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NEW WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WITH FIRST DERIVATIVES OF BOUNDED VARIATION

HUSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

ABSTRACT. In this paper, some new weighted Ostrowski type integral inequalities for mappings whose first derivatives are of bounded variation are obtained and midpoint quadrature formula is provided.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[17]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$ Then $f(x)$ is said to be of bounded variation if the sum*

$$\sum_{i=1}^m|\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^n|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b(f):=\sup\left\{\sum(P):P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [9], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [7], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

Theorem 2. *Let $I_k: a=x_0<x_1<...<x_k=b$ be a division of the interval $[a,b]$ and $\alpha_i\,(i=0,1,...,k+1)$ be $k+2$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,...,k)$, $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality:*

$$\left|\int_a^b f(x)dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\right| \tag{1.2}$$

$$\leq\left[\frac{1}{2}\upsilon(h)+\max\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|,i=0,1,...,k-1\right]\bigvee_a^b(f)$$

$$\leq\upsilon(h)\bigvee_a^b(f)$$

where $\upsilon(h):=\max\{h_i|\ i=0,...,n-1\}$, $h_i:=x_{i+1}-x_i\,(i=0,1,...,k-1)$ and $\bigvee_a^b(f)$ is the total variation of f on the interval $[a,b]$.

For some recent results connected with functions of bounded variation see [1],[2],[4]-[6],[8],[10],[11]-[14],[18]-[22].

The aim of this paper is to obtain some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

2. Main Results

Theorem 3. *Let $w:[a,b]\to\mathbb{R}$ be nonnegative and continous and let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on $[a,b]$. If f' is of bounded variation on $[a,b]$, then we have the weighted inequality*

$$\left|\left(\int_a^b(u-x)w(u)du\right)f'(x)+\left(\int_a^b w(u)du\right)f(x)-\int_a^b w(t)f(t)dt\right| \tag{2.1}$$

$$\leq\left(\int_a^x(u-x)w(u)du\right)\bigvee_a^x(f')+\left(\int_x^b(u-x)w(u)du\right)\bigvee_x^b(f')$$

for any $x\in[a,b]$, where $\bigvee_c^d(f')$ denotes the total variation of f' on $[c,d]$.

Proof. Define the mapping $P_w(x,t)$ by,

$$P_w(x,t)=\begin{cases}\int_a^t (u-t)\,w(u)du, & a\le t<x\\ \int_b^t (u-t)\,w(u)du & x\le t\le b.\end{cases}$$

Integrating by parts, we get

$$\begin{aligned}
&\int_a^b P_w(x,t)df'(t) \qquad (2.2)\\
&= \int_a^x\left(\int_a^t (u-t)\,w(u)du\right)df'(t)+\int_x^b\left(\int_b^t (u-t)\,w(u)du\right)df'(t)\\
&= \left(\int_a^t (u-t)\,w(u)du\right)f'(t)\Big|_a^x+\int_a^x\left(\int_a^t w(u)du\right)f'(t)dt\\
&\quad+\left(\int_b^t (u-t)\,w(u)du\right)f'(t)\Big|_x^b+\int_x^b\left(\int_b^t w(u)du\right)f'(t)dt\\
&= \left(\int_a^x (u-x)\,w(u)du\right)f'(x)-\left(\int_b^x (u-x)\,w(u)du\right)f'(x)\\
&\quad+\left(\int_a^t w(u)du\right)f(t)\Big|_a^x-\int_a^x w(t)f(t)dt\\
&\quad+\left(\int_b^t w(u)du\right)f(t)\Big|_x^b-\int_x^b w(t)f(t)dt\\
&= \left(\int_a^b (u-x)\,w(u)du\right)f'(x)+\left(\int_b^b w(u)du\right)f(t)-\int_a^b w(t)f(t)dt.
\end{aligned}$$

It is well known [3, see p. 159] that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exist and [3, see p. 177]

$$\left|\int_a^b g(t)df(t)\right|\le\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \qquad (2.3)$$

Taking modulus in (2.2) and by using (2.3), we have

$$
\begin{aligned}
&\left|\left(\int_a^b (u-x)\,w(u)du\right) f'(x)+\left(\int_b^b w(u)du\right) f(t)-\int_a^b w(t)f(t)dt\right| \\
&= \left|\int_a^b P_w(x,t)df'(t)\right| \\
&\leq \left|\int_a^x \left(\int_a^t (u-t)\,w(u)du\right) df'(t)\right| + \left|\int_x^b \left(\int_b^t (u-t)\,w(u)du\right) df'(t)\right| \\
&\leq \sup_{t\in[a,x]}\left|\int_a^t (u-t)\,w(u)du\right| \bigvee_a^x (f') + \sup_{t\in[x,b]}\left|\int_b^t (u-t)\,w(u)du\right| \bigvee_x^b (f') \\
&\leq \sup_{t\in[a,x]}\left\{\int_a^t (t-u)\,w(u)du\right\} \bigvee_a^x (f') + \sup_{t\in[x,b]}\left\{\int_t^b (u-t)\,w(u)du\right\} \bigvee_x^b (f') \\
&= \left(\int_a^x (x-u)\,w(u)du\right) \bigvee_a^x (f') + \left(\int_x^b (u-x)\,w(u)du\right) \bigvee_x^b (f')
\end{aligned}
$$

which is the desired result. □

Corollary 1. *Under assumption of Theorem 3 with $w \equiv 1$, then we have the inequality*

(2.4)

$$
\begin{aligned}
&\left|\left(\frac{a+b}{2}-x\right) f'(x)+f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{b-a}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{2}\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f')
\end{aligned}
$$

for any $x \in [a,b]$.

Proof. Choosing $w \equiv 1$ in Theorem 3, we have

$$\begin{aligned}
&\left|(b-a)\left(\frac{a+b}{2}-x\right)f'(x)+(b-a)\,f(x)-\int_a^b w(t)f(t)dt\right| \\
\leq\ & \frac{1}{2}\left[(x-a)^2\bigvee_a^x(f')+(b-x)^2\bigvee_x^b(f')\right] \\
\leq\ & \frac{1}{2}\max\left\{(x-a)^2,(b-x)^2\right\}\bigvee_a^b(f') \\
=\ & \frac{1}{2}\left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2+\frac{b-a}{2}\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f') \\
=\ & \frac{(b-a)^2}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{2}\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f')
\end{aligned}$$

which is required result. □

Remark 1. *If we choose $x=\frac{a+b}{2}$ in (2.4), then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\bigvee_a^b(f').$$

which was given by Liu in [16].

Under assumption of of Theorem 3, we have the following corollaries:

Corollary 2. *Let $f\in C^{(2)}[a,b]$. Then we have the inequality*

(2.5)

$$\begin{aligned}
&\left|\left(\int_a^b(u-x)\,w(u)du\right)f'(x)+\left(\int_a^b w(u)du\right)f(x)-\int_a^b w(t)f(t)dt\right| \\
\leq\ & \left(\int_a^x(u-x)\,w(u)du\right)\|f''\|_{[a,x],1}+\left(\int_x^b(u-x)\,w(u)du\right)\|f''\|_{[x,b],1}
\end{aligned}$$

for all $x\in[a,b]$, where $\|.\|_{[c,d],1}$ is the L_1-norm, namely

$$\|f''\|_{[c,d],1}=\int_c^d|f''(t)|\,dt.$$

Remark 2. *If we choose $w\equiv 1$ and $x=\frac{a+b}{2}$ in (2.5), then we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\|f''\|_{[a,b],1}$$

which was given by Liu in [15].

Corollary 3. *Let $f' : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constants L_1, $L_2 > 0$. Then, we have the inequality*

(2.6)
$$\begin{aligned}
&\left| \left(\int_a^b (u-x)\, w(u)du \right) f'(x) + \left(\int_a^b w(u)du \right) f(x) - \int_a^b w(t)f(t)dt \right| \\
\leq & \left(\int_a^x (u-x)\, w(u)du \right) (x-a)\, L_1 + \left(\int_x^b (u-x)\, w(u)du \right) (b-x)\, L_2 \\
\leq & \left(\int_a^b (u-x)\, w(u)du \right) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] L
\end{aligned}$$

for all $x \in [a,b]$ and $L = \max\{L_1, L_2\}$.

3. Application to Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$ with $h_i := x_{i+1} - x_i$. Then the following Theorem holds:

Theorem 4. *Let $f : Q \to \mathbb{R}$ is of bounded variatin on Q and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the quadrature formula:*

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1} \left[\left(\frac{x_i + x_{i+1}}{2} - x \right) f'(\xi_i)h_i + f(\xi_i)h_i \right] + R(I_n, f, \xi).$$

The remainder term $R(I_n, f, \xi)$ satisfies

(3.1)
$$\begin{aligned}
&|R(I_n, f, \xi)| \\
\leq & \frac{1}{2} \left[\frac{1}{4} + \max_{i \in \{0,...,n-1\}} \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right|^2 + \frac{1}{2} \max_{i \in \{0,...,n-1\}} \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right] \bigvee_a^b (f').
\end{aligned}$$

Proof. Applying Corollary 1 for interval $[x_i, x_{i+1}]$, we have

(3.2)
$$\begin{aligned}
&\left| \left(\frac{x_i + x_{i+1}}{2} - x \right) f'(\xi_i)h_i + f(\xi_i)h_i - \int_{x_i}^{x_{i+1}} f(t)dt \right| \\
\leq & \frac{1}{2} \left[\frac{1}{4} + \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right)^2 + \frac{1}{2} \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right] \bigvee_{x_i}^{x_{i+1}} (f').
\end{aligned}$$

Summing the inequality (3.2) over i from 0 to $n-1$, then we have

$$
\begin{aligned}
&|R(I_n, f, \xi)| \\
\leq\ & \sum_{i=0}^{n-1} \frac{1}{2}\left[\frac{1}{4} + \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right)^2 + \frac{1}{2}\left|\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f') \\
\leq\ & \max_{i\in\{0,\dots,n-1\}} \frac{1}{2}\left[\frac{1}{4} + \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right)^2 + \frac{1}{2}\left|\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right|\right] \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f') \\
\leq\ & \frac{1}{2}\left[\frac{1}{4} + \max_{i\in\{0,\dots,n-1\}}\left|\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right|^2 + \frac{1}{2}\max_{i\in\{0,\dots,n-1\}}\left|\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i}\right|\right] \bigvee_{a}^{b}(f').
\end{aligned}
$$

This completes the proof of the Theorem. □

Remark 3. *If we choose* $\xi_i \in \frac{x_i + x_{i+1}}{2}$ $(i = 0, ..., n-1)$ *in Theorem 4, we have the midpoint rule*

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1} f(\xi_i)h_i + R_M(I_n, f)$$

and the remainder term satisfies

$$|R_M(I_n, f)| \leq \frac{1}{8}\bigvee_a^b (f').$$

References

[1] M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] T.M. Apostol, *Mathematical Analysis*, Second Edition, Addision-Wesley Publishing Company, 1975.

[4] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.

[5] P. Cerone, W.S. Cheung, and S.S. Dragomir, *On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation,* Computers and Mathematics with Applications 54 (2007) 183–191.

[6] P. Cerone, S. S. Dragomir, and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turk J Math, 24 (2000), 147-163.

[7] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull.Austral. Math. Soc., 60(1) (1999), 495-508.

[8] S. S. Dragomir, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. 22(2000) 13-19.

[9] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. 4 (2001), no. 1, 59–66.

[10] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. 5 (2014) No. 1, 89-97

[11] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation.* Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

[12] S.S. Dragomir and E. Momoniat, *A Three Point Quadrature Rule for Functions of Bounded Variation and Applications*, RGMIA Research Report Collection, 14(2011), Article 33, 16 pp.

[13] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 93.

[14] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, 2012.

[15] Z. Liu, *Some Companion of an Ostrowski type inequality and application,* JIPAM, 10(2) 2009, Article 52, 12 pp.

[16] Z. Liu, *Some Ostrowski type inequalities*, Mathematical and Computer Modelling 48 (2008) 949–960.

[17] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[18] K-L Tseng, G-S Yang, and S. S. Dragomir, *Generalizations of Weighted Trapezoidal Inequality for Mappings of Bounded Variation and Their Applications*, Mathematical and Computer Modelling 40 (2004) 77-84.

[19] K-L Tseng, *Improvements of some inequalites of Ostrowski type and their applications*, Taiwan. J. Math. 12 (9) (2008) 2427–2441.

[20] K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Applied Mathematics and Computation 217 (2010) 2348–2355.

[21] K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Weighted Ostrowski Integral Inequality for Mappings of Bounded Variation*, Taiwanese J. of Math., Vol. 15, No. 2, pp. 573-585, April 2011.

[22] K-L Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Applied Mathematics and Computation 218 (2012) 5841–5847.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com