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A NEW COMPANION OF OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we obtain a new companion of Ostrowski type inequalities for functions of two independent variables with bounded variation and give an application.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[26]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [18], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [5], Barnett et. al. proved following inequalities related functions of bounded variation:

Theorem 2. *Assume that the function $f:[a,b]\to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

(1.2)

$$|\Psi_f(t)|\leq\frac{1}{b-a}\left[(t-a)\bigvee_a^t(f)+(b-t)\bigvee_t^b(f)\right]$$

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$$\leq \begin{cases} \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \\ \left[\left(\frac{t-a}{b-a}\right)^q+\left(\frac{b-t}{b-a}\right)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^t(f)\right)^p+\left(\bigvee_t^b(f)\right)^p+\right]^{\frac{1}{p}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right| \end{cases}$$

where

$$\Psi_f(t):=f(t)-\frac{f(a)(t-a)+(b-t)f(b)}{b-a}.$$

First inequality is sharp and the constant $\frac{1}{2}$ *is also the best possible in both branches in (1.2).*

2. Preliminaries and Lemmas

In 1910, Fréchet [23] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $g(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b,\ \text{and}\ c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}g(x_i,y_j)=g(x_{i-1},y_{j-1})-g(x_{i-1},y_j)-g(x_i,y_{j-1})+g(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^n\sum_{j=1}^m f\left(\xi_i,\eta_j\right)\Delta_{11}g(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^n\sum_{j=1}^m f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}g(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [15] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [14], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a \leq x \leq b,\ c \leq y \leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i (i = 0,1,2,...,m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j (j = 0,1,2,...,n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)$$

$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j)$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$,or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 3. *(Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i, y_i) $(i = 0,1,2,...,m)$ be any set of points satisfying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left|\Delta_{11}f(x_i,y_j)\right|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f):=\bigvee_{c}^{d}\bigvee_{a}^{b}(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q.

In [24], authors proved the following Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f\in RS(g)$ on Q, then $g\in RS(f)$ on Q, and we have*

$$\begin{aligned}&\int_c^d\int_a^b f(t,s)d_td_sg(t,s)+\int_c^d\int_a^b g(t,s)d_td_sf(t,s) \qquad (2.3)\\ =\;& f(b,d)g(b,d)-f(b,c)g(b,c)-f(a,d)g(a,d)+f(a,c)g(a,c).\end{aligned}$$

Lemma 2. *Assume that $\Omega\in RS(g)$ on Q and g is of bounded variation on Q, then*

$$\left|\int_c^d\int_a^b\Omega(x,y)d_xd_yg(x,y)\right|\leq\sup_{(x,y)\in Q}\left|\Omega(x,y)\right|\bigvee_Q(g). \qquad (2.4)$$

In [24], Jawarneh and Noorani proved the following Ostrowski type inequality or functions of two variables with bounded variation:

Theorem 3. *Let $f:Q\to\to\mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y)\in Q$, we have inequality*

(2.5)

$$\begin{aligned}&\left|(b-a)(d-c)f(x,y)-\int_c^d\int_a^b f(t,s)dtds\right|\\ \leq\;&\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\\ &\times\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_Q(f)\end{aligned}$$

where $\bigvee_Q(f)$ denotes the total (double) variation of f on Q.

In [7], Budak and Sarıkaya have proved the following genaralization of the inequality (2.5):

Theorem 4. *Let $f : Q \to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x, y) \in Q$, we have inequality*

$$
\begin{aligned}
&\left| (b-a)(d-c) + \left[\lambda\eta \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right. \right. \\
&+ (1-\lambda)\eta \frac{f(a,y)+f(b,y)}{2} + \lambda(1-\eta)\frac{f(x,c)+f(x,d)}{2} \\
&\left. (1-\lambda)(1-\eta) f(x,y) \right] - \int_a^b \int_c^d f(t,s)dsdt \Bigg| \\
\leq\ & \max\left\{ \lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)a+\lambda b}{2}\right), \left(\frac{(2-\lambda)b+\lambda a}{2} - x\right) \right\} \\
&\times \max\left\{ \eta\frac{d-c}{2}, \left(y - \frac{(2-\eta)c+\eta d}{2}\right), \left(\frac{(2-\eta)d+\eta c}{2} - y\right) \right\} \bigvee_a^b \bigvee_c^d (f)
\end{aligned}
\tag{2.6}
$$

for any $\lambda, \eta \in [0,1]$ and $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$, $c + \eta\frac{d-c}{2} \leq y \leq d - \eta\frac{d-c}{2}$, where $\bigvee_a^b \bigvee_c^d (f)$ denotes he total variation of f on Q.

In [10], authors have proved the following Ostrowski type inequality for mappings of bounded variation.

Theorem 5. *If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then we have*

$$
\begin{aligned}
&\left| \frac{1}{4}\left[f(x,y) + f(x, c+d-y) + f(a+b-x, y) \right. \right. \\
&\left. + f(a+b-x, c+d-y) \right] - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \Bigg| \\
\leq\ & \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \left[\frac{1}{4} + \left| \frac{y - \frac{3c+d}{4}}{d-c} \right| \right] \bigvee_Q (f)
\end{aligned}
\tag{2.7}
$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and $y \in \left[c, \frac{c+d}{2}\right]$, where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q.

A companion of Ostrowski type inequalities for functions of two variables with bounded variation were given by Budak and Sarıkaya in [10]. Then, in [11], authors gave the generalization of inequalities in [10].

Recently, many of inequalities for functions of single variable with bounded variation have been proved. For more information and recent developments on inequalities for mappings of single variable with bounded variation, please refer to

([1]-[4],[8],[12],[13],[16],[17],[19]-[22],[25],[27]-[32]). In the literature, there are a few study for functions of two variables with bounded variation(see [6],[7],[9]-[11],[24]).

The aim of this paper is to establish new Ostrowski type inequalities for functions of two independent variables with bounded variation similar to inequalities in (1.2).

3. Main Results

First, we give the following notations to simplify presentetion of some intervals and functionals $\Psi_f(t,s)$ and $GS(f;u)$;

$$\begin{aligned} Q_1 &= [a,x]\times[c,y]\,,\ Q_2=[a,x]\times[y,d]\,,\\ Q_3 &= [x,b]\times[c,y]\,,\ Q_4=[x,b]\times[y,d]\,, \end{aligned}$$

$$\begin{aligned} &\Psi_f(x,y)=\frac{1}{(b-a)(d-c)}\\ &\times\left[(b-a)(d-c)f(x,y)-(b-a)(y-c)f(x,c)-(b-a)(d-y)f(x,d)\right.\\ &-(x-a)(d-c)f(a,y)-(b-x)(d-c)f(b,y)+(x-a)(y-c)f(a,c)\\ &\left.+(x-a)(d-y)f(a,d)+(b-x)(y-c)f(b,c)+(b-x)(d-y)f(b,d)\right], \end{aligned}$$

$$\begin{aligned} &GS(f;u)\\ =&\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\frac{f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)}{4}d_yd_xu(x,y)\\ &-\frac{u\left(\frac{a+b}{2},\frac{c+d}{2}\right)-u\left(\frac{a+b}{2},c\right)-u\left(a,\frac{c+d}{2}\right)+u(a,c)}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt. \end{aligned}$$

We may state the following results.

Theorem 6. *If the function $f:Q=[a,b]\times[c,d]\to R$ is of bounded variation on Q, then we have*

$$\begin{aligned} &|\Psi_f(x,y)|\\ \le\ &\frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\bigvee_{Q_1}(f)+(x-a)(d-y)\bigvee_{Q_2}(f)\right.\\ &\left.+(b-x)(y-c)\bigvee_{Q_3}(f)+(b-x)(d-y)\bigvee_{Q_4}(f)\right] \end{aligned} \tag{3.1}$$

$$\leq \begin{cases} \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\bigvee_{Q}(f), \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right. \\ \left.+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+\left[\bigvee_{Q_3}(f)\right]^{\beta}+\left[\bigvee_{Q_4}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}, \ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),\bigvee_{Q_3}(f),\bigvee_{Q_4}(f)\right\} \end{cases}$$

for all $(x,y)\in Q$.

Proof. Let us consider the mappings T defined by

$$T(x,t;y,s)=\begin{cases} (x-a)(y-c) & \text{if } (t,s)\in Q_1 \\ (x-a)(y-d) & \text{if } (t,s)\in Q_2 \\ (x-b)(y-c) & \text{if } (t,s)\in Q_3 \\ (x-b)(y-d) & \text{if } (t,s)\in Q_4. \end{cases}$$

Integrating by parts the mappings T on Q,

$$\begin{aligned} &\int_a^b\int_c^d T(x,t;y,s)d_sd_tf(t,s) \\ = \ & (x-a)(y-c)\int_a^x\int_c^y d_sd_tf(t,s)+(x-a)(y-d)\int_a^x\int_y^d d_sd_tf(t,s) \\ & +(x-b)(y-c)\int_x^b\int_c^y d_sd_tf(t,s)+(x-b)(y-d)\int_x^b\int_y^d d_sd_tf(t,s) \\ = \ & (x-a)(y-c)\left[f(x,y)-f(x,c)-f(a,y)+f(a,c)\right] \\ & +(x-a)(y-d)\left[f(x,d)-f(x,y)-f(a,d)+f(a,y)\right] \\ & +(x-b)(y-c)\left[f(b,y)-f(b,c)-f(x,y)+f(x,c)\right] \\ & +(x-b)(y-d)\left[f(b,d)-f(b,y)-f(x,d)+f(x,y)\right] \\ = \ & (b-a)(d-c)\Psi_f(x,y). \end{aligned}$$

Hence,

$$\Psi_f(x,y)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d T(x,t;y,s)d_sd_tf(t,s). \tag{3.2}$$

On the other hand, taking modulus in (3.2), then we have

(3.3)

$$\begin{aligned}
&|\Psi_f(x,y)| \\
=\ & \frac{1}{(b-a)(d-c)}\left|\int_a^b\int_c^d T(x,t;y,s)d_sd_tf(t,s)\right| \\
\leq\ & \frac{1}{(b-a)(d-c)} \\
&\times\left|(x-a)(y-c)\int_a^x\int_c^y d_sd_tf(t,s)+(x-a)(y-d)\int_a^x\int_y^d d_sd_tf(t,s)\right. \\
&\left.+(x-b)(y-c)\int_x^b\int_c^y d_sd_tf(t,s)+(x-b)(y-d)\int_x^b\int_y^d d_sd_tf(t,s)\right| \\
\leq\ & \frac{1}{(b-a)(d-c)} \\
&\times\left[(x-a)(y-c)\left|\int_a^x\int_c^y d_sd_tf(t,s)\right|+(x-a)(d-y)\left|\int_a^x\int_y^d d_sd_tf(t,s)\right|\right. \\
&\left.+(b-x)(y-c)\left|\int_x^b\int_c^y d_sd_tf(t,s)\right|+(b-x)(d-y)\left|\int_x^b\int_y^d d_sd_tf(t,s)\right|\right].
\end{aligned}$$

Using Lemma 2 in (3.3), then we get

(3.4)

$$\begin{aligned}
&|\Psi_f(x,y)| \\
\leq\ & \frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\bigvee_{Q_1}(f)+(x-a)(d-y)\bigvee_{Q_2}(f)\right. \\
&\left.+(b-x)(y-c)\bigvee_{Q_3}(f)+(b-x)(d-y)\bigvee_{Q_4}(f)\right]:=N(x,y)
\end{aligned}$$

which completes the proof of first inequality in (3.1).

$$\begin{aligned}
&N(x,y)\\
\leq\ & \frac{1}{(b-a)(d-c)}\max_{x,y}\{(x-a)(y-c),(x-a)(d-y),\\
&(b-x)(y-c),(b-x)(d-y)\}\\
&\times\left\{\bigvee_{Q_1}(f)+\bigvee_{Q_2}(f)+\bigvee_{Q_3}(f)+\bigvee_{Q_4}(f)\right\}\\
=\ & \frac{1}{(b-a)(d-c)}\max_{x}\Big\{(x-a)\max_{y}\{y-c,d-y\},\\
&(b-x)\max_{y}\{y-c,d-y\}\Big\}\bigvee_{Q}(f).
\end{aligned}$$

Since $\max_{y}$ is independent of x, we have

$$\begin{aligned}
&N(x,y)\\
\leq\ & \frac{1}{(b-a)(d-c)}\max_{x}\{x-a,b-x\}\max_{y}\{y-c,d-y\}\bigvee_{Q}(f)\\
=\ & \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\bigvee_{Q}(f).
\end{aligned}$$

This fnishes the proof of the first branch in the second inequality in (3.1).

For $\frac{1}{\alpha}+\frac{1}{\beta}=1$, using Hölder's discrete inequality in (3.4), then we have

$$\begin{aligned}
&N(x,y)\\
\leq\ & \frac{1}{(b-a)(d-c)}\left[(x-a)^{\alpha}(y-c)^{\alpha}+(x-a)^{\alpha}(d-y)^{\alpha}\right.\\
&\left.+(b-x)^{\alpha}(y-c)^{\alpha}+(b-x)^{\alpha}(d-y)^{\alpha}\right]^{\frac{1}{\alpha}}\\
&\times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+\left[\bigvee_{Q_3}(f)\right]^{\beta}+\left[\bigvee_{Q_4}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}\\
=\ & \left[\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right.\\
&\left.+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\\
&\times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+\left[\bigvee_{Q_3}(f)\right]^{\beta}+\left[\bigvee_{Q_4}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}
\end{aligned}$$

which completes the proof of the second branch in the second inequality in (3.1).

Finally,

$$
\begin{aligned}
& N(x,y) \\
\leq\ & \frac{1}{(b-a)(d-c)}\max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),\bigvee_{Q_3}(f),\bigvee_{Q_4}(f)\right\} \\
& \times\left[(x-a)(y-c)+(x-a)(d-y)+(b-x)(y-c)+(b-x)(d-y)\right] \\
=\ & \max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),\bigvee_{Q_3}(f),\bigvee_{Q_4}(f)\right\}.
\end{aligned}
$$

The proof is completely completed. □

Corollary 1. *Under assumption Theorem 6, choosing* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$*, we have*

$$
\begin{aligned}
& \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
& \left.-\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| \\
\leq\ & \frac{1}{4}\bigvee_{Q}(f).
\end{aligned}
$$

Theorem 7. *Let* $u:Q=[a,b]\times[c,d]\to R$ *be a mapping of bounded variation on* Q *and* $f:Q\to R$ *be continuous and of bounded variation on* Q*. Then we have the inequality:*

$$
|GS(f;u)|\leq\frac{1}{4}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)\bigvee_{a}^{\frac{a+b}{2}}\bigvee_{c}^{\frac{c+d}{2}}(u).
$$

Proof. Using Lemma 2, we have

$$
\begin{aligned}
&|GS(f;u)| \\
= &\left| \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[\frac{1}{4} \left[f(x,y) + f(x,c+d-y) + f(a+b-x,y) \right.\right.\right. \\
&\left.\left.\left. + f(a+b-x,c+d-y) \right] - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right] d_y d_x u(x,y) \right| \\
\leq &\sup_{(x,y)\in Q_1} \left| \frac{1}{4} \left[f(x,y) + f(x,c+d-y) + f(a+b-x,y) \right.\right. \\
&\left.\left. + f(a+b-x,c+d-y) \right] - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \bigvee_a^{\frac{a+b}{2}} \bigvee_c^{\frac{c+d}{2}} (u).
\end{aligned}
$$

Since f is of bounded variation, using Theorem 5, we have

$$
\begin{aligned}
&\left| \frac{f(x,y) + f(x,c+d-y) + f(a+b-x,y) + f(a+b-x,c+d-y)}{4} \right. \\
&\left. - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \\
\leq &\left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \left[\frac{1}{4} + \left| \frac{y - \frac{3c+d}{4}}{d-c} \right| \right] \bigvee_a^b \bigvee_c^d (f).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|GS(f;u)| &\leq \sup_{(x,y)\in Q_1} \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \left[\frac{1}{4} + \left| \frac{y - \frac{3c+d}{4}}{d-c} \right| \right] \bigvee_a^b \bigvee_c^d (f) \bigvee_a^{\frac{a+b}{2}} \bigvee_c^{\frac{c+d}{2}} (u) \\
&\leq \frac{1}{4} \bigvee_a^b \bigvee_c^d (f) \bigvee_a^{\frac{a+b}{2}} \bigvee_c^{\frac{c+d}{2}} (u).
\end{aligned}
$$

This completes the proof. □

Corollary 2. *If* $\bigvee_a^{\frac{a+b}{2}} \bigvee_c^{\frac{c+d}{2}} (u) = \bigvee_a^{\frac{a+b}{2}} \bigvee_{\frac{c+d}{2}}^{d} (u) = \bigvee_{\frac{a+b}{2}}^{b} \bigvee_c^{\frac{c+d}{2}} (u) = \bigvee_{\frac{a+b}{2}}^{b} \bigvee_{\frac{c+d}{2}}^{d} (u)$ *in Theorem 7, then we have*

$$
|GS(f;u)| \leq \frac{1}{16} \bigvee_a^b \bigvee_c^d (f) \bigvee_a^b \bigvee_c^d (u).
$$

4. Application to Qudrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$.

Then the following Theorem holds.

Theorem 8. *Let f and u be as in Theorem 7. Then we have the quadrature formula:*

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} \frac{f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)}{4} d_y d_x u(x,y)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \frac{u\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)-u\left(\frac{x_i+x_{i+1}}{2},y_j\right)-u\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+u\left(x_i,y_j\right)}{h_i l_j}$$

$$\times \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt + R(I_n,J_m,f,u)$$

The remainder term $R(I_n,J_m,f,u)$ satisfies

$$|R(I_n,J_m,f,u)| \leq \frac{1}{4} \max_{\substack{i=\overline{0,n-1}\\ j=\overline{0,m-1}}} \left\{ \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u) \right\} \bigvee_a^b \bigvee_c^d (f).$$

Proof. Applying Theorem 7 to bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have the inequality

$$\text{(4.1)}\quad \left| \int_{x_i}^{\frac{x_i+x_{i+1}}{2}}\int_{y_j}^{\frac{y_j+y_{j+1}}{2}} \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)\right.\right.$$

$$\left. +f(a+b-x,c+d-y)\right] d_y d_x u(x,y)$$

$$-\frac{u\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)-u\left(\frac{x_i+x_{i+1}}{2},y_j\right)-u\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+u\left(x_i,y_j\right)}{h_i l_j}$$

$$\left.\times \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt\right|$$

$$\leq \frac{1}{4} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u).$$

Summing the inequality (4.1) over i from 0 to $n-1$ and j from 0 to $m-1$, then we get

$$\begin{aligned}
&\left|R(I_n,J_m,f,u)\right| \\
&\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}}\bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}}(u) \\
&\leq \frac{1}{4}\max_{\substack{i=\overline{0,n-1}\\ j=\overline{0,m-1}}}\left\{\bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}}\bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}}(u)\right\}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
&= \frac{1}{4}\max_{\substack{i=\overline{0,n-1}\\ j=\overline{0,m-1}}}\left\{\bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}}\bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}}(u)\right\}\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

This completes the proof. □

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