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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HA-CONVEX FUNCTIONS

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Abstract. Some new inequalities of Hermite-Hadamard type for HA-convex functions defined on positive intervals are given.

1. Introduction

Following [1] (see also [41]) we say that the function $f : I \subset R \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t) f(x) + t f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)x+ty) \leq \frac{f(x) f(y)}{(1-t) f(y) + t f(x)}, \tag{1.2}$$

we call the class of functions satisfying (1.1) as *HA-convex functions*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

The following simple but important fact is as follows:

Criterion 1. *If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

For a convex function $h : [c,d] \to \mathbb{R}$, the following inequality is well known in the literature as the Hermite-Hadamard inequality

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d h(t)\,dt \leq \frac{h(c)+h(d)}{2} \tag{1.3}$$

for any convex function $h : [c,d] \to \mathbb{R}$.

For related results, see [1]-[18], [21]-[26], [27]-[37] and [38]-[49].

If we write the Hermite-Hadamard inequality for the convex function $g(t) = f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b}, \frac{1}{a}\right]$, then we have

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt \leq \frac{f(b)+f(a)}{2}. \tag{1.4}$$

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Using the change of variable $s=\frac{1}{t}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt=\int_a^b \frac{f(s)}{s^2} ds$$

and by (1.4) we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(s)}{s^2} ds \leq \frac{f(b)+f(a)}{2}. \tag{1.5}$$

The inequality (1.5) has been obtained in a different manner in [41] by I. Işcan.

The *identric mean* $I(a,b)$ is defined by

$$I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b):=\frac{b-a}{\ln b-\ln a}.$$

In the recent paper [25] we established the following inequalities for *HA*-convex functions:

Theorem 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b)) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{(L(a,b)-a)\,bf(b)+(b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}, \tag{1.6}$$

and

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{bf(b)+af(a)}{2}. \tag{1.7}$$

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for *HA*-convex functions. Some applications for special means are also given.

2. Further Results

We start with the following characterization of *HA*-convex functions.

Theorem 3. *Let $f,h:[a,b]\subset(0,\infty)\to\mathbb{R}$ so that $h(t)=tf(t)$ for $t\in[a,b]$. Then f is HA-convex on the interval $[a,b]$ if and only if h is convex on $[a,b]$.*

Proof. Assume that f is *HA*-convex on the interval $[a,b]$. Then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, $g(t)=f\left(\frac{1}{t}\right)$ is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. By replacing t with $\frac{1}{t}$ we have $f(t)=g\left(\frac{1}{t}\right)$.

If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of g on $\left[\frac{1}{b}, \frac{1}{a}\right]$, we have

$$\begin{aligned}
h\left((1-\lambda)x+\lambda y\right) &= \left[(1-\lambda)x+\lambda y\right] f\left((1-\lambda)x+\lambda y\right) \\
&= \left[(1-\lambda)x+\lambda y\right] g\left(\frac{1}{(1-\lambda)x+\lambda y}\right) \\
&= \left[(1-\lambda)x+\lambda y\right] g\left(\frac{(1-\lambda)x\frac{1}{x}+\lambda y\frac{1}{y}}{(1-\lambda)x+\lambda y}\right) \\
&\le \left[(1-\lambda)x+\lambda y\right] \frac{(1-\lambda)xg\left(\frac{1}{x}\right)+\lambda yg\left(\frac{1}{y}\right)}{(1-\lambda)x+\lambda y} \\
&= (1-\lambda)xg\left(\frac{1}{x}\right)+\lambda yg\left(\frac{1}{y}\right) \\
&= (1-\lambda)xf(x)+\lambda yf(y) = (1-\lambda)h(x)+\lambda h(y),
\end{aligned}$$

which shows that h is convex on $[a,b]$.

We have $f(t) = \frac{h(t)}{t}$ for $t \in [a,b]$. If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of h on $[a,b]$, we have

$$\begin{aligned}
f\left(\frac{xy}{\lambda x+(1-\lambda)y}\right) &= \frac{h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)}{\frac{xy}{\lambda x+(1-\lambda)y}} \\
&= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right) \\
&= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{1}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}}\right) \\
&= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{(1-\lambda)\frac{1}{x}x+\lambda\frac{1}{y}y}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}}\right) \\
&\le \frac{\lambda x+(1-\lambda)y}{xy} \frac{(1-\lambda)\frac{1}{x}h(x)+\lambda\frac{1}{y}h(y)}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}} \\
&= (1-\lambda)\frac{1}{x}h(x)+\lambda\frac{1}{y}h(y) = (1-\lambda)f(x)+\lambda f(y),
\end{aligned}$$

which shows that f is *HA*-convex on the interval $[a,b]$. □

Remark 1. *If f is HA-convex on the interval $[a,b]$, then by Theorem 3 the function $h(t) = tf(t)$ is convex on $[a,b]$ and by Hermite-Hadamard inequality (1.3) we get the inequality (1.7). This gives a direct proof of (1.7) and it is simpler than in* [25].

In 1994, [11] (see also [32, p. 22]) we proved the following refinement of Hermite-Hadamard inequality. For a direct proof that is different from the one in [11], see the recent paper [24].

Lemma 1. *Let $p:[c,d]\to\mathbb{R}$ be a convex function on $[c,d]$. Then for any division $c=y_0<y_1<...<y_{n-1}<y_n=d$ with $n\geq 1$ we have the inequalities*

$$
\begin{aligned}
p\left(\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\sum_{i=0}^{n-1}(y_{i+1}-y_i)\,p\left(\frac{y_{i+1}+y_i}{2}\right) \\
&\leq \frac{1}{d-c}\int_c^d p(y)\,dy \leq \frac{1}{d-c}\sum_{i=0}^{n-1}(y_{i+1}-y_i)\frac{p(y_i)+p(y_{i+1})}{2} \\
&\leq \frac{1}{2}\left[p(c)+p(d)\right].
\end{aligned} \tag{2.1}
$$

We can state the following result:

Theorem 4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ we have the inequalities*

$$
\begin{aligned}
\frac{a+b}{2}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)}\sum_{i=0}^{n-1}\left(x_{i+1}^2-x_i^2\right)f\left(\frac{x_{i+1}+x_i}{2}\right) \\
&\leq \frac{1}{b-a}\int_a^b xf(x)\,dx \\
&\leq \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\frac{x_if(x_i)+x_{i+1}f(x_{i+1})}{2} \\
&\leq \frac{1}{2}\left[af(a)+bf(b)\right].
\end{aligned} \tag{2.2}
$$

Follows by Lemma 1 for the convex function $p(x)=xf(x)$, $x\in[a,b]$.

If we take $n=2$ and $x\in[a,b]$, then by (2.2) we have

$$
\begin{aligned}
&\frac{a+b}{2}f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2(b-a)}\left[\left(x^2-a^2\right)f\left(\frac{x+a}{2}\right)+\left(b^2-x^2\right)f\left(\frac{x+b}{2}\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b tf(t)\,dt \\
&\leq \frac{1}{2(b-a)}\left[(b-a)\,xf(x)+(x-a)\,af(a)+(b-x)\,bf(b)\right] \\
&\leq \frac{1}{2}\left[af(a)+bf(b)\right].
\end{aligned} \tag{2.3}
$$

If in this inequality we choose $x = \frac{a+b}{2}$, then we get the inequality

$$\begin{aligned}
&\frac{a+b}{2} f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2(b-a)}\left[\frac{b+3a}{4} f\left(\frac{b+3a}{4}\right)+\frac{a+3b}{4} f\left(\frac{a+3b}{4}\right)\right] \\
&\leq \frac{1}{b-a} \int_a^b t f(t)\, dt \\
&\leq \frac{1}{2}\left[\frac{a+b}{2} f\left(\frac{a+b}{2}\right)+\frac{a f(a)+b f(b)}{2}\right] \leq \frac{1}{2}[a f(a)+b f(b)] .
\end{aligned} \tag{2.4}$$

If we take in (2.3) $x = \frac{2ab}{a+b}$, then we get

$$\begin{aligned}
&\frac{a+b}{2} f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{4(a+b)^2}\left[a^2(a+3b) f\left(\frac{a(a+3b)}{2(a+b)}\right)+b^2(3a+b) f\left(\frac{b(3a+b)}{2(a+b)}\right)\right] \\
&\leq \frac{1}{b-a} \int_a^b t f(t)\, dt \\
&\leq \frac{1}{a+b}\left[a b f\left(\frac{2ab}{a+b}\right)+\frac{a^2 f(a)+b^2 f(b)}{2}\right] \leq \frac{1}{2}[a f(a)+b f(b)] .
\end{aligned} \tag{2.5}$$

We also have:

Theorem 5. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a HA-convex function on the interval $[a, b]$. Then for any division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ with $n \geq 1$ we have the inequalities*

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{b-a} \sum_{j=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1} x_j}\right) f\left(\frac{2 x_{j+1} x_j}{x_{j+1}+x_j}\right) \\
&\leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\
&\leq \frac{ab}{b-a} \sum_{i=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1} x_j}\right) \frac{f(x_j)+f(x_{j+1})}{2} \leq \frac{f(b)+f(a)}{2} .
\end{aligned} \tag{2.6}$$

Proof. Consider the convex function $p(x) = f\left(\frac{1}{x}\right)$ that is convex on the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$. The division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ with $n \geq 1$ produces the division $y_i = \frac{1}{x_{n-i}}$, $i \in \{0, ..., n\}$ of the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Using the inequality (2.1) we get

$$f\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right) \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right) f\left(\frac{1}{\frac{\frac{1}{x_{n-i-1}}+\frac{1}{x_{n-i}}}{2}}\right) \tag{2.7}$$
$$\le \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt$$
$$\le \frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right)\frac{f\left(\frac{1}{\frac{1}{x_{n-i-1}}}\right)+f\left(\frac{1}{\frac{1}{x_{n-i}}}\right)}{2}$$
$$\le \frac{1}{2}\left[f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)\right]$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right) f\left(\frac{2x_{n-i-1}x_{n-i}}{x_{n-i}+x_{n-i-1}}\right) \tag{2.8}$$
$$\le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt$$
$$\le \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right)\frac{f\left(x_{n-i-1}\right)+f\left(x_{n-i}\right)}{2}$$
$$\le \frac{1}{2}\left[f\left(b\right)+f\left(a\right)\right].$$

By re-indexing the sums and taking into account that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt = \int_a^b \frac{f\left(x\right)}{x^2}dx$$

we obtain the desired result (2.6). □

Remark 2. *If we take $n=2$ and $x\in[a,b]$, then by (2.6) we have, after appropriate calculations, that*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{x}\left[\frac{(x-a)\,bf\left(\frac{2ax}{a+x}\right)+(b-x)\,af\left(\frac{2xb}{x+b}\right)}{b-a}\right] \tag{2.9}$$
$$\le \frac{ab}{b-a}\int_a^b \frac{f\left(x\right)}{x^2}dx$$
$$\le \frac{1}{2}\left[f\left(x\right)+\frac{(x-a)\,bf\left(a\right)+(b-x)\,af\left(b\right)}{x\left(b-a\right)}\right]$$
$$\le \frac{f\left(b\right)+f\left(a\right)}{2}.$$

If we take in (2.9) $x=\frac{2ab}{a+b}\in[a,b]$*, then we get*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right]\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.10}$$

If we take in (2.9) $x=\frac{a+b}{2}\in[a,b]$*, then we get*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{bf\left(\frac{a(a+b)}{3a+b}\right)+af\left(\frac{b(a+b)}{a+3b}\right)}{a+b}\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{bf(a)+af(b)}{a+b}\right]\leq\frac{f(b)+f(a)}{2}. \tag{2.11}$$

3. Related Results

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f:I\to\mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $f'_-(x)\leq f'_+(x)\leq f'_-(y)\leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f:I\to\mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi:I\to[-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right)\subset\mathbb{R}$ and

$$f(x)\geq f(a)+(x-a)\varphi(a)\ \text{ for any } x,a\in I. \tag{3.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+\in\partial f$ and if $\varphi\in\partial f$, then

$$f'_-(x)\leq\varphi(x)\leq f'_+(x)\ \text{ for any } x\in\mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f=\{f'\}$.

Lemma 2. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be an HA-convex function on the interval* $[a,b]$*. Then* f *has lateral derivatives in every point of* (a,b) *and*

$$f(t)-f(s)\geq sf'_\pm(s)\left(1-\frac{s}{t}\right) \tag{3.2}$$

for any $s\in(a,b)$ *and* $t\in[a,b]$*.*

Also, we have

$$f(t)-f(a)\geq af'_+(a)\left(1-\frac{a}{t}\right) \tag{3.3}$$

and

$$f(t)-f(b)\geq bf'_-(b)\left(1-\frac{b}{t}\right) \tag{3.4}$$

for any $t \in [a,b]$ provided the lateral derivatives $f'_+(a)$ and $f'_-(b)$ are finite.

Proof. If f is *HA*-convex function on the interval $[a,b]$, then the function $h(t) = tf(t)$ is convex on $[a,b]$, therefore the function f has lateral derivatives in each point of (a,b) and

$$h'_\pm(t) = f(t) + tf'_\pm(t)$$

for any $t \in (a,b)$. Also, if $f'_+(a)$ and $f'_-(b)$ are finite then

$$h'_+(a) = f(a) + af'_+(a) \text{ and } h'_-(b) = f(b) + bf'_-(b).$$

Writing the gradient inequality for the convex function h, namely

$$h(t) - h(s) \geq h'_\pm(s)(t-s)$$

for any $s \in (a,b)$ and $t \in [a,b]$, we have

$$tf(t) - sf(s) \geq \left[f(s) + sf'_\pm(s)\right](t-s) = f(s)(t-s) + sf'_\pm(s)(t-s)$$

that is equivalent to

$$tf(t) - tf(s) \geq sf'_\pm(s)(t-s)$$

for any $s \in (a,b)$ and $t \in [a,b]$.

Now, by dividing with $t > 0$ we get the desired result (3.2).

The rest follows by the corresponding properties of convex function h. □

We use the following results obtained by the author in [19] and [20]

Lemma 3. *Let $h : [\alpha,\beta] \to \mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\begin{aligned} (3.5)\qquad &\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ &\leq \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha) \end{aligned}$$

and

$$\begin{aligned} (3.6)\qquad &\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ &\leq \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha). \end{aligned}$$

The constant $\frac{1}{8}$ is best possible in (3.5) and (3.6).

The following result holds:

Theorem 6. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned} (3.7)\qquad &\frac{1}{16}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b^2-a^2) \\ &\leq \frac{af(a)+bf(b)}{2} - \frac{1}{b-a}\int_a^b tf(t)\,dt \\ &\leq \frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b) - af'_+(a)\right](b-a) \end{aligned}$$

and

$$\frac{1}{16}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]\left(b^{2}-a^{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} tf(t)\,dt-\frac{a+b}{2}f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a). \tag{3.8}$$

Proof. Making use of inequality (3.5) in Lemma 3 for the convex function $h(t)=tf(t)$ we have

$$\frac{1}{8}\left[\frac{a+b}{2}f'_{+}\left(\frac{a+b}{2}\right)-\frac{a+b}{2}f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \leq \frac{af(a)+bf(b)}{2}-\frac{1}{b-a}\int_{a}^{b} tf(t)\,dt \leq \frac{1}{8}\left[f(b)+bf'_{-}(b)-f(a)-af'_{+}(a)\right](b-a),$$

which proves the inequality (3.7).

The inequality (3.8) follows by (3.6). □

Corollary 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable HA-convex function on the interval $[a,b]$. Then we have*

$$0 \leq \frac{af(a)+bf(b)}{2}-\frac{1}{b-a}\int_{a}^{b} tf(t)\,dt \leq \frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a) \tag{3.9}$$

and

$$0 \leq \frac{1}{b-a}\int_{a}^{b} tf(t)\,dt-\frac{a+b}{2}f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a). \tag{3.10}$$

We remark that from (3.9) we have

$$\frac{(3a+b)f(a)+(a+3b)f(b)}{8}-\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a) \leq \frac{1}{b-a}\int_{a}^{b} tf(t)\,dt \leq \frac{af(a)+bf(b)}{2} \tag{3.11}$$

and from (3.10) we have

$$\frac{a+b}{2}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} tf(t)\,dt \leq \frac{a+b}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{8}\left[f(b)-f(a)\right](b-a) +\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a). \tag{3.12}$$

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

The following result also holds:

Theorem 7. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$.*

(i) If $bf(b) - af(a) \neq \int_a^b f(s)\,ds$ and

$$\alpha_f := \frac{\int_a^b s^2 f'(s)\,ds}{\int_a^b sf'(s)\,ds} = \frac{b^2 f(b) - a^2 f(a) - 2\int_a^b sf(s)\,ds}{bf(b) - af(a) - \int_a^b f(s)\,ds} \in [a,b] \tag{3.13}$$

then

$$f(\alpha_f) \geq \frac{1}{b-a}\int_a^b f(s)\,ds. \tag{3.14}$$

(ii) If $f(b) \neq f(a)$ and

$$\beta_f = \frac{\int_a^b sf'(s)\,ds}{\int_a^b f'(s)\,ds} = \frac{bf(b) - af(a) - \int_a^b f(s)\,ds}{f(b) - f(a)} \in [a,b] \tag{3.15}$$

then

$$f(\beta_f) \geq \frac{1}{\ln b - \ln a}\int_a^b f(s)\,ds. \tag{3.16}$$

(iii) If $af(b) \neq bf(a)$ and

$$\gamma_f := \frac{(f(b) - f(a))\,ab}{af(b) - bf(a)} \in [a,b] \tag{3.17}$$

then

$$f(\gamma_f) \geq \frac{2ab}{b-a}\int_a^b \frac{f(s)}{s^2}ds. \tag{3.18}$$

Proof. We know that if $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$ then the functions is differentiable except for at most countably many points. Then, from (3.2) we have

$$f(t) - f(s) \geq sf'(s)\left(1 - \frac{s}{t}\right) \tag{3.19}$$

for any $t \in [a,b]$ and almost every $s \in (a,b)$.

(i) If we take the Lebesgue integral mean in (3.19), then we get

$$f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \geq \frac{1}{b-a}\int_a^b sf'(s)\,ds - \frac{1}{t}\frac{1}{b-a}\int_a^b s^2 f'(s)\,ds \tag{3.20}$$

for any $t \in [a,b]$.

If we take $t = \alpha_f$ in (3.20) then we get the desired inequality (3.14).

(ii) If we divide the inequality (3.19) by s then we get

$$\frac{1}{s}f(t)-\frac{f(s)}{s}\geq f'(s)-\frac{1}{t}sf'(s) \tag{3.21}$$

for any $t\in[a,b]$ and almost every $s\in(a,b)$.

If we take the Lebesgue integral mean in (3.21), then we get

$$\begin{aligned}&f(t)\frac{1}{b-a}\int_a^b\frac{1}{s}ds-\frac{1}{b-a}\int_a^b\frac{f(s)}{s}ds\\&\geq\frac{1}{b-a}\int_a^b f'(s)\,ds-\frac{1}{t}\frac{1}{b-a}\int_a^b sf'(s)\,ds\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{f(t)}{L(a,b)}-\frac{1}{b-a}\int_a^b\frac{f(s)}{s}ds\\&\geq\frac{f(b)-f(a)}{b-a}-\frac{1}{t}\frac{bf(b)-af(a)-\int_a^b f(s)\,ds}{b-a}\end{aligned} \tag{3.22}$$

for any $t\in[a,b]$

If we take $t=\beta_f$ in (3.22) then we get the desired result (3.16).

(iii) If we divide the inequality (3.19) by s^2 then we get

$$\frac{1}{s^2}f(t)-\frac{f(s)}{s^2}\geq\frac{f'(s)}{s}-\frac{1}{t}f'(s) \tag{3.23}$$

for any $t\in[a,b]$ and almost every $s\in(a,b)$.

If we take the Lebesgue integral mean in (3.23), then we get

$$\begin{aligned}&f(t)\frac{1}{b-a}\int_a^b\frac{1}{s^2}ds-\frac{1}{b-a}\int_a^b\frac{f(s)}{s^2}ds\\&\geq\frac{1}{b-a}\int_a^b\frac{f'(s)}{s}ds-\frac{1}{t}\frac{1}{b-a}\int_a^b f'(s)\,ds,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&f(t)\frac{1}{ab}-\frac{1}{b-a}\int_a^b\frac{f(s)}{s^2}ds\\&\geq\frac{1}{b-a}\left[\frac{f(b)}{b}-\frac{f(a)}{a}+\int_a^b\frac{f(s)}{s^2}ds\right]-\frac{1}{t}\frac{f(b)-f(a)}{b-a}\end{aligned}$$

or, to

$$\begin{aligned}&f(t)\frac{1}{ab}-\frac{2}{b-a}\int_a^b\frac{f(s)}{s^2}ds\\&\geq\frac{1}{b-a}\frac{af(b)-bf(a)}{ba}-\frac{1}{t}\frac{f(b)-f(a)}{b-a}.\end{aligned}$$

□

Remark 3. *We observe that a sufficient condition for (3.13) and (3.15) to hold is that f is increasing on $[a,b]$. If $f(a)<0<f(b)$, then the inequality (3.17) also holds.*

We also have the following result:

Theorem 8. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then we have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{a+b-t}dt\leq\frac{af(a)+bf(b)}{a+b}. \tag{3.24}$$

Proof. Since the function $h(t)=tf(t)$ is convex, then we have

$$\frac{x+y}{2}f\left(\frac{x+y}{2}\right)\leq\frac{xf(x)+yf(y)}{2}$$

for any $x,y\in[a,b]$.

If we divide this inequality by $xy>0$ we get

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}\left(\frac{f(x)}{y}+\frac{f(y)}{x}\right), \tag{3.25}$$

for any $x,y\in[a,b]$.

If we replace x by $(1-t)a+tb$ and y by $ta+(1-t)b$ in (3.25), then we get

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{(1-t)a+tb}+\frac{1}{ta+(1-t)b}\right)f\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{2}\left(\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\right),\end{aligned} \tag{3.26}$$

for any $t\in[0,1]$.

Integrating (3.26) on $[0,1]$ over t we get

$$\begin{aligned}&\frac{1}{2}\left(\int_0^1\frac{1}{(1-t)a+tb}dt+\int_0^1\frac{1}{ta+(1-t)b}dt\right)f\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{2}\left(\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}dt\right).\end{aligned} \tag{3.27}$$

Observe that, by the appropriate change of variable,

$$\int_0^1\frac{1}{(1-t)a+tb}dt=\int_0^1\frac{1}{ta+(1-t)b}dt=\frac{1}{b-a}\int_a^b\frac{du}{u}=\frac{\ln b-\ln a}{b-a}$$

and

$$\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt=\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}=\frac{1}{b-a}\int_a^b\frac{f(u)}{a+b-u}du$$

and by (3.27) we get the first inequality in (3.24).

From the convexity of h we also have

$$((1-t)a+tb)f((1-t)a+tb)\leq(1-t)af(a)+tbf(b)$$

and

$$(ta+(1-t)b)f(ta+(1-t)b)\leq taf(a)+(1-t)bf(b)$$

for any $t\in[0,1]$.

Add these inequalities to get

$$\begin{aligned}&((1-t)a+tb)f((1-t)a+tb)+(ta+(1-t)b)f(ta+(1-t)b)\\&\leq af(a)+bf(b)\end{aligned}$$

for any $t\in[0,1]$.

If we divide this inequality by $((1-t)a+tb)(ta+(1-t)b)$, then we get

$$\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\le\frac{af(a)+bf(b)}{((1-t)a+tb)(ta+(1-t)b)} \tag{3.28}$$

for any $t\in[0,1]$.

If we integrate the inequality (3.28) over t on $[0,1]$, then we obtain

$$\begin{aligned}\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}dt \\ \le[af(a)+bf(b)]\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}.\end{aligned} \tag{3.29}$$

Since

$$\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}=\frac{1}{b-a}\int_a^b\frac{du}{u(a+b-u)}$$

and

$$\frac{1}{u(a+b-u)}=\frac{1}{a+b}\left(\frac{1}{u}+\frac{1}{a+b-u}\right),$$

then

$$\begin{aligned}\int_a^b\frac{du}{u(a+b-u)}&=\frac{1}{a+b}\int_a^b\left(\frac{1}{u}+\frac{1}{a+b-u}\right)du\\&=\frac{2}{a+b}(\ln b-\ln a).\end{aligned}$$

By (3.29) we then have

$$\frac{2}{b-a}\int_a^b\frac{f(u)}{a+b-u}du\le2\left[\frac{af(a)+bf(b)}{a+b}\right]\frac{\ln b-\ln a}{b-a},$$

which proves the second inequality in (3.24). □

4. Applications

We consider the *arithmetic mean* $A(a,b)=\frac{a+b}{2}$, the *geometric mean* $G(a,b)=\sqrt{ab}$ and *harmonic mean* $H(a,b)=\frac{2ab}{a+b}$ for the positive numbers $a,b>0$.

If we use the inequalities (2.6) for the HA-convex function $f(t)=t$ on the interval $[a,b]\subset(0,\infty)$ then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\ge1$ we have the inequalities

$$\begin{aligned}\frac{2ab}{a+b}\le\frac{2ab}{b-a}\sum_{j=0}^{n-1}\frac{x_{j+1}-x_j}{x_{j+1}+x_j}\le\frac{G^2(a,b)}{L(a,b)}\\ \le\frac{ab}{2(b-a)}\sum_{i=0}^{n-1}\frac{x_{j+1}^2-x_j^2}{x_{j+1}x_j}\le A(a,b).\end{aligned} \tag{4.1}$$

In particular, we have

$$\begin{aligned}H(a,b)\le2ab\left(\frac{1}{a+3b}+\frac{1}{3a+b}\right)\le\frac{G^2(a,b)}{L(a,b)}\\ \le\frac{H(a,b)+A(a,b)}{2}(\le A(a,b)).\end{aligned} \tag{4.2}$$

Consider the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$. Observe that $g(t)=f\left(\frac{1}{t}\right)=-t\ln t$, which shows that f is HA-concave on $(0,\infty)$.

If we write the inequality (2.4) for the HA-concave function $f(t)=\frac{\ln t}{t}$ on $(0,\infty)$, then we have for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ that

$$A(a,b)\geq\prod_{i=0}^{n-1}\left(\frac{x_{i+1}+x_i}{2}\right)^{\frac{x_{i+1}-x_i}{b-a}}\geq I(a,b)\geq\prod_{i=0}^{n-1}(x_ix_{i+1})^{\frac{x_{i+1}-x_i}{2(b-a)}}\geq G(a,b). \tag{4.3}$$

In particular, we have

$$A(a,b)\geq\left(\frac{b+3a}{4}\right)^{\frac{1}{2(b-a)}}\left(\frac{a+3b}{4}\right)^{\frac{1}{2(b-a)}}\geq I(a,b)\geq\sqrt{A(a,b)\,G(a,b)}\,(\geq G(a,b)). \tag{4.4}$$

The interested reader may apply the above inequalities for other HA-convex functions such as $f(t)=\frac{h(t)}{t}$, $t>0$ with h any convex function on an interval $I\subset(0,\infty)$ etc...The details are omitted.

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