frac{2f\left(\sqrt{ab}\right)}{\ln b-\ln a}\int_a^{\sqrt{ab}}\frac{g(x)}{x}dx-\frac{2}{\ln b-\ln a}\int_a^{\sqrt{ab}}\frac{g(x)\,f(x)}{x}dx. \quad (2.2)
\end{aligned}$$

Similarly, we have

$$-I_2=\frac{2f\left(\sqrt{ab}\right)}{\ln b-\ln a}\int_{\sqrt{ab}}^{b}\frac{g(x)}{x}dx-\frac{2}{\ln b-\ln a}\int_{\sqrt{ab}}^{b}\frac{g(x)\,f(x)}{x}dx. \quad (2.3)$$

Adding (2.2) and (2.3) and multiplying the result by $\frac{\ln b-\ln a}{2}$, we get the required identity. This completes the proof of the Lemma. □

Lemma 2. *For $u,v>0$, we have*

$$\zeta(u,v)\triangleq\int_0^1 u^{(1-t)/2}v^{(1+t)/2}dt=\begin{cases}\sqrt{v}L\left(\sqrt{u},\sqrt{v}\right), & u\neq v,\\ u, & u=v,\end{cases}$$

$$\xi(u,v)\triangleq\frac{1}{2}\int_0^1 t u^{(1-t)/2}v^{(1+t)/2}dt=\begin{cases}\frac{v-\sqrt{v}L\left(\sqrt{u},\sqrt{v}\right)}{\ln v-\ln u}, & u\neq v,\\ \frac{1}{4}u, & u=v,\end{cases}$$

and

$$\varsigma(u,v)\triangleq\frac{1}{2}\int_0^1 t^2 u^{(1-t)/2}v^{(1+t)/2}dt=\begin{cases}\frac{4\sqrt{v}L\left(\sqrt{u},\sqrt{v}\right)-4v+v(\ln v-\ln u)}{(\ln v-\ln u)^2}, & u\neq v,\\ \frac{1}{6}u, & u=v.\end{cases}$$

Proof. The proof follows from a straightforward computation. □

We now establish new Fejér type inequalities for GA-convex functions, which provide weighted generalization of some of the results established in recent literature concerning GA-convex functions.

Theorem 3. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$ such that $f' \in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds*

$$\begin{aligned}
&\left| f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx \right| \\
&\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \left\{ [\zeta(a,b) - 2\xi(a,b)]^{1-1/q} \right. \\
&\times \left([\zeta(a,b) + 2\varsigma(a,b)] A\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q \right) - 2\xi(a,b)\left|f'(a)\right|^q \right)^{1/q} \\
&+ [\zeta(b,a) - 2\xi(b,a)]^{1-1/q} \left([\zeta(b,a) + 2\varsigma(b,a)] \right. \\
&\left. \times A\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q \right) - 2\xi(b,a)\left|f'(b)\right|^q \right)^{1/q} \Bigg\}, \quad (2.4)
\end{aligned}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ and $\zeta(\cdot,\cdot)$, $\xi(\cdot,\cdot)$, $\varsigma(\cdot,\cdot)$ are defined in Lemma 2.

Proof. From Lemma 1 and Hölder's inequality , we have

$$\begin{aligned}
&\left| f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx \right| \\
&\leq \frac{\ln b - \ln a}{2}\int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x}dx \right) \left[U(t)\left|f'(U(t))\right| + L(t)\left|f'(L(t))\right| \right] dt \\
&\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \int_0^1 \left[(1-t)U(t)\left|f'(U(t))\right| + (1-t)L(t)f'(L(t)) \right] dt \\
&\leq \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \left\{ \left(\int_0^1 (1-t)U(t) \right)^{1-1/q} \left(\int_0^1 (1-t)U(t)\left|f'(U(t))\right|^q \right)^{1/q} \right. \\
&\left. + \left(\int_0^1 (1-t)L(t) \right)^{1-1/q} \left(\int_0^1 (1-t)L(t)\left|f'(L(t))\right|^q \right)^{1/q} \right\}. \quad (2.5)
\end{aligned}$$

By the GA-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 2, we have

$$\begin{aligned}
&\int_0^1 (1-t)U(t)\left|f'(U(t))\right|^q \leq \left|f'(a)\right|^q \int_0^1 (1-t)\left(\frac{1-t}{2}\right) a^{(1-t)/2}b^{(1+t)/2}dt \\
&+ \left|f'(b)\right|^q \int_0^1 (1-t)\left(\frac{1+t}{2}\right) a^{(1-t)/2}b^{(1+t)/2}dt \\
&= \left[\frac{1}{2}\zeta(a,b) - 2\xi(a,b) + \varsigma(a,b)\right]\left|f'(a)\right|^q + \left[\frac{1}{2}\zeta(a,b) + \varsigma(a,b)\right]\left|f'(b)\right|^q \quad (2.6)
\end{aligned}$$

and

$$\int_0^1 (1-t)\,L(t)\left|f'(L(t))\right|^q \le \left|f'(a)\right|^q \int_0^1 (1-t)\left(\frac{1+t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt$$
$$+\left|f'(b)\right|^q \int_0^1 (1-t)\left(\frac{1-t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt$$
$$=\left[\frac{1}{2}\zeta(b,a)+\varsigma(b,a)\right]\left|f'(a)\right|^q+\left[\frac{1}{2}\zeta(b,a)-2\xi(b,a)+\varsigma(b,a)\right]\left|f'(b)\right|^q. \quad (2.7)$$

Using (2.6) and (2.7) in (2.5), we get the required result. This completes the proof of the theorem. □

Corollary 1. *Suppose the assumptions of Theorem 3 are satisfied and $q=1$, then the following inequality holds*

$$\left|f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\le 2\|g\|_\infty \left\{\left[L(a,b)-\sqrt{a}A\left(\sqrt{a},\sqrt{b}\right)-\frac{1}{4}a(\ln b-\ln a)\right]\left|f'(a)\right|\right.$$
$$\left.+\left[L(a,b)-\sqrt{b}A\left(\sqrt{a},\sqrt{b}\right)+\frac{1}{4}b(\ln b-\ln a)\right]\left|f'(b)\right|\right\}, \quad (2.8)$$

where $\|g\|_\infty=\sup_{x\in[a,b]} g(x)<\infty$.

Corollary 2. *If $g(x)=\frac{1}{\ln b-\ln a}$, for all $x\in[a,b]$ in Theorem 3, then*

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\le \frac{(\ln b-\ln a)}{4}\left\{[\zeta(a,b)-2\xi(a,b)]^{1-1/q}\left([\zeta(a,b)+2\varsigma(a,b)]A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right.\right.$$
$$\left.-2\xi(a,b)\left|f'(a)\right|^q\right)^{1/q}+[\zeta(b,a)-2\xi(b,a)]^{1-1/q}$$
$$\left.\times\left([\zeta(b,a)+2\varsigma(b,a)]A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-2\xi(b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}, \quad (2.9)$$

where $\zeta(\cdot,\cdot)$, $\xi(\cdot,\cdot)$ and $\varsigma(\cdot,\cdot)$ are defined in Lemma 2.

Corollary 3. *If $q=1$ in Corollary 2, then we get the following inequality*

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\le \frac{2}{\ln b-\ln a}\left\{\left[L(a,b)-\sqrt{a}A\left(\sqrt{a},\sqrt{b}\right)-\frac{1}{4}a(\ln b-\ln a)\right]\left|f'(a)\right|\right.$$
$$\left.+\left[L(a,b)-\sqrt{b}A\left(\sqrt{a},\sqrt{b}\right)+\frac{1}{4}b(\ln b-\ln a)\right]\left|f'(b)\right|\right\}. \quad (2.10)$$

Theorem 4. *Let $f: I\subseteq \mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and a, $b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and*

geometrically symmetric to $\sqrt{ab}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|^q$ *is GA-convex on* $[a,b]$ *for* $q>1$, *then the following inequality holds*

$$\left|f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{(\ln b-\ln a)^{2-1/q}\|g\|_\infty}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]\left|f'(a)\right|^q\right.\right.$$
$$\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'(b)\right|^q\right)^{1/q}$$
$$+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]\left|f'(a)\right|^q\right.$$
$$\left.\left.+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'(b)\right|^q\right)^{1/q}\right\}, \quad (2.11)$$

where $\|g\|_\infty=\sup_{x\in[a,b]} g(x)<\infty$.

Proof. From Lemma 1 and Hölder's inequality , we have

$$\left|f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{\ln b-\ln a}{2}\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x}dx\right)\left[U(t)\left|f'(U(t))\right|+L(t)\left|f'(L(t))\right|\right]dt$$
$$\leq \frac{(\ln b-\ln a)^2\|g\|_\infty}{4}\int_0^1\left[(1-t)U(t)\left|f'(U(t))\right|+(1-t)L(t)f'(L(t))\right]dt$$
$$\leq \frac{(\ln b-\ln a)^2\|g\|_\infty}{4}\left(\int_0^1 (1-t)^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 [U(t)]^q\left|f'(U(t))\right|^q dt\right)^{1/q}+\left(\int_0^1 [L(t)]^q\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (2.12)$$

Since

$$\int_0^1 [U(t)]^q\left|f'(U(t))\right|^q=\int_0^1 a^{[q(1-t)]/2}b^{[q(1+t)]/2}\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|$$
$$\leq\left|f'(a)\right|^q\int_0^1\left(\frac{1-t}{2}\right)a^{[q(1-t)]/2}b^{[q(1+t)]/2}dt$$
$$+\left|f'(b)\right|^q\int_0^1\left(\frac{1+t}{2}\right)a^{[q(1-t)]/2}b^{[q(1+t)]/2}dt$$
$$=\frac{b^{q/2}\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]}{q(\ln b-\ln a)}\left|f'(a)\right|^q$$
$$+\frac{b^{q/2}\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]}{q(\ln b-\ln a)}\left|f'(b)\right|^q \quad (2.13)$$

and

$$\begin{aligned}\int_0^1 [L(t)]^q \left|f'(L(t))\right|^q &= \int_0^1 a^{[q(1+t)]/2} b^{q(1-t)/2} \left|f'\left(a^{(1+t)/2} b^{(1-t)/2}\right)\right| \\ &\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{[q(1+t)]/2} b^{[q(1-t)]/2} dt \\ &\quad + \left|f'(b)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{[q(1+t)]/2} b^{[q(1-t)]/2} dt \\ &= \frac{a^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q \\ &\qquad + \frac{a^{q/2}\left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q. \quad (2.14)\end{aligned}$$

The inequality (2.11) is proved by applying (2.13) and (2.14) in (2.12). □

Corollary 4. *If the assumptions of Theorem 4 are satisfied and if* $g(x) = \frac{1}{\ln b - \ln a}$ *for all* $x \in [a, b]$*, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx\right| \\ &\leq \frac{(\ln b - \ln a)^{1-1/q}}{4 \cdot q^{1/q}} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2}\right] \left|f'(a)\right|^q\right.\right. \\ &\qquad + \left.\left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q\right)^{1/q} \\ &\qquad + a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right] \left|f'(a)\right|^q\right. \\ &\qquad\qquad + \left.\left.\left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q\right)^{1/q}\right\}. \quad (2.15)\end{aligned}$$

Theorem 5. *Let* $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a < b$ *and let* $g : [a, b] \to [0, \infty)$ *be continuous positive mapping and geometrically symmetric to* $\sqrt{ab}$ *such that* $f' \in L([a, b])$*. If* $\left|f'\right|^q$ *is GA-convex on* $[a, b]$ *for* $q > 1$*, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx\right| \\ &\leq \frac{(\ln b - \ln a)^{2-1/q}}{2 \cdot (4q)^{1/q}} \|g\|_\infty \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{\left[L\left(a^q, b^q\right) - a^q\right] \left|f'(a)\right|^q\right. \\ &\qquad + \left.\left[b^q - L\left(a^q, b^q\right)\right] \left|f'(b)\right|^q\right\}^{1/q}, \quad (2.16)\end{aligned}$$

where $\|g\|_\infty = \sup_{x \in [a,b]} g(x) < \infty$.

Proof. From Lemma 1 and Hölder's inequality , we have

$$\begin{aligned}
&\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)g(x)}{x} dx \right| \\
&\leq \frac{\ln b - \ln a}{2} \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x} dx \right) \left[U(t) \left| f'(U(t)) \right| + L(t) \left| f'(L(t)) \right| \right] dt \\
&\leq \frac{(\ln b - \ln a)^2}{4} \|g\|_\infty \int_0^1 \left[(1-t) U(t) \left| f'(U(t)) \right| + (1-t) L(t) f'(L(t)) \right] dt \\
&\qquad \leq \frac{(\ln b - \ln a)^2}{4} \|g\|_\infty \left(\int_0^1 (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\
&\times \left\{ \left(\int_0^1 [U(t)]^q \left| f'(U(t)) \right|^q dt \right)^{1/q} + \left(\int_0^1 [L(t)]^q \left| f'(L(t)) \right|^q dt \right)^{1/q} \right\}. \qquad (2.17)
\end{aligned}$$

By the power-mean inequality ($a^r + b^r \leq 2^{1-r}(a+b)^r$ for $a > 0, b > 0$ and $r < 1$), we have

$$\begin{aligned}
&\left(\int_0^1 [U(t)]^q \left| f'(U(t)) \right|^q dt \right)^{1/q} + \left(\int_0^1 [L(t)]^q \left| f'(L(t)) \right|^q dt \right)^{1/q} \\
&\leq 2^{1-1/q} \left(\int_0^1 [U(t)]^q \left| f'(U(t)) \right|^q dt + \int_0^1 [L(t)]^q \left| f'(L(t)) \right|^q dt \right)^{1/q}. \qquad (2.18)
\end{aligned}$$

Since $\left| f' \right|^q$ is GA-convex on $[a,b]$ for $q > 1$

$$\begin{aligned}
&\int_0^1 [U(t)]^q \left| f'(U(t)) \right|^q dt + \int_0^1 [L(t)]^q \left| f'(L(t)) \right|^q dt \\
&\leq \left| f'(a) \right|^q \left[\int_0^1 \left(\frac{1-t}{2} \right) a^{[q(1-t)]/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1+t}{2} \right) a^{[q(1+t)]/2} b^{q(1-t)} dt \right] \\
&+ \left| f'(b) \right|^q \left[\int_0^1 \left(\frac{1+t}{2} \right) a^{[q(1-t)]/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1-t}{2} \right) a^{[q(1+t)]/2} b^{q(1-t)} dt \right] \\
&\qquad = \left[\frac{2L(a^q, b^q) - 2a^q}{q(\ln b - \ln a)} \right] \left| f'(a) \right|^q + \left[\frac{2b^q - 2L(a^q, b^q)}{q(\ln b - \ln a)} \right] \left| f'(b) \right|^q. \qquad (2.19)
\end{aligned}$$

Using (2.18) in (2.19), we get

$$\begin{aligned}
&\left(\int_0^1 [U(t)]^q \left| f'(U(t)) \right|^q dt \right)^{1/q} + \left(\int_0^1 [L(t)]^q \left| f'(L(t)) \right|^q dt \right)^{1/q} \\
&\leq 2^{1-2/q} \left(\left[\frac{L(a^q, b^q) - a^q}{q(\ln b - \ln a)} \right] \left| f'(a) \right|^q + \left[\frac{b^q - L(a^q, b^q)}{q(\ln b - \ln a)} \right] \left| f'(b) \right|^q \right)^{1/q} \qquad (2.20)
\end{aligned}$$

Applying (2.20) in (2.17), we obtain the required inequality (2.16). □

Corollary 5. *If the assumptions of Theorem 5 are satisfied and if* $g(x) = \frac{1}{\ln b - \ln a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \right| \le \frac{(\ln b - \ln a)^{1-1/q}}{2\cdot (4q)^{1/q}} \|g\|_\infty \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{ [L(a^q,b^q) - a^q] \left| f'(a) \right|^q + [b^q - L(a^q,b^q)] \left| f'(b)\right|^q \right\}^{1/q}, \quad (2.21)$$

Theorem 6. *Let* $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a,\, b \in I^\circ$ *with* $a < b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and geometrically symmetric to* $\sqrt{ab}$ *such that* $f' \in L([a,b])$*. If* $\left|f'\right|$ *is GA-convex on* $[a,b]$*, then the following inequality holds for* $q > 1$

$$\left| f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx \right| \le \frac{(\ln b - \ln a)^2 \left[L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q} \|g\|_\infty}{8}$$
$$\times \left\{ \left(b^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + a^{1/2}\left[{}_2F_1(-q,1;q+2;-1)\cdot \frac{1}{q+1}\right]^{1/q}\right)\left|f'(a)\right| + \left(a^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + b^{1/2}\left[{}_2F_1(-q,1;q+2;-1)\cdot \frac{1}{q+1}\right]^{1/q}\right)\left|f'(b)\right| \right\}, \quad (2.22)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ *and* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

Proof. From Lemma 1 and the GA-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\left| f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx \right|$$
$$\le \frac{\ln b - \ln a}{2}\int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x}dx\right)\left[U(t)\left|f'(U(t))\right| + L(t)\left|f'(L(t))\right|\right]dt$$
$$\le \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \int_0^1 \left[(1-t)U(t)\left|f'(U(t))\right| + (1-t)L(t)f'(L(t))\right]dt$$
$$\le \frac{(\ln b - \ln a)^2}{4}\|g\|_\infty \left\{ \int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[(1-t)\left(\frac{1-t}{2}\right)\left|f'(a)\right| + (1-t)\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt + \int_0^1 a^{(1+t)/2}b^{(1-t)/2} \times \left[(1-t)\left(\frac{1+t}{2}\right)\left|f'(a)\right| + (1-t)\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right\}. \quad (2.23)$$

Using Hölder integral inequality, we have

$$\int_0^1 a^{(1-t)/2} b^{(1+t)/2} \left[(1-t) \left(\frac{1-t}{2} \right) \left| f'(a) \right| + (1-t) \left(\frac{1+t}{2} \right) \left| f'(b) \right| \right] dt$$
$$\leq \left(\int_0^1 a^{[q(1-t)]/[2(q-1)]} b^{[q(1+t)]/[2(q-1)]} dt \right)^{1-1/q}$$
$$\times \left\{ \left[\int_0^1 (1-t)^q \left(\frac{1-t}{2} \right)^q dt \right]^{1/q} \left| f'(a) \right| + \left[\int_0^1 (1-t)^q \left(\frac{1+t}{2} \right)^q dt \right]^{1/q} \left| f'(b) \right| \right\}$$
$$= \frac{1}{2} \left[b^{q/[2(q-1)]} L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]} \right) \right]^{1-1/q} \left\{ \left(\frac{1}{2q+1} \right)^{1/q} \left| f'(a) \right| \right.$$
$$\left. + \left[{}_2F_1(-q,1;q+2;-1) \cdot \frac{1}{q+1} \right]^{1/q} \left| f'(b) \right| \right\}. \quad (2.24)$$

Similarly, we one have

$$\int_0^1 a^{(1+t)/2} b^{(1-t)/2} \left[t \left(\frac{1+t}{2} \right) \left| f'(a) \right| + t \left(\frac{1-t}{2} \right) \left| f'(b) \right| \right] dt$$
$$\leq \left(\int_0^1 a^{[q(1+t)]/[2(q-1)]} b^{[q(1-t)]/[2(q-1)]} dt \right)^{1-1/q}$$
$$\times \left\{ \left[\int_0^1 (1-t)^q \left(\frac{1+t}{2} \right)^q dt \right]^{1/q} \left| f'(a) \right| + \left[\int_0^1 (1-t)^q \left(\frac{1-t}{2} \right)^q dt \right]^{1/q} \left| f'(b) \right| \right\}$$
$$= \frac{1}{2} \left[a^{q/[2(q-1)]} L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]} \right) \right]^{1-1/q} \left\{ \left(\frac{1}{2q+1} \right)^{1/q} \left| f'(b) \right| \right.$$
$$\left. + \left[{}_2F_1(-q,1;q+2;-1) \cdot \frac{1}{q+1} \right]^{1/q} \left| f'(a) \right| \right\}. \quad (2.25)$$

Using (2.24) and (2.25) in (2.23), we obtain the required inequality (2.22). □

Corollary 6. *Under the assumptions of Theorem 6, if* $g(x) = \frac{1}{\ln b - \ln a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right|$$
$$\leq \frac{(\ln b - \ln a) \left[L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]} \right) \right]^{1-1/q}}{2}$$
$$\times \left\{ \left(b^{1/2} \left(\frac{1}{2q+1} \right)^{1/q} + a^{1/2} \left[{}_2F_1(-q,1;q+2;-1) \cdot \frac{1}{q+1} \right]^{1/q} \right) \left| f'(a) \right| \right.$$
$$\left. + \left(a^{1/2} \left(\frac{1}{2q+1} \right)^{1/q} + b^{1/2} \left[{}_2F_1(-q,1;q+2;-1) \cdot \frac{1}{q+1} \right]^{1/q} \right) \left| f'(b) \right| \right\}, \quad (2.26)$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

3. Applications to Special Means

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a geometrically-arithmetically convex

function and a geometrically symmetric function to construct inequalities for special means.

For positive numbers $a > 0$ and $b > 0$ with $a \neq b$

$$A(a,b) = \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2ab}{a+b}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ L(a,b), & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively. For further information on means, we refer the readers to [1, 19, 20] and the references therein.

Now let $f(x) = x^r$ for $x > 0$, $r \in \mathbb{R}$ with $r \neq 0$. Then

$$\begin{aligned} \left|f'\left(x^{\lambda}y^{1-\lambda}\right)\right|^q &= |r|^q \left[x^{q(r-1)}\right]^{\lambda}\left[y^{q(r-1)}\right]^{1-\lambda} \\ &\leq |r|^q \left[\lambda x^{q(r-1)} + (1-\lambda)\, y^{q(r-1)}\right] \end{aligned}$$

for $\lambda \in [0,1]$, $x,\ y > 0$ and $q \geq 1$. That is $\left|f'(x)\right|^q = |r|^q\, x^{q(r-1)}$ is geometrically-arithmetically convex on $[a,b]$ for $q \geq 1$ and $r \neq 1$, where $a,\ b > 0$.

Let $g : [a,b] \to \mathbb{R}_0$ be defined as

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, x \in [a,b].$$

It is obvious that

$$g\left(\frac{ab}{x}\right) = g(x)$$

for all $x \in [a,b]$. Hence $g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2$, $x \in [a,b]$ is geometrically symmetric with respect to $x = \sqrt{ab}$.

Now applications of our results are given in the following theorems to come.

Theorem 7. *Let* $0 < a < b$, $r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$ *and* $q \geq 1$. *Then*

$$\begin{aligned}
&\Big| 2G^{r-2}(a,b)\, L\left(a^2, b^2\right) + 2L\left(a^r, b^r\right) - 2G^r(a,b) \\
&\qquad - G^2(a,b)\, L\left(a^{r-2}, b^{r-2}\right) + \frac{L\left(a^{r+2}, b^{r+2}\right)}{G^2(a,b)} \Big| \\
&\leq \frac{(b-a)|r|}{2G(a,b)} \left\{ \left[\sqrt{b} L\left(\sqrt{a}, \sqrt{b}\right) - G(a,b) \right]^{1-1/q} \right. \\
&\times \left(\left[\frac{4\sqrt{b}L\left(\sqrt{a}, \sqrt{b}\right) - 4b}{\ln b - \ln a} - G(a,b) + 2b \right] A\left(a^{q(r-1)}, b^{q(r-1)}\right) \right. \\
&- \left. \left[b - \sqrt{b}L\left(\sqrt{a}, \sqrt{b}\right) \right] a^{q(r-1)} \right)^{1/q} + \left[G(a,b) - \sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) \right]^{1-1/q} \\
&\times \left(\left[\frac{4\sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) - 4a}{\ln b - \ln a} + G(a,b) - 2a \right] A\left(a^{q(r-1)}, b^{q(r-1)}\right) \right. \\
&\left. \left. + \left[a - \sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) \right] b^{q(r-1)} \right)^{1/q} \right\}. \qquad (3.1)
\end{aligned}$$

Proof. Applying Theorem 3 to the functions

$$f(x) = x^r \text{ for } x > 0, \; r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x} \right)^2, x \in [a, b]$$

we get the desired result. □

Corollary 7. *Suppose the assumptions of Theorem 7 are satisfied and if* $r = -1$, *the the following inequality holds*

$$\begin{aligned}
&\Big| \frac{2L\left(a^2, b^2\right) - 2G^2(a,b) + L(a,b)\, G(a,b)}{G^3(a,b)} \\
&\qquad - G^2(a,b)\, L\left(a^{-3}, b^{-3}\right) + 2L\left(a^{-1}, b^{-1}\right) \Big| \\
&\leq \frac{(b-a)}{2G(a,b)} \left\{ \left[\sqrt{b} L\left(\sqrt{a}, \sqrt{b}\right) - G(a,b) \right]^{1-1/q} \right. \\
&\times \left(\left[\frac{4\sqrt{b}L\left(\sqrt{a}, \sqrt{b}\right) - 4b}{\ln b - \ln a} - G(a,b) + 2b \right] A\left(a^{-2q}, b^{-2q}\right) \right. \\
&- \left. \left[b - \sqrt{b}L\left(\sqrt{a}, \sqrt{b}\right) \right] a^{-2q} \right)^{1/q} + \left[G(a,b) - \sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) \right]^{1-1/q} \\
&\times \left(\left[\frac{4\sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) - 4a}{\ln b - \ln a} + G(a,b) - 2a \right] A\left(a^{-2q}, b^{-2q}\right) \right. \\
&\left. \left. + \left[a - \sqrt{a}L\left(\sqrt{a}, \sqrt{b}\right) \right] b^{-2q} \right)^{1/q} \right\}. \qquad (3.2)
\end{aligned}$$

Corollary 8. *Under the assumptions of Theorem 7, the following inequality holds true for* $q = 1$

$$\begin{aligned}&\Big|2G^{r-2}(a,b)\,L\left(a^2,b^2\right)+2L\left(a^r,b^r\right)-2G^r(a,b)\\&\qquad -G^2(a,b)\,L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^2(a,b)}\Bigg|\\&\leq\frac{(b-a)|r|}{2G(a,b)}\left[\frac{8A\left(\sqrt{a},\sqrt{b}\right)L\left(\sqrt{a},\sqrt{b}\right)-8A(a,b)}{\ln b-\ln a}+2(b-a)\right.\\&\left.(r-2)(b-a)\,G^2(a,b)\,L_{r-3}^{r-3}(a,b)-\left(r-\frac{3}{2}\right)(b-a)\,G(a,b)\,L_{r-\frac{5}{2}}^{r-\frac{5}{2}}(a,b)\right].\end{aligned}\tag{3.3}$$

Corollary 9. *If we take* $r = -1$ *in Corollary 8, then the following inequality holds valid*

$$\begin{aligned}&\Bigg|\frac{2L\left(a^2,b^2\right)-2G^2(a,b)+L(a,b)\,G(a,b)}{G^3(a,b)}\\&\qquad -G^2(a,b)\,L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\Big|\\&\leq\frac{(b-a)}{2}\left[\frac{8A\left(\sqrt{a},\sqrt{b}\right)L\left(\sqrt{a},\sqrt{b}\right)-8A(a,b)}{(\ln b-\ln a)\,G(a,b)}+\frac{2(b-a)}{G(a,b)}\right.\\&\qquad\left.-3(b-a)\,G(a,b)\,L_{-4}^{-4}(a,b)+\frac{5}{2}(b-a)\,L_{\frac{7}{2}}^{\frac{7}{2}}(a,b)\right].\end{aligned}\tag{3.4}$$

Theorem 8. *Let* $0 < a < b$, $r \in \mathbb{R}\setminus\{-2,0,1,2\}$ *and* $q > 1$. *Then*

$$\begin{aligned}&\Big|2G^{r-2}(a,b)\,L\left(a^2,b^2\right)+2L\left(a^r,b^r\right)-2G^r(a,b)\\&\qquad -G^2(a,b)\,L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^2(a,b)}\Bigg|\\&\leq\frac{(\ln b-\ln a)^{1-1/q}(b-a)|r|}{4\cdot q^{1/q}G(a,b)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\Big\{b^{1/2}\Big(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]a^{q(r-1)}\\&\qquad+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{q(r-1)}\Big)^{1/q}\\&\qquad+a^{1/2}\Big(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]a^{q(r-1)}\\&\qquad\qquad+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{q(r-1)}\Big)^{1/q}\Big\}.\end{aligned}\tag{3.5}$$

Proof. Applying Theorem 4 to the functions

$$f(x) = x^r \text{ for } x > 0, r \in \mathbb{R}\setminus\{-2,0,1,2\}$$

and

$$g(x)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^2, x\in[a,b]$$

we get the desired result. □

Corollary 10. *Suppose the assumptions of Theorem 8 are fulfilled and if* $r=-1$, *the following inequality holds true*

$$\Bigg|\frac{2L\left(a^{2},b^{2}\right)-2G^{2}\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^{3}\left(a,b\right)} -G^{2}\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\Bigg|$$
$$\leq\frac{\left(\ln b-\ln a\right)^{1-1/q}\left(b-a\right)}{4\cdot q^{1/q}G\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]a^{-2q}+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}\right.$$
$$\left.+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]a^{-2q}+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}\right\}. \quad (3.6)$$

Theorem 9. *Let* $0<a<b$, $r\in\mathbb{R}\backslash\{-2,0,1,2\}$ *and* $q>1$. *Then*

$$\Bigg|2G^{r-2}\left(a,b\right)L\left(a^{2},b^{2}\right)+2L\left(a^{r},b^{r}\right)-2G^{r}\left(a,b\right) -G^{2}\left(a,b\right)L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^{2}\left(a,b\right)}\Bigg|\leq\frac{\left(b-a\right)^{2}|r|}{2^{2/q+1}L\left(a,b\right)G\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{rL\left(a^{qr},b^{qr}\right)-\left(r-1\right)L\left(a^{q(r-1)},b^{q(r-1)}\right)L\left(a^{q},b^{q}\right)\right\}^{1/q}. \quad (3.7)$$

Proof. Applying Theorem 5 to the functions

$$f(x)=x^{r}\text{ for }x>0,r\in\mathbb{R}\backslash\{-2,0,1,2\}$$

and

$$g\left(x\right)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^{2},x\in[a,b]$$

we get the desired result. □

Corollary 11. *Suppose the assumptions of Theorem 9 are satisfied and if* $r=-1$, *the the following inequality holds valid*

$$\Bigg|\frac{2L\left(a^{2},b^{2}\right)-2G^{2}\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^{3}\left(a,b\right)} -G^{2}\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\Bigg|$$
$$\leq\frac{\left(b-a\right)^{2}}{2^{2/q+1}G\left(a,b\right)L\left(a,b\right)}\left(\frac{2q-1}{q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[2L\left(a^{-2q},b^{-2q}\right)L\left(a^{-1},b^{-1}\right)-L\left(a^{-q},b^{-q}\right)\right]^{1/q}. \quad (3.8)$$

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