Received 29/04/15

A COMPANION OF GENERALIZATION OF OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

HUSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

Abstract. In this paper, a companion of generalization of Ostrowski type inequalities for functions of two variables with bounded variation is given and quadrature formula is provided.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[24]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [16], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [17], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

2000 *Mathematics Subject Classification.* 26D15, 26B30, 26D10, 41A55.
Key words and phrases. Ostrowski type inequalities, Bounded variation, Riemann-Stieltjes.

Theorem 2. *Assume that the function $f:[a,b]\to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

(1.2)

$$\begin{aligned}
&\frac{1}{2}\left|f(x)+f(a+b-x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]
\end{aligned}$$

$$\leq\begin{cases}
\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f), \\
\left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
\times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^b(f)\right]^{\beta}\right]^{\frac{1}{\beta}} & ,\ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
\left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}
\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ where $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$. The constant $\frac{1}{4}$ is the best possible in the first branch of second inequality in (1.2).

In [3], Alomari proved the following generalized inequalities:

Theorem 3. *Let $f:[a,b]\to R$ be a mapping of bounded variation on $[a,b]$. Then for all $\lambda\in[0,1]$ and $a+\lambda\frac{b-a}{2}\leq x\leq\frac{a+b}{2}$, we have the inequality*

(1.3)

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right]-\int_a^b f(t)dt\right|$$

$$\leq\begin{cases}
\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{a+b}{2}-x\right)\right\}\bigvee_a^b(f), \\
\frac{b-a}{2}\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\},
\end{cases}$$

where $\bigvee_a^b(f)$ denotes to the total variation of f over $[a,b]$. The constant $\frac{1}{4}$ in second inequality is the best possible in the sense that it cannot be replaced by smaller one.

2. Preliminaries and Lemmas

In 1910, Fréchet [21] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}\alpha(x_i,y_j)=\alpha(x_{i-1},y_{j-1})-\alpha(x_{i-1},y_j)-\alpha(x_i,y_{j-1})+\alpha(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}\alpha(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([13]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a\le x\le b,\ c\le y\le d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i=0,1,2,...,m),\ a=x_0<x_1<...<x_m=b; \\ y &= y_j(j=0,1,2,...,n),\ c=y_0<y_1<...<y_n=d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell.* We employ the notation

$$\Delta_{11}f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j),$$

$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x},y)$ $[f(x,\overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$, or as $+\infty$ if $f(\overline{x},y)$ $[f(x,\overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \left|\Delta_{11} f(x_i,y_j)\right|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i\overline{\epsilon_j}\left|\Delta_{11} f(x_i,y_j)\right|$$

is bounded for all nets and all possible choices of $\epsilon_i=\pm 1$ and $\overline{\epsilon_j}=\pm 1$.

Definition 3. *(Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x},y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x,\overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i,y_i) $(i=0,1,2,...,m)$ be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0<x_1<...<x_m=b; \\ c &= y_0<y_1<...<y_m=d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m}\left|\Delta f(x_i,y_i)\right|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of two variables, as follows:

Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\Delta_{11} f(x_i,y_j)\right|$ corresponding to the partition P of Q. The number

$$\bigvee_Q(f):=\bigvee_c^d\bigvee_a^b(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [22], Jawarneh and Noorani proved foolowing Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(\alpha)$ on Q, then $\alpha \in RS(f)$ on Q, and we have*

$$\int_c^d \int_a^b f(t,s) d_t d_s \alpha(t,s) + \int_c^d \int_a^b \alpha(t,s) d_t d_s f(t,s) \tag{2.3}$$
$$= f(b,d)\alpha(b,d) - f(b,c)\alpha(b,c) - f(a,d)\alpha(a,d) + f(a,c)\alpha(a,c).$$

Lemma 2. *Assume that $g \in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left| \int_c^d \int_a^b g(x,y) d_x d_y \alpha(x,y) \right| \leq \sup_{(x,y)\in Q} |g(x,y)| \bigvee_Q (\alpha). \tag{2.4}$$

In [22], Jawarneh and Noorani proved the following Ostrowski type inequality or functions of two variables with bounded variation:

Theorem 4. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\left| (b-a)(d-c) f(x,y) - \int_c^d \int_a^b f(t,s) dt ds \right| \tag{2.5}$$
$$\leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]$$
$$\times \left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right] \bigvee_Q (f)$$

where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q.

Dragomir proved a companion of Ostrowski type inequalities for functions of bounded variation in [17]. Then, Alomari proved a companion of generalization of Ostrowski type inequalities of bounded variation in [3]. Recently, Budak and Sarikaya [10] have proved a companion of Ostrowski type inequalities for functions of two variables with bounded variation. For more information and recent developments on integral inequalities for mappings of bounded variation (single variable and two variables), please refer to ([1]-[11], [14]-[20],[22],[23],[25]-[30]).

In this paper we establish a companion of generalization of Ostrowski type inequalities for functions of two variables with bounded variation and apply it quadrature formula.

3. Main Results

First, we give the following notations to simplify presentetion of some intervals.

$$\begin{aligned} Q_1 &= [a,x]\times[c,y],\ Q_2=[a,x]\times[y,c+d-y],\ Q_3=[a,x]\times[c+d-y,d],\\ Q_4 &= [x,a+b-x]\times[c,y],\ Q_5=[x,a+b-x]\times[y,c+d-y],\\ Q_6 &= [x,a+b-x]\times[c+d-y,d],\ Q_7=[a+b-x,b]\times[c,y],\\ Q_8 &= [a+b-x,b]\times[y,c+d-y],\ Q_9=[a+b-x,b]\times[c+d-y,d]. \end{aligned}$$

To prove our theorem, we need the following lemma:

Lemma 3. *Let $f:Q=[a,b]\times[c,d]\to\mathbb{R}$ is of bounded variatin on Q. Then, we have the following equality*

(3.1)

$$\begin{aligned} &\frac{1}{4}\left[\lambda\eta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right.\\ &+\lambda(1-\eta)\left[f(a,y)+f(a,c+d-y)+f(b,y)+f(b,c+d-y)\right]\\ &+(1-\lambda)\eta\left[f(x,c)+f(x,d)+f(a+b-x,c)+f(a+b-x,d)\right]\\ &(1-\lambda)(1-\eta)\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]]\\ &-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\\ =\ &\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right.\\ &+\int_a^x\int_c^{c+d-y}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\\ &+\int_a^x\int_{c+d-y}^d\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\\ &+\int_x^{a+b-x}\int_c^y\left(t-\frac{a+b}{2}\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\\ &+\int_x^{a+b-x}\int_y^{c+d-y}\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\\ &+\int_x^{a+b-x}\int_{c+d-y}^d\left(t-\frac{a+b}{2}\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s) \end{aligned}$$

$$+\int_{a+b-x}^{b}\int_{c}^{y}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)$$
$$+\int_{a+b-x}^{b}\int_{y}^{c+d-y}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)$$
$$\left.+\int_{a+b-x}^{b}\int_{c+d-y}^{d}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right]$$

for any $\lambda,\eta\in[0,1]$ *and* $a+\lambda\frac{b-a}{2}\leq x\leq\frac{a+b}{2}$, $c+\eta\frac{d-c}{2}\leq y\leq\frac{c+d}{2}$.

Proof. Using Lemma 1, we have

$$\begin{aligned}
&\int_{a}^{x}\int_{c}^{y}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\\
=&\left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(x,y)\\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,c)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(a,y)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a,c)-\int_{a}^{x}\int_{c}^{y}f(t,s)dsdt,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_{a}^{x}\int_{y}^{c+d-y}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\\
=&\left(x-a-\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(x,c+d-y)\\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(x,y)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(a,c+d-y)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(a,y)-\int_{a}^{x}\int_{c}^{c+d-y}f(t,s)dsdt,
\end{aligned}$$

$$\int_a^x \int_{c+d-y}^d \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) d_s d_t f(t,s)$$
$$= \left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(x,d)$$
$$+\left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right) f(x,c+d-y)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(a,d)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right) f(a,c+d-y) - \int_a^x \int_{c+d-y}^d f(t,s)dsdt,$$

$$\int_x^{a+b-x} \int_c^y \left(t-\frac{a+b}{2}\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) d_s d_t f(t,s)$$
$$= \left(\frac{a+b}{2}-x\right)\left(y-c-\eta\frac{d-c}{2}\right) f(a+b-x,y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\eta\frac{d-c}{2}\right) f(a+b-x,c)$$
$$+\left(\frac{a+b}{2}-x\right)\left(y-c-\eta\frac{d-c}{2}\right) f(x,y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\eta\frac{d-c}{2}\right) f(x,c) - \int_x^{a+b-x} \int_c^y f(t,s)dsdt,$$

$$\int_x^{a+b-x} \int_y^{c+d-y} \left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right) d_s d_t f(t,s)$$
$$= \left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right) f(a+b-x,c+d-y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right) f(a+b-x,y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right) f(x,c+d-y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right) f(x,y) - \int_x^{a+b-x} \int_y^{c+d-y} f(t,s)dsdt,$$

$$\int_{x}^{a+b-x}\int_{c+d-y}^{d}\left(t-\frac{a+b}{2}\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)$$

$$\begin{aligned}
&= \left(\frac{a+b}{2}-x\right)\left(\eta\frac{d-c}{2}\right)f(a+b-x,d) \\
&+\left(\frac{a+b}{2}-x\right)\left(y-c-\eta\frac{d-c}{2}\right)f(a+b-x,c+d-y) \\
&+\left(\frac{a+b}{2}-x\right)\left(\eta\frac{d-c}{2}\right)f(x,d) \\
&+\left(\frac{a+b}{2}-x\right)\left(y-c-\eta\frac{d-c}{2}\right)f(x,c+d-y)-\int_{x}^{a+b-x}\int_{c+d-y}^{d}f(t,s)dsdt,
\end{aligned}$$

$$\int_{a+b-x}^{b}\int_{c}^{y}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)$$

$$\begin{aligned}
&= \left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(b,y) \\
&+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(b,c) \\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(a+b-x,y) \\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a+b-x,c)-\int_{a+b-x}^{d}\int_{c}^{y}f(t,s)dsdt,
\end{aligned}$$

$$\int_{a+b-x}^{b}\int_{y}^{c+d-y}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)$$

$$\begin{aligned}
&= \left(\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(b,c+d-y) \\
&+\left(\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(b,y) \\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(a+b-x,c+d-y) \\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\frac{c+d}{2}-y\right)f(a+b-x,y)-\int_{a+b-x}^{d}\int_{y}^{c+d-y}f(t,s)dsdt,
\end{aligned}$$

$$\int_{a+b-x}^{b}\int_{c+d-y}^{d}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_s d_t f(t,s)$$

$$\begin{aligned}
&= \left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(b,d)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(b,c+d-y)\\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a+b-x,d)\\
&+\left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(a+b-x,c+d-y)-\int_{a+b-x}^{d}\int_{c+d-y}^{d}f(t,s)dsdt.
\end{aligned}$$

Summing the above equalities we deduce (3.1). □

Remark 1. *If we choose $\lambda=0$ and $\eta=0$ in Lemma 3, then equality (3.1) reduces Lemma 3 in* [10].

Theorem 5. *If the function $f:Q=[a,b]\times[c,d]\to R$ is of bounded variation on Q, then we have the inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[\lambda\eta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right.\\
&+\lambda(1-\eta)\left[f(a,y)+f(a,c+d-y)+f(b,y)+f(b,c+d-y)\right]\\
&+(1-\lambda)\eta\left[f(x,c)+f(x,d)+f(a+b-x,c)+f(a+b-x,d)\right]\\
&\left.(1-\lambda)(1-\eta)\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]\right]\\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\Bigg|\\
\leq\;& \frac{1}{(b-a)(d-c)}\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{a+b}{2}-x\right)\right\}\\
&\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{c+d}{2}-y\right)\right\}\bigvee_Q(f)
\end{aligned}\tag{3.2}$$

for any $\lambda,\eta\in[0,1]$ and $a+\lambda\frac{b-a}{2}\leq x\leq\frac{a+b}{2}$, $c+\eta\frac{d-c}{2}\leq y\leq\frac{c+d}{2}$., where $\bigvee_Q(f)$ denotes the total (double) variation of f on Q.

Proof. Taking modulus in (3.1) and applying Lemma 2, we have

$$\Bigg|\frac{1}{4}\left[\lambda\eta\left[f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)\right]\right.$$

$$+\lambda\left(1-\eta\right)\left[f\left(a,y\right)+f\left(a,c+d-y\right)+f\left(b,y\right)+f\left(b,c+d-y\right)\right]$$

$$+\left(1-\lambda\right)\eta\left[f\left(x,c\right)+f\left(x,d\right)+f\left(a+b-x,c\right)+f\left(a+b-x,d\right)\right]$$

$$\left.\left(1-\lambda\right)\left(1-\eta\right)\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]\right]$$

$$-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$

$$\leq \frac{1}{\left(b-a\right)\left(d-c\right)}\left[\left|\int_a^x\int_c^y\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|\right.$$

$$+\left|\int_a^x\int_c^{c+d-y}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_a^x\int_{c+d-y}^d\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_x^{a+b-x}\int_c^y\left(t-\frac{a+b}{2}\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_x^{a+b-x}\int_y^{c+d-y}\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_x^{a+b-x}\int_{c+d-y}^d\left(t-\frac{a+b}{2}\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_{a+b-x}^b\int_c^y\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|$$

$$+\left|\int_{a+b-x}^b\int_y^{c+d-y}\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)\right|$$

$$\left.+\left|\int_{a+b-x}^b\int_{c+d-y}^d\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s)\right|\right]$$

$$
\begin{aligned}
&\leq \frac{1}{(b-a)(d-c)}\left[\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q_1}(f)\right.\\
&+\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\left(\frac{c+d}{2}-y\right)\bigvee_{Q_2}(f)\\
&+\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q_3}(f)\\
&+\left(\frac{a+b}{2}-x\right)\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q_4}\\
&+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_5}+\left(\frac{a+b}{2}-x\right)\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q6}\\
&+\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q_7}(f)\\
&+\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\left(\frac{c+d}{2}-y\right)\bigvee_{Q_8}(f)\\
&\left.+\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\}\bigvee_{Q_9}(f)\right]\\
:&= M(x,y)
\end{aligned}
$$

In last inequality, we have

$$
\begin{aligned}
M(x,y) &\leq \frac{1}{(b-a)(d-c)}\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{a+b}{2}-x\right)\right\}\\
&\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{c+d}{2}-y\right)\right\}\\
&\times\left[\bigvee_{Q_1}(f)+\bigvee_{Q_2}(f)+\bigvee_{Q_3}(f)+...+\bigvee_{Q_9}(f)\right]\\
&= \frac{1}{(b-a)(d-c)}\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{a+b}{2}-x\right)\right\}\\
&\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{c+d}{2}-y\right)\right\}\bigvee_{Q}(f)
\end{aligned}
$$

which completes the proof. □

Remark 2. *Under assumption Theorem 5 with $\lambda = 0$ and $\eta = 0$, we have the inequality*

$$\begin{aligned} &\left|\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]\right. \\ &\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4}+\left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right]\bigvee_Q(f) \end{aligned} \tag{3.3}$$

which is proved by Budak and Sarikaya in [10].

Corollary 1. *Under assumption Theorem 5 with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we get the inequality*

$$\begin{aligned} &\left|\frac{1}{4}\left[\lambda\eta\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]+2\lambda(1-\eta)\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right]\right.\right. \\ &+2(1-\lambda)\eta\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]+4(1-\lambda)(1-\eta)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg] \\ &\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \frac{1}{4}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\bigvee_Q(f). \end{aligned} \tag{3.4}$$

for all $\lambda, \eta \in [0,1]$.

Remark 3. *In Corollary 1, if we choose*

1) If we choose $\lambda = 0$ and $\eta = 0$, we have the "midpoint inequality"

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_Q(f) \tag{3.5}$$

which is given by Jawarneh and Noorani in [22]. *The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see* [10].

2) If we choose $\lambda = 1$ and $\eta = 1$, we get the "trapezoid inequality"

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_Q(f) \tag{3.6}$$

which is proved by Jawarneh and Noorani in [22]. *The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see* [7].

Remark 4. *If we choose $\lambda = 1$ and $\eta = 0$ in Corollary 1, we get the inequality*
(3.7)
$$\left| \frac{1}{2}\left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)\right] - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \le \frac{1}{4}\bigvee_Q(f).$$
The constant $\frac{1}{4}$ is the best possible.

Proof. For sharpness of constant, assume that 3.7 holds with a constant A>0, that is,
(3.8)
$$\left| \frac{1}{2}\left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)\right] - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \le A\bigvee_Q(f).$$
If we choose $f : Q \to \mathbb{R}$ with
$$f(x,y) = \begin{cases} 1 \text{ if } & x = a, b \text{ and } y = \frac{c+d}{2} \\ 0 \text{ if } & (x,y) \in [a,b]\times[c,d] \setminus \left\{ \left(a, \frac{c+d}{2}\right), \left(b, \frac{c+d}{2}\right)\right\} \end{cases}$$
then f is of bounded variation on Q, and
$$\frac{1}{2}\left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)\right] = 1, \quad \int_a^b\int_c^d f(t,s)dsdt = 0, \text{ and } \bigvee_Q(f) = 4,$$
giving in (3.8), $1 \le 4A$, thus $A \ge \frac{1}{4}$ which implies the constant $\frac{1}{4}$ is the best possible. This completes the proof. □

Remark 5. *In Corollary 1, taking $\lambda = 0$ and $\eta = 1$, we have the inequality*
(3.9)
$$\left| \frac{1}{2}\left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)\right] - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \le \frac{1}{4}\bigvee_Q(f).$$
The constant $\frac{1}{4}$ is the best possible.

Proof. Similarly, assume that 3.9 holds with a constant B>0, that is,
(3.10)
$$\left| \frac{1}{2}\left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)\right] - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \le \frac{1}{4}\bigvee_Q(f).$$
Here, if we take $f : Q \to \mathbb{R}$ with
$$f(x,y) = \begin{cases} 1 \text{ if } & x = \frac{a+b}{2} \text{ and } y = c, d \\ 0 \text{ if } & (x,y) \in [a,b]\times[c,d] \setminus \left\{ \left(\frac{a+b}{2}, c\right), \left(\frac{a+b}{2}, d\right)\right\} \end{cases}$$
then f is of bounded variation on Q, and
$$\frac{1}{2}\left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)\right] = 1, \quad \int_a^b\int_c^d f(t,s)dsdt = 0, \text{ and } \bigvee_Q(f) = 4.$$
putting in (3.10), $1 \le 4B$, thus $B \ge \frac{1}{4}$ which completes the proof. □

Remark 6. *Similarly, if we choose $\lambda = \frac{1}{3}$ and $\eta = \frac{1}{3}$ in Corollary 1, we have the "Simpson's rule inequality"*

(3.11)
$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{36}\right.\\&+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)}{9}\\&\left.+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\\leq\;&\frac{1}{9}\bigvee_Q(f).\end{aligned}$$

which is proved by Jawarneh and Noorani in [22].

Remark 7. *If we choose $\lambda = \frac{1}{2}$ and $\eta = \frac{1}{2}$ in Corollary 1, we have the inequality*

(3.12)
$$\begin{aligned}&\left|\frac{1}{4}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\right.\right.\\&\frac{1}{2}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]\\&\left.\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\\leq\;&\frac{1}{16}\bigvee_Q(f).\end{aligned}$$

which is given by Budak and Sarikaya in [7]. *The constant $\frac{1}{16}$ is the best possible.*

4. Some Composite Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$,

$$\upsilon(h) := \max\{h_i|\ i = 0, ..., n-1\},$$

$$\upsilon(l) := \max\{l_j|\ j = 0, ..., m-1\}.$$

Then, the following theorem holds.

Theorem 6. *Let f as Theorem 5. Then we have the quadrature formula:*

$$\int_a^b\int_c^d f(t,s)dsdt \tag{4.1}$$

$$= \frac{\lambda\eta}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_il_j$$

$$+\frac{\lambda(1-\eta)}{2}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right]h_il_j$$

$$+\frac{(1-\lambda)\eta}{2}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)h_il_j$$

$$+(1-\lambda)(1-\eta)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_il_j+R(I_n,J_m,f).$$

The remainder $R(\xi,\eta,I_n,J_m,f)$ satisfies the inequality

$$|R(\xi,\eta,I_n,J_m,f)|\le\frac{1}{4}\upsilon(h)\upsilon(l)\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f). \tag{4.2}$$

Proof. Aplying Corollary 1 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

(4.3)

$$\Bigg|\frac{\lambda\eta}{4}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_il_j$$

$$+\frac{\lambda(1-\eta)}{2}\left[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right]h_il_j$$

$$+\frac{(1-\lambda)\eta}{2}\left[f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right]h_il_j$$

$$+(1-\lambda)(1-\eta)f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_il_j-\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}f(t,s)dsdt\Bigg|$$

$$\le\frac{1}{4}h_il_j\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)$$

for any $i\in\{0,...,n-1\}$ and $j\in\{0,...,m-1\}$.

Summing the inequality (4.3) over i from 0 to $n-1$ and j from 0 to $m-1$,then we get

$$\begin{aligned}
|R(I_n, J_m, f)| &\leq \frac{1}{4}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j \bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
&\leq \frac{1}{4}\upsilon(h)\upsilon(l)\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
&= \frac{1}{4}\upsilon(h)\upsilon(l)\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}+\left|\eta-\frac{1}{2}\right|\right]\bigvee_{a}^{b}\bigvee_{c}^{d}(f).
\end{aligned}$$

This completes the proof. □

Remark 8. *In Theorem 6, if we choose*
1) $\lambda = 0$ *and* $\eta = 0$*, then we have the trapezoid rule*

$$\int_a^b\int_c^d f(t,s)dsdt = \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_i l_j + R_T(I_n,J_m,f)$$

with

$$|R_T(I_n,J_m,f)| \leq \frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f),$$

2) $\lambda = 1$ *and* $\eta = 1$*, then we have the midpoint rule*

$$\int_a^b\int_c^d f(t,s)dsdt = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right) + R_M(I_n,J_m,f)$$

with

$$|R_M(I_n,J_m,f)| \leq \frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f),$$

3) $\lambda = \frac{1}{3}$ *and* $\eta = \frac{1}{3}$*, then we have the Simpson's rule*

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \\
=\;& \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right)h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2}, y_j\right)+f\left(\frac{x_i+x_{i+1}}{2}, y_{j+1}\right)\right]h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i, \frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1}, \frac{y_j+y_{j+1}}{2}\right)\right]h_i l_j \\
&+\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_i l_j \\
&+R_S(I_n,J_m,f)
\end{aligned}$$

with

$$|R_S(I_n,J_m,f)| \leq \frac{1}{9}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

References

[1] M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] M. W. Alomari, *A Generalization of Dragomir's generalization of Ostrowski Integral Inequality and applications in numerical integration*, Ukrainian Mathematical Journal, 2012, 64(4).

[4] M. W. Alomari, *A companion of Ostrowski's inequality for the Riemann–Stieltjes integral* $\int_a^b f(t)du(t)$*, where* f *is of bounded variation and* u *is of* $r-H-$*Hölder type and applications,* Applied Mathematics and Computation 219 (2013) 4792–4799.

[5] N.S. Barnett, S.S. Dragomir, I. Gomm, *A companion for the Ostrowski and the generalized trapezoid inequalities,* Mathematical and Computer Modelling 50 (2009) 179 187.

[6] H. Budak and M.Z. Sarıkaya, *On Ostrowski type inequalities for functions of two variables with bounded variation*,RGMIA Research Report Collection, 17(2014), Article 153, 18 pp.

[7] H. Budak and M.Z. Sarıkaya, *On generalization Ostrowski type inequalities for functions of two variables with bounded variation,* RGMIA Research Report Collection, 17(2014), Article 154, 13 pp.

[8] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.

[9] H. Budak and M.Z. Sarıkaya, *On generalization trapezoid inequality for functions of two variables with bounded variation and applications*, RGMIA Research Report Collection, 17(2014), Article 156, 10 pp.

[10] H. Budak and M.Z. Sarıkaya, *A companion of Ostrowski type inequalities for functions of two variables with bounded Variation*, RGMIA Research Report Collection, 17(2014), Article 157, 16 pp.

[11] P. Cerone, S. S. Dragomir, and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turk J Math, 24 (2000), 147-163.

[12] Clarkson J.A.and Adams C. R., *On definitions of bounded variation for functions of two variables*,Bull. Amer. Math. Soc. 35 (1933), 824-854.

[13] J.A. Clarkson, *On double Riemann-Stieltjes integrals*, Bull. Amer. Math. Soc. 39 (1933), 929-936.

[14] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull.Austral. Math. Soc., 60(1) (1999), 495-508.

[15] S. S. Dragomir, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. 22(2000) 13-19.

[16] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. 4 (2001), no. 1, 59–66.

[17] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. 5 (2014) No. 1, 89-97

[18] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation.* Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

[19] S.S. Dragomir and E. Momoniat, *A Three point quadrature rule for functions of bounded bariation and applications*, RGMIA Research Report Collection, 14(2011), Article 33, 16 pp.

[20] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 93.

[21] Fréchet, M., *Extension au cas des intégrals multiples d'une définition de l'intégrale due á Stieltjes*, Nouvelles Annales de Math ematiques 10 (1910), 241-256.

[22] Y. Jawarneh and M.S.M Noorani, *Inequalities of Ostrowski and Simpson Type for Mappings of Two Variables with Bounded Variation and Applications*, TJMM, 3 (2011), No. 2, 81-94

[23] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, 2012.

[24] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[25] K-L Tseng, G-S Yang, and S. S. Dragomir, *Generalizations of weighted trapezoidal inequality for mappings of bounded variation and their applications*, Mathematical and Computer Modelling 40 (2004) 77-84.

[26] K-L Tseng, *Improvements of some inequalites of Ostrowski type and their applications*, Taiwan. J. Math. 12 (9) (2008) 2427–2441.

[27] K-L Tseng, S-R Hwang, and S. S. Dragomir, *Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and applications,* Computers and Mathematics with Applications, 55(2008), 1785-1793.

[28] K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Applied Mathematics and Computation 217 (2010) 2348–2355.

[29] K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Weighted Ostrowski integral inequality for mappings of bounded variation*, Taiwanese J. of Math., Vol. 15, No. 2, pp. 573-585, April 2011.

[30] K-L Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Applied Mathematics and Computation 218 (2012) 5841–5847.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com