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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR LOG-CONVEX FUNCTIONS

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Abstract. Some new inequalities of Hermite-Hadamard type for log-convex functions defined on real intervals are given.

1. Introduction

A function $f : I \to [0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let us recall the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I, $a, b \in I$ and $a < b$.

For related results, see [1]-[22], [25]-[28], [29]-[39] and [40]-[51].

Note that if we apply the above inequality for the log-convex functions $f : I \to (0, \infty)$, we have that

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{b-a} \int_a^b \ln f(x)\, dx \leq \frac{\ln f(a) + \ln f(b)}{2}, \tag{1.3}$$

from which we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x)\, dx\right] \leq \sqrt{f(a) f(b)}, \tag{1.4}$$

which is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result [31].

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Theorem 1. *Let $f : I \to [0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \sqrt{f(x)f(a+b-x)}dx \le \sqrt{f(a)f(b)}. \tag{1.5}$$

The inequality between the first and second term in (1.5) may be improved as follows [31]. A different upper bound for the middle term in (1.5) can be also provided.

Theorem 2. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\ &\le \frac{1}{b-a}\int_a^b \sqrt{f(x)f(a+b-x)}dx \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx \le L(f(a), f(b)), \end{aligned} \tag{1.6}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$L(p,q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \ne q \text{ and } L(p,p) := p.$$

The last inequality in (1.6) was obtained in a different context in [41].

As shown in [57], the following result also holds:

Theorem 3. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.7}$$

The following result improving the classical first Hermite-Hadamard inequality for differentiable log-convex functions also hold [15]:

Theorem 4. *Let $f : I \to (0, \infty)$ be a differentiable log-convex function on the interval of real numbers $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ with $a < b$. Then the following inequalities hold:*

$$\begin{aligned} &\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \\ &\ge L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \ge 1. \end{aligned} \tag{1.8}$$

The second Hermite-Hadamard inequality can be improved as follows [15].

Theorem 5. *Let $f : I \to \mathbb{R}$ be as in Theorem 4. Then we have the inequality:*

$$\frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f(x)\,dx} \geq 1+\log\left[\frac{\int_a^b f(x)\,dx}{\int_a^b f(x)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2}-x\right)\right]dx}\right] \tag{1.9}$$

$$\geq 1+\log\left[\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)}\right] \geq 1.$$

Motivated by the above results, we establish in this paper some new inequalities for log-convex functions, some of them improving earlier results. Applications for special means are also provided.

2. New Inequalities

The following refinement of the Hermite-Hadamard inequality holds:

Lemma 1. *Let $h : [a,b] \to \mathbb{R}$ be a convex function and $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ an arbitrary division of $[a,b]$ with $n \geq 2$. Then*

$$\begin{aligned} h\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\sum_{i=0}^{n-1} h\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i) \\ &\leq \frac{1}{b-a}\int_a^b h(x)\,dx \\ &\leq \frac{1}{b-a}\sum_{i=0}^{n-1}\frac{h(x_i)+h(x_{i+1})}{2}(x_{i+1}-x_i) \leq \frac{h(a)+h(b)}{2}. \end{aligned} \tag{2.1}$$

The inequality (2.1) was obtained in 1994 as a particular case of a more general result, see [14] and also mentioned in [34, p. 22]. For a direct proof, see the recent paper [27].

Theorem 6. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function on $[a,b]$ and $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ an arbitrary division of $[a,b]$ with $n \geq 1$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \prod_{i=1}^{n-1}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}} \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right) \\ &\leq \prod_{i=1}^{n-1}\left[\sqrt{f(x_i)f(x_{i+1})}\right]^{\frac{x_{i+1}-x_i}{b-a}} \leq \sqrt{f(a)f(b)}. \end{aligned} \tag{2.2}$$

Proof. If we write the inequality (2.1) for the function $h = \ln f$ then we get

$$\begin{aligned}\ln f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\ln f\left(\frac{x_i+x_{i+1}}{2}\right)\\ &\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx\\ &\leq \frac{1}{b-a}\sum_{i=0}^{n-1}\frac{\ln f(x_i)+\ln f(x_{i+1})}{2}(x_{i+1}-x_i) \leq \frac{\ln f(a)+\ln f(b)}{2}.\end{aligned}$$

This inequality is equivalent to

$$\begin{aligned}\ln f\left(\frac{a+b}{2}\right) &\leq \ln\left(\prod_{i=1}^{n-1}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}}\right)\\ &\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx\\ &\leq \ln\left(\prod_{i=1}^{n-1}\left[\sqrt{f(x_i)f(x_{i+1})}\right]^{\frac{x_{i+1}-x_i}{b-a}}\right) \leq \ln\sqrt{f(a)f(b)}.\end{aligned} \tag{2.3}$$

This inequality is of interest in itself.

If we take the exponential in (2.3) we get the desired result (2.2). □

Corollary 1. *Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$ and $x \in [a, b]$, then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \left[f\left(\frac{a+x}{2}\right)\right]^{\frac{x-a}{b-a}}\left[f\left(\frac{x+b}{2}\right)\right]^{\frac{b-x}{b-a}}\\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right)\\ &\leq \left[\sqrt{f(a)}\right]^{\frac{x-a}{b-a}}\sqrt{f(x)}\left[\sqrt{f(b)}\right]^{\frac{b-x}{b-a}} \leq \sqrt{f(a)f(b)}\end{aligned} \tag{2.4}$$

and, equivalently

$$\begin{aligned}\ln f\left(\frac{a+b}{2}\right) &\leq \frac{x-a}{b-a}\ln f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}\ln f\left(\frac{x+b}{2}\right)\\ &\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx\\ &\leq \frac{1}{2}\left[\ln f(x)+\frac{(x-a)\ln f(a)+(b-x)\ln f(b)}{b-a}\right]\\ &\leq \frac{\ln f(a)+\ln f(b)}{2}.\end{aligned} \tag{2.5}$$

Remark 1. *If we take in (2.5) $x=\frac{a+b}{2}$, then we get*

$$\ln f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left[\ln f\left(\frac{3a+b}{4}\right)+\ln f\left(\frac{a+3b}{4}\right)\right] \tag{2.6}$$
$$\le\frac{1}{b-a}\int_a^b\ln f(x)\,dx$$
$$\le\frac{1}{2}\left[\ln f\left(\frac{a+b}{2}\right)+\frac{\ln f(a)+\ln f(b)}{2}\right]\le\frac{\ln f(a)+\ln f(b)}{2}.$$

From the second inequality in (2.6) we get

$$0\le\frac{1}{b-a}\int_a^b\ln f(x)\,dx-\ln f\left(\frac{a+b}{2}\right)$$
$$\le\frac{\ln f(a)+\ln f(b)}{2}-\frac{1}{b-a}\int_a^b\ln f(x)\,dx,$$

which shows that the integral term in (1.3) is closer to the left side than to the right side of that inequality.

We also have the particular inequalities:

$$\ln f\left(\frac{a+b}{2}\right) \tag{2.7}$$
$$\le\frac{1}{\sqrt{b}+\sqrt{a}}\left[\sqrt{a}\ln f\left(\frac{\sqrt{a}\left(\sqrt{a}+\sqrt{b}\right)}{2}\right)+\sqrt{b}\ln f\left(\frac{\sqrt{b}\left(\sqrt{a}+\sqrt{b}\right)}{2}\right)\right]$$
$$\le\frac{1}{b-a}\int_a^b\ln f(y)\,dy$$
$$\le\frac{1}{2}\left[\frac{\sqrt{b}\ln f(b)+\sqrt{a}\ln f(a)}{\sqrt{b}+\sqrt{a}}+\ln f\left(\sqrt{ab}\right)\right]\le\frac{\ln f(a)+\ln f(b)}{2}$$

and

$$\ln f\left(\frac{a+b}{2}\right) \tag{2.8}$$
$$\le\frac{1}{a+b}a\ln f\left(a\frac{3a+b}{2(a+b)}\right)+\frac{1}{a+b}b\ln f\left(b\frac{a+3b}{2(a+b)}\right)$$
$$\le\frac{1}{b-a}\int_a^b\ln f(y)\,dy$$
$$\le\frac{1}{2}\left[\frac{b\ln f(b)+a\ln f(a)}{a+b}+\ln f\left(\frac{2ab}{a+b}\right)\right]\le\frac{\ln f(a)+\ln f(b)}{2}.$$

The following reverses of the Hermite-Hadamard inequality hold [23] and [24]:

Lemma 2. *Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$0\leq\frac{1}{8}\left[h_{+}\left(\frac{a+b}{2}\right)-h_{-}\left(\frac{a+b}{2}\right)\right](b-a)\leq\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)\,dx\leq\frac{1}{8}\left[h_{-}(b)-h_{+}(a)\right](b-a)\tag{2.9}$$

and

$$0\leq\frac{1}{8}\left[h_{+}\left(\frac{a+b}{2}\right)-h_{-}\left(\frac{a+b}{2}\right)\right](b-a)\leq\frac{1}{b-a}\int_a^b h(x)\,dx-h\left(\frac{a+b}{2}\right)\leq\frac{1}{8}\left[h_{-}(b)-h_{+}(a)\right](b-a).\tag{2.10}$$

The constant $\frac{1}{8}$ is best possible in all inequalities from (2.9) and (2.10).

In the case of log-convex functions we have:

Theorem 7. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then*

$$1\leq\exp\left(\frac{1}{8}\left[\frac{f_{+}\left(\frac{a+b}{2}\right)-f_{-}\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\right)\leq\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)}\leq\exp\left(\frac{1}{8}\left[\frac{f_{-}(b)}{f(b)}-\frac{f_{+}(a)}{f(a)}\right](b-a)\right)\tag{2.11}$$

and

$$1\leq\exp\left(\frac{1}{8}\left[\frac{f_{+}\left(\frac{a+b}{2}\right)-f_{-}\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\right)\leq\frac{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)}{f\left(\frac{a+b}{2}\right)}\leq\exp\left(\frac{1}{8}\left[\frac{f_{-}(b)}{f(b)}-\frac{f_{+}(a)}{f(a)}\right](b-a)\right).\tag{2.12}$$

Proof. If we write the inequality (2.9) for the convex function $h=\ln f$

$$0\leq\frac{1}{8}\left[\frac{f_{+}\left(\frac{a+b}{2}\right)-f_{-}\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\leq\frac{\ln f(a)+\ln f(b)}{2}-\frac{1}{b-a}\int_a^b\ln f(x)\,dx\leq\frac{1}{8}\left[\frac{f_{-}(b)}{f(b)}-\frac{f_{+}(a)}{f(a)}\right](b-a)$$

that is equivalent to

$$\begin{aligned}0 &\le \ln\left[\exp\left(\frac{1}{8}\left[\frac{f_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\right)\right]\\ &\le \ln\left(\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right)}\right)\\ &\le \ln\left[\exp\left(\frac{1}{8}\left[\frac{f_-(b)}{f(b)}-\frac{f_+(a)}{f(a)}\right](b-a)\right)\right].\end{aligned}$$

By taking the exponential in this inequality we get the desired result (2.11).

The inequality (2.12) follows from (2.10). □

We also have the following result:

Theorem 8. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $a=x_0<x_1<...<x_{n-1}<x_n=b$ an arbitrary division of $[a,b]$ with $n\ge 1$. Then*

$$\begin{aligned}\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] &\le \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\sqrt{f(x)f(x_i+x_{i+1}-x)}dx\\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}\tag{2.13}$$

Proof. Observe that we have

$$\begin{aligned}&\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right]\\ &=\exp\left[\frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\ln f(x)\,dx\right]\\ &=\exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln f(x)\,dx\right)\right].\end{aligned}\tag{2.14}$$

Since $\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}=1$, then by Jensen's inequality for the convex function exp we have

$$\begin{aligned}&\exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln f(x)\,dx\right)\right]\\ &\le\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\exp\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln f(x)\,dx\right).\end{aligned}\tag{2.15}$$

Utilising the inequality (1.6) on each of the intervals $[x_i,x_{i+1}]$ for $i\in\{0,...,n-1\}$ we have

$$\begin{aligned}&\exp\left[\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln f(x)\,dx\right]\\ &\le\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\sqrt{f(x)f(x_i+x_{i+1}-x)}dx\\ &\le\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx,\end{aligned}\tag{2.16}$$

for any $i \in \{0, ..., n-1\}$.

If we multiply the inequality (2.16) by $\frac{x_{i+1}-x_i}{b-a}$ and sum over i from 0 to $n-1$ then we get

$$\begin{aligned}(2.17)\quad & \sum_{i=0}^{n-1} \frac{x_{i+1}-x_i}{b-a} \exp\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right) \\ & \le \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \sqrt{f(x) f(x_i+x_{i+1}-x)}dx \le \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx \\ & = \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

Making use of (2.14), (2.15) and (2.17) we get the desired result (2.13). □

Corollary 2. *Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$ and $y \in [a, b]$, then*

$$\begin{aligned}(2.18)\quad & \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\ & \le \frac{1}{b-a}\left[\int_a^y \sqrt{f(x) f(a+y-x)}dx + \int_y^b \sqrt{f(x) f(b+y-x)}dx\right] \\ & \le \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

We define the *p-logarithmic mean* as

$$L_p(a, b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{with } a \ne b \\ a, \text{ if } a = b \end{cases}$$

for $p \ne 0, -1$ and $a, b > 0$.

The following result also holds:

Theorem 9. *Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$. Then for any $p > 0$ we have the inequality*

$$\begin{aligned}(2.19)\quad f\left(\frac{a+b}{2}\right) & \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\ & \le \left(\frac{1}{b-a}\int_a^b f^p(x) f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\ & \le \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \\ & \le \begin{cases} [L_{2p-1}(f(a), f(b))]^{1-\frac{1}{2p}} [L(f(a), f(b))]^{\frac{1}{2p}}, & p \ne \frac{1}{2}; \\ L(f(a), f(b)), & p = \frac{1}{2}. \end{cases}\end{aligned}$$

If $p \in \left(0, \frac{1}{2}\right)$, *then we have*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.20}$$

Proof. If f is a log-convex function on $[a,b]$ then f^{2p} is log-convex on $[a,b]$ for $p>0$ and by (1.6) we have

$$f^{2p}\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f^{2p}(x)\,dx\right] \leq \frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx \leq \frac{1}{b-a}\int_a^b f^{2p}(x)\,dx \leq L\left(f^{2p}(a), f^{2p}(b)\right). \tag{2.21}$$

Taking the power $\frac{1}{2p}$ in (2.21) we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \left[L\left(f^{2p}(a), f^{2p}(b)\right)\right]^{\frac{1}{2p}}. \tag{2.22}$$

Observe that, for $p \neq \frac{1}{2}$,

$$\begin{aligned}\left[L\left(f^{2p}(a), f^{2p}(b)\right)\right]^{\frac{1}{2p}} &= \left[\frac{f^{2p}(a)-f^{2p}(b)}{\ln f^{2p}(a)-\ln f^{2p}(b)}\right]^{\frac{1}{2p}} \\ &= \left[\frac{f^{2p}(a)-f^{2p}(b)}{2p\,(f(a)-f(b))}\,\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}\right]^{\frac{1}{2p}} \\ &= \left[\frac{f^{2p}(a)-f^{2p}(b)}{2p\,(f(a)-f(b))}\right]^{\frac{1}{2p}}\left[\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}\right]^{\frac{1}{2p}} \\ &= \left[L_{2p-1}(f(a), f(b))\right]^{1-\frac{1}{2p}}\left[L(f(a), f(b))\right]^{\frac{1}{2p}}\end{aligned}$$

and by (2.22) we get the desired result (2.19).

The last inequality in (2.20) follows by the following integral inequality for power $q \in (0,1)$, namely

$$\frac{1}{b-a}\int_a^b f^q(x)\,dx \leq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^q,$$

that follows by Jensen's inequality for concave functions. □

Remark 2. *If we take in (2.19) $p=1$, then we get*

$$f\left(\frac{a+b}{2}\right)\le\exp\left[\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right]\le\left(\frac{1}{b-a}\int_a^b f(x)\,f(a+b-x)\,dx\right)^{\frac{1}{2}}\le\left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\le[A(f(a),f(b))]^{\frac{1}{2}}[L(f(a),f(b))]^{\frac{1}{2}}. \tag{2.23}$$

If we take $p=\frac{1}{4}$ in (2.20), then we get

$$f\left(\frac{a+b}{2}\right)\le\exp\left[\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right]\le\left(\frac{1}{b-a}\int_a^b\sqrt[4]{f(x)\,f(a+b-x)\,dx}\right)^2\le\left(\frac{1}{b-a}\int_a^b\sqrt{f(x)}dx\right)^2\le\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.24}$$

This improves the inequality (1.7).

3. Related Inequalities

In this section we establish some related results for log-convex functions.

Theorem 10. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then for any $x\in[a,b]$ we have*

$$f(b)(b-x)+f(a)(x-a)-\int_a^b f(y)\,dy\ge\int_a^b f(y)\ln f(y)\,dy-\ln f(x)\int_a^b f(y)\,dy. \tag{3.1}$$

In particular,

$$\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy\ge\frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy-\ln f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(y)\,dy, \tag{3.2}$$

$$\frac{f(b)\sqrt{b}+f(a)\sqrt{a}}{\sqrt{b}+\sqrt{a}}-\frac{1}{b-a}\int_a^b f(y)\,dy\ge\frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy-\ln f\left(\sqrt{ab}\right)\frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.3}$$

and

$$\begin{aligned}(3.4)\qquad &\frac{f(b)\,b+f(a)\,a}{a+b}-\frac{1}{b-a}\int_a^b f(y)\,dy\\ &\geq\frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy-\ln f\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b f(y)\,dy.\end{aligned}$$

Proof. Since the function $\ln f$ is convex on $[a,b]$, then by the gradient inequality we have

$$\ln f(x)-\ln f(y)\geq\frac{f'_+(y)}{f(y)}(x-y) \tag{3.5}$$

for any $x\in[a,b]$ and $y\in(a,b)$.

If we multiply (3.5) by $f(y)>0$ and integrate on $[a,b]$ over y we get

$$\begin{aligned}&\ln f(x)\int_a^b f(y)\,dy-\int_a^b f(y)\ln f(y)\,dy\\ &\geq\int_a^b f'_+(y)(x-y)\,dy=f(y)(x-y)\big|_a^b+\int_a^b f(y)\,dy\\ &=f(b)(x-b)+f(a)(a-x)+\int_a^b f(y)\,dy,\end{aligned}$$

which is equivalent to (3.1).

The inequality (3.2) follows by (3.1) on taking $x=\frac{a+b}{2}$.

If we take in (3.1) $x=\sqrt{ab}$, then we get

$$\begin{aligned}&f(b)\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)+f(a)\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)-\int_a^b f(y)\,dy\\ &\geq\int_a^b f(y)\ln f(y)\,dy-\ln f\left(\sqrt{ab}\right)\int_a^b f(y)\,dy,\end{aligned}$$

which is equivalent to (3.3).

If we take in (3.1) $x=\frac{2ab}{a+b}$, then we get

$$\begin{aligned}&f(b)\,b\left(\frac{b-a}{a+b}\right)+f(a)\,a\left(\frac{b-a}{a+b}\right)-\int_a^b f(y)\,dy\\ &\geq\int_a^b f(y)\ln f(y)\,dy-\ln f\left(\frac{2ab}{a+b}\right)\int_a^b f(y)\,dy,\end{aligned}$$

which is equivalent to (3.4). □

Corollary 3. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then*

$$\begin{aligned}(3.6)\qquad &\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy\\ &\geq\int_a^b f(y)\ln f(y)\,dy-\int_a^b f(y)\,dy\frac{1}{b-a}\int_a^b \ln f(y)\,dy\geq 0.\end{aligned}$$

Proof. If we take the integral mean over x in (3.1), then we get

$$\frac{1}{b-a}\int_a^b [f(b)(b-x)+f(a)(x-a)]\,dx-\int_a^b f(y)\,dy$$
$$\geq \int_a^b f(y)\ln f(y)\,dy-\int_a^b f(y)\,dy\frac{1}{b-a}\int_a^b \ln f(x)$$

and since

$$\frac{1}{b-a}\int_a^b [f(b)(b-x)+f(a)(x-a)]\,dx=\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy$$

then the first inequality in (3.6) is proved.

Since ln is an increasing function on $(0,\infty)$, then we have

$$(f(x)-f(y))(\ln f(x)-\ln f(y))\geq 0$$

for any $x,y\in[a,b]$, showing that the functions f and $\ln f$ are synchronous on $[a,b]$.

By making use of the Čebyšev integral inequality for synchronous functions $g,h:[a,b]\to\mathbb{R}$, namely

$$\frac{1}{b-a}\int_a^b g(x)\,h(x)\,dx\geq\frac{1}{b-a}\int_a^b g(x)\,dx\frac{1}{b-a}\int_a^b h(x)\,dx,$$

then we have

$$\frac{1}{b-a}\int_a^b f(x)\ln f(x)\,dx\geq\frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b \ln f(x)\,dx,$$

which proves the last part of (3.6). □

The inequality (3.6) improves the well know result for convex functions

$$\frac{f(b)+f(a)}{2}\geq\frac{1}{b-a}\int_a^b f(y)\,dy.$$

We have:

Corollary 4. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. If $f(a)\neq f(b)$ and*

$$\alpha_f:=\frac{\int_a^b f'(y)\,y\,dy}{\int_a^b f'(y)\,dy}=\frac{bf(b)-af(a)-\int_a^b f(y)\,dy}{f(b)-f(a)}\in[a,b], \tag{3.7}$$

then

$$\ln f(\alpha_f)\geq\frac{\int_a^b f(y)\ln f(y)\,dy}{\int_a^b f(y)\,dy}. \tag{3.8}$$

Proof. Follows from (3.1) by observing that

$$f(b)(b-\alpha_f)+f(a)(\alpha_f-a)=\int_a^b f(y)\,dy.$$

□

Remark 3. *We observe that if $f:[a,b]\to(0,\infty)$ is nondecreasing with $f(a)\neq f(b)$ the condition (3.7) is satisfied.*

We also have:

Corollary 5. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then*

$$f(b)\left(b-\frac{\int_a^b yf(y)\,dy}{\int_a^b f(y)\,dy}\right)+f(a)\left(\frac{\int_a^b yf(y)\,dy}{\int_a^b f(y)\,dy}-a\right)-\int_a^b f(y)\,dy \tag{3.9}$$
$$\geq\int_a^b f(y)\ln f(y)\,dy-\int_a^b f(y)\,dy\ln f\left(\frac{\int_a^b yf(y)\,dy}{\int_a^b f(y)\,dy}\right)\geq 0.$$

Proof. The first inequality follows by (3.1) on taking

$$x=\frac{\int_a^b yf(y)\,dy}{\int_a^b f(y)\,dy}\in[a,b]$$

since $f(y)>0$ for any $y\in[a,b]$.

By Jensen's inequality for the convex function $\ln f$ and the positive weight f we have

$$\frac{\int_a^b f(y)\ln f(y)\,dy}{\int_a^b f(y)\,dy}\geq f\left(\frac{\int_a^b f(y)\,ydy}{\int_a^b f(y)\,dy}\right),$$

which proves the second inequality in (3.9). □

4. Applications

The function $f:(0,\infty)\to(0,\infty)$, $f(t)=\frac{1}{t}$ is log-convex on $(0,\infty)$. If we use the inequality (2.2) for this function, then we have

$$A(a,b)\geq\prod_{i=1}^{n-1}\left[A(x_i,x_{i+1})\right]^{\frac{x_{i+1}-x_i}{b-a}}\geq I(a,b) \tag{4.1}$$
$$\geq\prod_{i=1}^{n-1}\left[G(x_i,x_{i+1})\right]^{\frac{x_{i+1}-x_i}{b-a}}\geq G(a,b),$$

for any $a=x_0<x_1<...<x_{n-1}<x_n=b$ an arbitrary division of $[a,b]$ with $n\geq 1$.

In particular, we have

$$A(a,b)\geq\left[A(a,x)\right]^{\frac{x-a}{b-a}}\left[A(x,b)\right]^{\frac{b-x}{b-a}} \tag{4.2}$$
$$\geq I(a,b)\geq\sqrt{a^{\frac{x-a}{b-a}}xb^{\frac{b-x}{b-a}}}\geq G(a,b)$$

for any $x\in[a,b]$.

If we use the inequalities (2.11) and (2.12) for $f:(0,\infty)\to(0,\infty)$, $f(t)=\frac{1}{t}$, then we have

$$(1\leq)\,\frac{I(a,b)}{G(a,b)}\leq\exp\left(\frac{1}{8}\frac{(b-a)^2}{ab}\right) \tag{4.3}$$

and

$$(1\leq)\,\frac{A(a,b)}{I(a,b)}\leq\exp\left(\frac{1}{8}\frac{(b-a)^2}{ab}\right). \tag{4.4}$$

If we use the inequality (3.6) for $f:(0,\infty)\to(0,\infty)$, $f(t)=\frac{1}{t}$, then we have

$$L(a,b)-H(a,b)\geq(b-a)\,H(a,b)\ln\left(\frac{I(a,b)}{G(a,b)}\right)(\geq 0). \tag{4.5}$$

The interested reader may apply the above inequalities for other log-convex functions such as $f(t) = \frac{1}{t^p}$, $p > 0, t > 0$, $f(t) = \exp g(t)$, with g any convex function on an interval, etc...The details are omitted.

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