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ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we obtain new weighted Ostrowski type inequalities for functions of two independent variables with bounded variation.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[19]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [15], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [22], Tseng et al. gave the following weighted Ostrowski type inequalities for functions of bounded variation:

Theorem 2. *Let us have $0\leq\alpha\leq1$, let $w:[a,b]\to[0,\infty)$ continuous and positive on (a,b) and $h:[a,b]\to\mathbb{R}$ be differentiable such that $h'(t)=w(t)$ on $[a,b]$. Let $a_1=h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, $b_1=h^{-1}\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right)$. If $f:[a,b]\to\mathbb{R}$ be mapping of bounded variation on $[a,b]$, then for all $x\in[a_1,b_1]$,*

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we have the inequality

$$\left|\int_a^b w(t)f(t)dt-\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b w(t)dt\right|\leq K\bigvee_a^b(f) \tag{1.2}$$

where

$$K:=\begin{cases}\frac{1-\alpha}{2}\int_a^b w(t)dt+\left|h(x)-\frac{h(a)+h(b)}{2}\right|, & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \max\left\{\frac{1-\alpha}{2}\int_a^b w(t)dt+\left|h(x)-\frac{h(a)+h(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w(t)dt\right\}, & \text{if } \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}\int_a^b w(t)dt, & \text{if } \frac{2}{3}\leq\alpha\leq 1\end{cases}$$

and $\bigvee_a^b(f)$ *denotes the total variation of* f *on interval* $[a,b]$. *In (1.2), the constant* $\frac{1-\alpha}{2}$ *for* $0\leq\alpha\leq\frac{1}{2}$ *and the constant* $\frac{\alpha}{2}$ *for* $\frac{2}{3}\leq\alpha\leq 1$ *are the best possible.*

2. Preliminaries and Lemmas

In 1910, Fréchet [17] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $g(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}g(x_i,y_j)=g(x_{i-1},y_{j-1})-g(x_{i-1},y_j)-g(x_i,y_{j-1})+g(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}g(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}g(x_i,y_j),$$

where ξ_{ij}, η_{ij} are numbers satisfying $\xi_{ij} \in [x_{i-1}, x_i]$, $\eta_{ij} \in [y_{j-1}, y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x,y) d_y d_x g(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [13] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a \leq x \leq b,\ c \leq y \leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i (i = 0,1,2,...,m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j (j = 0,1,2,...,n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell.* We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)$$

$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j)$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$,or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function* $f(x,y)$ *is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function* $f(x,y)$ *is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ *and* $\overline{\epsilon_j} = \pm 1$.

Definition 3. *(Hardy-Krause). The function* $f(x,y)$ *is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition* $f(\overline{x}, y)$ *is of bounded variation in* y *(i.e.* $\phi(\overline{x})$ *is finite) for at least one* $\overline{x}$ *and* $f(x, \overline{y})$ *is of bounded variation in* y *(i.e.* $\psi(\overline{y})$ *is finite) for at least one* $\overline{y}$.

Definition 4. *(Arzelà). Let* (x_i, y_i) $(i = 0,1,2,...,m)$ *be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_Q (f) := \bigvee_c^d \bigvee_a^b (f) := \sup\left\{\sum(P) : P \in P(Q)\right\},$$

is called the total variation of f on Q.

In [18], authors proved the following Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(g)$ on Q, then $g \in RS(f)$ on Q, and we have*

$$\begin{aligned} (2.3) \qquad & \int_c^d\int_a^b f(t,s) d_t d_s g(t,s) + \int_c^d\int_a^b g(t,s) d_t d_s f(t,s) \\ = \quad & f(b,d)g(b,d) - f(b,c)g(b,c) - f(a,d)g(a,d) + f(a,c)g(a,c). \end{aligned}$$

Lemma 2. *Assume that $\Omega \in RS(g)$ on Q and g is of bounded variation on Q, then*

$$(2.4) \qquad \left|\int_c^d\int_a^b \Omega(x,y) d_x d_y g(x,y)\right| \le \sup_{(x,y)\in Q} |\Omega(x,y)| \bigvee_Q (g).$$

In [18], Jawarneh and Noorani obtained the following Ostrowski type inequality for functions of two variables with bounded variation:

Theorem 3. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned} (2.5) \qquad & \left|(b-a)(d-c) f(x,y) - \int_c^d\int_a^b f(t,s) dt ds\right| \\ \le \quad & \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \\ & \times \left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right] \bigvee_Q (f) \end{aligned}$$

where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q.

In [6], Budak and Sarikaya proved the following generalization of the inequality (2.5):

Theorem 4. *Let $f : Q \to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$
\begin{aligned}
(2.6) \qquad & \Bigg| (b-a)(d-c)\Big[(1-\lambda)(1-\eta) f(x,y) \\
& + \frac{(1-\lambda)\eta}{2}\left[f(a,y)+f(b,y)\right] + \frac{\lambda(1-\eta)}{2}\left[f(x,c)+f(x,d)\right] \\
& + \frac{\lambda\eta}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\Big] - \int_a^b\int_c^d f(t,s)\,ds\,dt \Bigg| \\
\leq \; & \max\left\{\lambda\frac{b-a}{2}, \left(x-\frac{(2-\lambda)a+\lambda b}{2}\right), \left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\} \\
& \times \max\left\{\eta\frac{d-c}{2}, \left(y-\frac{(2-\eta)c+\eta d}{2}\right), \left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\} \bigvee_a^b \bigvee_c^d (f)
\end{aligned}
$$

for any $\lambda, \eta \in [0,1]$ and $a+\lambda\frac{b-a}{2} \leq x \leq b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2} \leq y \leq d-\eta\frac{d-c}{2}$, where $\bigvee_a^b\bigvee_c^d(f)$ denotes he total variation of f on Q.

For more information and recent developments on inequalities for mappings of bounded variation, please refer to([1]-[11],[14]-[16],[18],[20]-[25]).

The aim of this paper is to establish new weighted Ostrowski type inequalities for functions of two independent variables with bounded variation.

3. Main Results

Let us have $0 \leq \alpha, \beta \leq 1$. Let $w_1 : [a,b] \to [0,\infty)$ continuous and positive on (a,b) and $h_1 : [a,b] \to \mathbb{R}$ be differentiable such that $h_1'(t) = w_1(t)$ on $[a,b]$. Similarly, let $w_2 : [c,d] \to [0,\infty)$ continuous and positive on (c,d) and $h_2 : [c,d] \to \mathbb{R}$ be differentiable such that $h_2'(t) = w_2(t)$ on $[c,d]$. Let $a_1 = h_1^{-1}\left(\left(1-\frac{\alpha}{2}\right)h_1(a) + \frac{\alpha}{2}h_1(b)\right)$, $b_1 = h_1^{-1}\left(\frac{\alpha}{2}h_1(a) + \left(1-\frac{\alpha}{2}\right)h_1(b)\right)$, $c_1 = h_2^{-1}\left(\left(1-\frac{\beta}{2}\right)h_2(c) + \frac{\beta}{2}h_2(d)\right)$ and $d_1 = h_2^{-1}\left(\frac{\beta}{2}h_2(c) + \left(1-\frac{\beta}{2}\right)h_2(d)\right)$.

Theorem 5. *If $f:[a,b]\times[c,d]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]\times[c,d]$, then we have the following inequality for all $(x,y)\in[a_1,b_1]\times[c_1,d_1]$,*

(3.1)
$$\begin{aligned}
&\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\left[(1-\alpha)(1-\beta)f(x,y)\right.\right.\\
&+(1-\alpha)\beta\frac{f(x,c)+f(x,d)}{2}+\alpha(1-\beta)\frac{f(a,y)+f(b,y)}{2}\Big]\\
&\left.+\alpha\beta\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\right|\\
\leq\ & KL\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

where

$$K=\begin{cases}\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|, & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \max\left\{\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w_1(t)dt\right\}, & \text{if } \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}\int_a^b w_1(t)dt, & \text{if } \frac{2}{3}\leq\alpha\leq 1\end{cases}$$

and

$$L=\begin{cases}\frac{1-\beta}{2}\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|, & \text{if } 0\leq\beta\leq\frac{1}{2}\\ \max\left\{\frac{1-\beta}{2}\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}\int_c^d w_2(t)dt\right\}, & \text{if } \frac{1}{2}<\beta<\frac{2}{3}\\ \frac{\beta}{2}\int_c^d w_2(t)dt, & \text{if } \frac{2}{3}\leq\beta\leq 1\end{cases}$$

and $\bigvee_a^b\bigvee_c^d(f)$ *denotes the total variation of f on interval $[a,b]\times[c,d]$.*

Proof. For $(x,y)\in[a_1,b_1]\times[c_1,d_1]$, we define the following mappings q,p by

$$q(t)=\begin{cases}h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right], & t\in[a,x)\\ h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right], & t\in[x,b]\end{cases}$$

and

$$p(s)=\begin{cases} h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right], & s\in[c,y) \\ h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right], & s\in[y,d]. \end{cases}$$

Using the $q(t)$ and $p(s)$ kernels, we have

$$\begin{aligned} & \int_a^b\int_c^d q(t)p(s)d_sd_tf(t,s) \\ = & \int_a^x\int_c^y \left(h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right]\right) \\ & \times\left(h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right]\right)d_sd_tf(t,s) \\ & +\int_a^x\int_y^d \left(h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right]\right) \\ & \times\left(h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right]\right)d_sd_tf(t,s) \\ & +\int_x^b\int_c^y \left(h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right) \\ & \times\left(h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right]\right)d_sd_tf(t,s) \\ & +\int_x^b\int_y^d \left(h_1(t)-\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right) \\ & \times\left(h_2(s)-\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right]\right)d_sd_tf(t,s) \\ = & I_1+I_2+I_3+I_4. \end{aligned}$$

Here, by integrating by parts we get

$$\begin{aligned} & I_1 \\ = & \int_a^x\int_c^y \left(h_1(t)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right]\right) \\ & \times\left(h_2(s)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right]\right)d_sd_tf(t,s) \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&= \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \\
&\quad \times \left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(x,y) \\
&\quad + \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right]\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,c) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y) - \left(1-\frac{\beta}{2}\right)h_2(c) - \frac{\beta}{2}h_2(d)\right] f(a,y) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(a,c) \\
&\quad - \int_a^x\int_c^y w_1(t)w_2(s)f(t,s)dsdt,
\end{aligned}
$$

By similar method, we have

$$
\begin{aligned}
&I_2 \\
&= \int_a^x\int_y^d \left(h_1(t) - \left[\left(1-\frac{\alpha}{2}\right)h_1(a) + \frac{\alpha}{2}h_1(b)\right]\right) \\
&\quad \times \left(h_2(s) - \left[\frac{\beta}{2}h_2(c) + \left(1-\frac{\beta}{2}\right)h_2(d)\right]\right) d_s d_t f(t,s) \\
&= \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right]\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(x,d) \\
&\quad - \left[h_1(x) - \left(1-\frac{\alpha}{2}\right)h_1(a) - \frac{\alpha}{2}h_1(b)\right] \\
&\quad \times \left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(x,y) \\
&\quad + \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left(\beta\frac{h_2(d)-h_2(c)}{2}\right) f(a,d) \\
&\quad - \left(\alpha\frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y) - \frac{\beta}{2}h_2(c) - \left(1-\frac{\beta}{2}\right)h_2(d)\right] f(a,y) \\
&\quad - \int_a^x\int_y^d w_1(t)w_2(s)f(t,s)dsdt,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&I_3 \\
&= \int_x^b\int_c^y \left(h_1(t) - \left[\frac{\alpha}{2}h_1(a) + \left(1-\frac{\alpha}{2}\right)h_1(b)\right]\right) \\
&\quad \times \left(h_2(s) - \left[\left(1-\frac{\beta}{2}\right)h_2(c) + \frac{\beta}{2}h_2(d)\right]\right) d_s d_t f(t,s)
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
= & \left(\alpha \frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y)-\left(1-\frac{\beta}{2}\right) h_2(c)-\frac{\beta}{2} h_2(d)\right] f(b, y) \\
& +\left(\alpha \frac{h_1(b)-h_1(a)}{2}\right)\left(\beta \frac{h_2(d)-h_2(c)}{2}\right) f(b, c) \\
& -\left[h_1(x)-\frac{\alpha}{2} h_1(a)-\left(1-\frac{\alpha}{2}\right) h_1(b)\right] \\
& \times\left[h_2(y)-\left(1-\frac{\beta}{2}\right) h_2(c)-\frac{\beta}{2} h_2(d)\right] f(x, y) \\
& -\left[h_1(x)-\frac{\alpha}{2} h_1(a)-\left(1-\frac{\alpha}{2}\right) h_1(b)\right]\left(\beta \frac{h_2(d)-h_2(c)}{2}\right) f(x, c) \\
& -\int_x^b \int_c^y w_1(t) w_2(s) f(t, s) d s d t,
\end{aligned}
$$

and,

$$
\begin{aligned}
& I_4 \\
= & \int_x^b \int_y^d\left(h_1(t)-\left[\frac{\alpha}{2} h_1(a)+\left(1-\frac{\alpha}{2}\right) h_1(b)\right]\right) \\
& \times\left(h_2(s)-\left[\frac{\beta}{2} h_2(c)+\left(1-\frac{\beta}{2}\right) h_2(d)\right]\right) d_s d_t f(t, s) \\
= & \left(\alpha \frac{h_1(b)-h_1(a)}{2}\right)\left(\beta \frac{h_2(d)-h_2(c)}{2}\right) f(b, d) \\
& -\left(\alpha \frac{h_1(b)-h_1(a)}{2}\right)\left[h_2(y)-\frac{\beta}{2} h_2(c)-\left(1-\frac{\beta}{2}\right) h_2(d)\right] f(b, y) \\
& -\left[h_1(x)-\frac{\alpha}{2} h_1(a)-\left(1-\frac{\alpha}{2}\right) h_1(b)\right]\left(\beta \frac{h_2(d)-h_2(c)}{2}\right) f(x, d) \\
& +\left[h_1(x)-\frac{\alpha}{2} h_1(a)-\left(1-\frac{\alpha}{2}\right) h_1(b)\right] \\
& \times\left[h_2(y)-\frac{\beta}{2} h_2(c)-\left(1-\frac{\beta}{2}\right) h_2(d)\right] f(x, y) \\
& -\int_x^b \int_y^d w_1(t) w_2(s) f(t, s) d s d t .
\end{aligned} \tag{3.5}
$$

Adding (3.2)-(3.5), we have

$$
\begin{aligned}
& \int_a^b \int_c^d q(t) p(s) d_s d_t f(t, s) \\
= & \left(\int_a^b w(t) d t\right)\left(\int_c^d g(t) d t\right)\left[(1-\alpha)(1-\beta) f(x, y)+(1-\alpha) \beta \frac{f(x, c)+f(x, d)}{2}\right.
\end{aligned}
$$

$$+\alpha(1-\beta)\frac{f(a,y)+f(b,y)}{2}+\alpha\beta\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\Bigg]$$
$$-\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt.$$

On the other hand, using Lemma 2 we have

$$\begin{aligned}
&\left|\int_a^b\int_c^d q(t)p(s)d_sd_tf(t,s)\right| \\
\leq\ & \sup_{t\in[a,b]}|q(t)|\sup_{s\in[c,d]}|p(s)|\bigvee_a^b\bigvee_c^d(f) \\
=\ & \max\left\{h_1(x)-\left[\left(1-\frac{\alpha}{2}\right)h_1(a)+\frac{\alpha}{2}h_1(b)\right],\left[\frac{\alpha}{2}h_1(a)+\left(1-\frac{\alpha}{2}\right)h_1(b)\right]-h_1(x),\frac{\alpha}{2}\left[h_1(b)-h_1(a)\right]\right\} \\
& \times\max\left\{h_2(y)-\left[\left(1-\frac{\beta}{2}\right)h_2(c)+\frac{\beta}{2}h_2(d)\right],\left[\frac{\beta}{2}h_2(c)+\left(1-\frac{\beta}{2}\right)h_2(d)\right]-h_2(y),\frac{\beta}{2}\left[h_2(d)-h_2(c)\right]\right\} \\
& \times\bigvee_a^b\bigvee_c^d(f) \\
=\ & \max\left\{\frac{1-\alpha}{2}\left[h_1(b)-h_1(a)\right]+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}\left[h_1(b)-h_1(a)\right]\right\} \\
& \times\max\left\{\frac{1-\beta}{2}\left[h_2(d)-h_2(c)\right]+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}\left[h_2(d)-h_2(c)\right]\right\}\bigvee_a^b\bigvee_c^d(f) \\
=\ & \max\left\{\frac{1-\alpha}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|,\frac{\alpha}{2}\int_a^b w_1(t)dt\right\} \\
& \times\max\left\{\frac{1-\beta}{2}\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|,\frac{\beta}{2}\int_c^d w_2(t)dt\right\}\bigvee_a^b\bigvee_c^d(f) \\
=\ & KL\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

This completes the proof of the Theorem 5. □

Remark 1. *If we choose* $w(t)\equiv g(s)\equiv 1$ $(h_1(t)=t$ *and* $h_2(s)=s)$, *and* $\alpha=\beta=0$, *then the inequality (3.1) reduces the inequality (2.5).*

Remark 2. *If we choose* $w(t) \equiv g(s) \equiv 1$ $(h_1(t) = t$ *and* $h_2(s) = s)$, $\alpha = \beta = \frac{1}{3}$, $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in the inequality (3.1), then we have the Simpson's inequality*

$$\begin{aligned} &\left| \frac{f(b,d) + f(b,c) + f(a,d) + f(a,c)}{36} \right. \\ &+ \frac{f\left(a, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(b, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right)}{9} \\ &\left. + \frac{4}{9} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,ds\,dt \right| \\ \leq \;& \frac{1}{9} \bigvee_a^b \bigvee_c^d (f) \end{aligned}$$

which is proved by Jawarneh and Noorani in [18].

Remark 3. *If we choose* $w_1(t) \equiv w_2(s) \equiv 1$ $(h_1(t) = t$ *and* $h_2(s) = s)$, $\alpha = 1$, $\beta = 0$, $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in the inequality (3.1), then we have*

$$\left| \frac{f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)}{2} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,ds\,dt \right| \leq \frac{1}{4} \bigvee_Q (f).$$

which is given by Budak and Sarikaya in [10]. *The constant* $\frac{1}{4}$ *is the best possible.*

Remark 4. *If we choose* $w_1(t) \equiv w_2(s) \equiv 1$ $(h_1(t) = t$ *and* $h_2(s) = s)$, $\alpha = 0$, $\beta = 1$, $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in the inequality (3.1), then we have*

$$\left| \frac{f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)}{2} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,ds\,dt \right| \leq \frac{1}{4} \bigvee_Q (f)$$

which is proved by Budak and Sarikaya in [10]. *The constant* $\frac{1}{4}$ *is the best possible.*

Remark 5. *If we choose* $w_1(t) \equiv w_2(s) \equiv 1$ $(h_1(t) = t$ *and* $h_2(s) = s)$, $\alpha = \beta = \frac{1}{2}$, $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in the inequality (3.1), then we have*

(3.6)

$$\begin{aligned} &\left| \frac{1}{4} \left[\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \right. \right. \\ &\frac{1}{2} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right] \\ &\left. \left. + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,ds\,dt \right| \\ \leq \;& \frac{1}{16} \bigvee_Q (f). \end{aligned}$$

which is given by Budak and Sarikaya in [6]. *The constant* $\frac{1}{16}$ *is the best possible.*

Corollary 1 (weighted Ostrowski inequality)**.** *Under the assumption Theorem 5, if we choose $\alpha=\beta=0$, for all $(x,y)\in[a,b]\times[c,d]$, then we have the following weighted Ostrowski inequality;*

$$\begin{aligned}&\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)f(x,y)-\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ &\left[\frac{1}{2}\int_a^b w_1(t)dt+\left|h_1(x)-\frac{h_1(a)+h_1(b)}{2}\right|\right]\\ &\times\left[\frac{1}{2}\int_c^d w_2(t)dt+\left|h_2(y)-\frac{h_2(c)+h_2(d)}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

Corollary 2 (weighted trapezoid inequality)**.** *Under the assumption Theorem 5, if we choose $\alpha=\beta=1$, then we have the following weighted trapezoid inequality;*

$$\begin{aligned}&\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)-\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ &\frac{1}{4}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

Corollary 3 (weighted Simpson's inequality)**.** *Under assumption Theorem 5, if we choose $\alpha=\beta=\frac{1}{3}$, $x=h_1^{-1}\left(\frac{h_1(a)+h_1(b)}{2}\right)$ and $y=h_2^{-1}\left(\frac{h_2(c)+h_2(d)}{2}\right)$, then we have the weighted Simpson's inequality*

$$\begin{aligned}&\left|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36}\right.\right.\\ &+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9}\\ &\left.\left.+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ &\frac{1}{9}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

4. Some Composite Quadrature Formula

Let us consider the arbitrary division $I_n: a=x_0<x_1<...<x_n=b$, and $J_m: c=y_0<y_1<...<y_m=d, l_1^i:=x_{i+1}-x_i$, and $l_2^j:=y_{j+1}-y_j$,

$$\upsilon(l_1):=\max\left\{l_1^i\middle|\ i=0,...,n-1\right\},$$

$$\upsilon(l_2) := \max\left\{ l_2^j \middle| \; j = 0, ..., m-1 \right\},$$

$$\upsilon(W_1) := \max\left\{ W_1^i \middle| \; i = 0, ..., n-1 \right\}, \; W_1^i := \int_{x_i}^{x_{i+1}} w_1(u)du = h_1(x_{i+1}) - h_1(x_i),$$

$$\upsilon(W_2) := \max\left\{ W_2^j \middle| \; j = 0, ..., m-1 \right\}, \; W_2^j := \int_{y_j}^{y_{j+1}} w_2(u)du = h_2(y_{j+1}) - h_2(y_j).$$

Let us have $\alpha, \beta, w_1, h_1, w_2,$ and h_2 defined as in Theorem 5. Let $a_1^i = h_1^{-1}\left(\left(1-\frac{\alpha}{2}\right)h_1(x_i) + \frac{\alpha}{2}h_1(x_{i+1})\right)$, $b_1^i = h_1^{-1}\left(\frac{\alpha}{2}h_1(x_i) + \left(1-\frac{\alpha}{2}\right)h_1(x_{i+1})\right)$, $c_1^j = h_2^{-1}\left(\left(1-\frac{\beta}{2}\right)h_2(y_{j+1}) + \frac{\beta}{2}h_2(y_{j+1})\right)$ and $d_1^j = h_2^{-1}\left(\frac{\beta}{2}h_2(y_j) + \left(1-\frac{\beta}{2}\right)h_2(y_{j+1})\right)$, $\xi_i \in \left[a_1^i, b_1^i\right]$, $(i = 0, ..., n-1)$ and $\eta_j \in \left[c_1^j, d_1^j\right]$ $(\, j = 0, ..., m-1)$.

Define the sum

$$\begin{aligned} & A(f, w_1, h_1, w_2, h_2, I_n, J_m) \\ = & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[(1-\alpha)(1-\beta) f\left(\xi_i, \eta_j\right)\right. \\ & + (1-\alpha)\beta\frac{f(\xi_i, y_j) + f(\xi_i, y_{j+1})}{2} + \alpha(1-\beta)\frac{f(x_i, \eta_j) + f(x_{i+1}, \eta_j)}{2} \\ & \left. + \alpha\beta\frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4}\right] W_1^i W_2^j. \end{aligned}$$

Theorem 6. *Let f defined as in Theorem 5 and let*

$$\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt = A(f, w_1, h_1, w_2, h_2, I_n, J_m) + R(f, w_1, h_1, w_2, h_2, I_n, J_m).$$

The remainder term $R(f, w_1, h_1, w_2, h_2, I_n, J_m)$ satisfies

$$\begin{aligned} & |R(f, w_1, h_1, w_2, h_2, I_n, J_m)| \\ \leq & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} K_i L_j \bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \leq M_1 N_1 \bigvee_a^b\bigvee_c^d(f) \\ \leq & M_2 N_2 \bigvee_a^b\bigvee_c^d(f) \leq M_3 N_3 \bigvee_a^b\bigvee_c^d(f) \end{aligned} \tag{4.1}$$

where

$$K_i = \begin{cases} \frac{1-\alpha}{2}W_1^i + \left|h_1(\xi_i) - \frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{2}W_1^i + \left|h_1(\xi_i) - \frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|, \frac{\alpha}{2}W_1^i\right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}W_1^i, & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases} \quad (i = 0, ..., n-1),$$

$$M_1=\begin{cases}\max\limits_{i=0,\dots,n-1}\left\{\frac{1-\alpha}{2}W_1^i+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|\right\}, & \textit{if } 0\le\alpha\le\frac{1}{2}\\ \max\limits_{i=0,\dots,n-1}\left\{\max\left\{\frac{1-\alpha}{2}\upsilon(W_1)+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|,\frac{\alpha}{2}\upsilon(W_1)\right\}\right\}, & \textit{if } \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}\upsilon(W_1), & \textit{if } \frac{2}{3}\le\alpha\le 1\end{cases},$$

$$M_2=\begin{cases}\frac{1-\alpha}{2}\upsilon(W_1)+\max\limits_{i=0,\dots,n-1}\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|, & \textit{if } 0\le\alpha\le\frac{1}{2}\\ \max\limits_{i=0,\dots,n-1}\left\{\max\left\{\frac{1-\alpha}{2}\upsilon(W_1)dt+\left|h_1(\xi_i)-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|,\frac{\alpha}{2}\upsilon(W_1)\right\}\right\}, & \textit{if } \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}\upsilon(W_1), & \textit{if } \frac{2}{3}\le\alpha\le 1\end{cases},$$

$$M_3=\begin{cases}(1-\alpha)\,\upsilon(W_1) & \textit{if } 0\le\alpha\le\frac{2}{3}\\ \frac{\alpha}{2}\upsilon(W_1), & \textit{if } \frac{2}{3}\le\alpha\le 1\end{cases}$$

and similarly

$$L_i=\begin{cases}\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|, & \textit{if } 0\le\beta\le\frac{1}{2}\\ \max\left\{\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}W_2^j\right\}, & \textit{if } \frac{1}{2}<\beta<\frac{2}{3}\\ \frac{\beta}{2}W_2^j, & \textit{if } \frac{2}{3}\le\beta\le 1.\end{cases}\quad (\,j=0,\dots,m-1)\,,$$

$$N_1=\begin{cases}\max\limits_{j=0,\dots,m-1}\left\{\frac{1-\beta}{2}W_2^j+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|\right\}, & \textit{if } 0\le\beta\le\frac{1}{2}\\ \max\limits_{j=0,\dots,m-1}\left\{\max\left\{\frac{1-\beta}{2}\upsilon(W_2)+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}\upsilon(W_2)\right\}\right\}, & \textit{if } \frac{1}{2}<\beta<\frac{2}{3}\\ \frac{\beta}{2}\upsilon(W_2), & \textit{if } \frac{2}{3}\le\beta\le 1.\end{cases},$$

$$N_2=\begin{cases}\frac{1-\beta}{2}\upsilon(W_2)+\max\limits_{j=0,\dots,m-1}\left\{\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|\right\}, & \textit{if } 0\le\beta\le\frac{1}{2}\\ \max\limits_{j=0,\dots,m-1}\left\{\max\left\{\frac{1-\beta}{2}\upsilon(W_2)+\left|h_2(\eta_j)-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|,\frac{\beta}{2}\upsilon(W_2)\right\}\right\}, & \textit{if } \frac{1}{2}<\beta<\frac{2}{3}\\ \frac{\beta}{2}\upsilon(W_2), & \textit{if } \frac{2}{3}\le\beta\le 1.\end{cases},$$

$$N_3=\begin{cases}(1-\beta)\,\upsilon(W_2), & \textit{if } 0\le\beta\le\frac{2}{3}\\ \frac{\beta}{2}\upsilon(W_2), & \textit{if } \frac{2}{3}\le\beta\le 1.\end{cases}.$$

Proof. Appliying Theorem 5 to the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

$$\begin{aligned}(4.2)\qquad &\Bigg|\Bigg[(1-\alpha)(1-\beta)f\left(\xi_i,\eta_j\right)\\ &+(1-\alpha)\beta\frac{f(\xi_i,y_j)+f(\xi_i,y_{j+1})}{2}\\ &+\alpha(1-\beta)\frac{f(x_i,\eta_j)+f(x_{i+1},\eta_j)}{2}\\ &+\alpha\beta\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4}\Bigg]W_1^iW_2^j\\ &-\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}w_1(t)w_2(s)f(t,s)dsdt\Bigg|\\ \le\;\; & K_iL_j\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f).\end{aligned}$$

for any $\xi_i \in \left[a_1^i, b_1^i\right]$, $(i = 0, ..., n-1)$ and $\eta_j \in \left[c_1^j, d_1^j\right]$ $(\, j = 0, ..., m-1)$.

Summing the inequality (4.2) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, then we get

$$
\begin{aligned}
(4.3) \qquad & |R(f, w_1, h_1, w_2, h_2, I_n, J_m)| \\
\leq \; & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} K_i L_j \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
\leq \; & \left(\max_{i=0,\ldots,n-1} K_i \right) \left(\max_{j=0,\ldots,m-1} L_j \right) \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
= \; & M_1 N_1 \bigvee_a^b \bigvee_c^d (f) \leq M_2 N_2 \bigvee_a^b \bigvee_c^d (f).
\end{aligned}
$$

This finishes the proofs of the first three inequalities in (4.1).

In the last inequality (4.3), we observe that

$$\left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \leq \frac{1-\alpha}{2} W_1^i,$$

and so

$$\max_{i=0,\ldots,n-1} \left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \leq \frac{1-\alpha}{2} \upsilon(W_1).$$

Similarly

$$\max_{j=0,\ldots,m-1} \left| h_2(\eta_j) - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \leq \frac{1-\beta}{2} \upsilon(W_2).$$

These show that $M_2 \leq M_3$ and $N_2 \leq N_3$.

The proof of the theorem is completely completed. □

Remark 6. *If we choose* $\alpha = \beta = 0$, $w_1(t) \equiv 1, h_1(t) = t$ *on* $[a, b]$ *and* $w_2(s) \equiv 1$, $h_2(s) = s$ *on* $[c, d]$ *in Theorem 6, then the inequalities (4.1) reduce to the inequalityies (4.2) in* [5].

Remark 7. *If we choose* $\alpha = \beta = \frac{1}{3}$, $w(t) \equiv 1$, $h_1(t) = t$ *on* $[a, b]$ *and* $g(s) \equiv 1$, $h_2(s) = s$ *on* $[c, d]$, $\xi_i = \frac{x_i x_{i+1}}{2}$ $(i = 0, ..., n-1)$ *and* $\eta_j = \frac{y_j + y_{j+1}}{2}$ $(\, j = 0, ..., m-1)$

in Theorem 6, then we have the Simpson's sum

$$\begin{aligned}
& A_S(f, I_n, J_m) \\
= & \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right) l_1^i l_2^j \\
& +\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2}, y_j\right)+f\left(\frac{x_i+x_{i+1}}{2}, y_{j+1}\right)\right] l_1^i l_2^j \\
& +\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i, \frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1}, \frac{y_j+y_{j+1}}{2}\right)\right] l_1^i l_2^j \\
& +\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i, y_j)+f(x_i, y_{j+1})+f(x_{i+1}, y_j)+f(x_{i+1}, y_{j+1})\right] l_1^i l_2^j
\end{aligned}$$

with

$$\int_a^b\int_c^d f(t,s)\,ds\,dt = A_S(f, I_n, J_m) + R_S(f, I_n, J_m)$$

and the remainder term $R_S(f, I_n, J_m)$ *satisfies*

$$|R_S(f, I_n, J_m)| \leq \frac{1}{9}\upsilon(l_1)\upsilon(l_2)\bigvee_a^b\bigvee_c^d(f)$$

which given by Budak and Sarikaya in [10].

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