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A NEW GENERALIZATION OF OSTROWSKI TYPE INEQUALITY FOR MAPPINGS OF BOUNDED VARIATION

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Abstract. In this paper, a new generalization of Ostrowski type integral inequality for mappings of bounded variation is obtained and the quadrature formula is also provided.

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$[16]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b (f) := \sup\left\{ \sum(P) : P \in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [9], Dragomir proved following Ostrowski type inequalities for functions of bounded variation:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

We introduce the notation $I_n: a=x_0<x_1<...<x_n=b$ for a division of the interval $[a,b]$ with $h_i:=x_{i+1}-x_i$ and $v(h)=\max\{h_i: i=0,1,...,n-1\}$ and let intermediate points $\xi_i\in[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$. Then we have

$$\int_a^b f(t)dt=A(f,I_n,\xi)+R(f,I_n,\xi) \tag{1.3}$$

where

$$A(f,I_n,\xi):=\sum_{i=0}^{n} f(\xi_i)h_i \tag{1.4}$$

and the remainder term satisfies

$$\begin{aligned}|R(f,I_n,\xi)| &\leq \left[\frac{1}{2}v(h)+\max_{i\in\{0,1,...,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\bigvee_a^b \\ &\leq v(h)\bigvee_a^b.\end{aligned} \tag{1.5}$$

In [7], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

Theorem 2. *Let $I_k: a=x_0<x_1<...<x_k=b$ be a division of the interval $[a,b]$ and $\alpha_i\,(i=0,1,...,k+1)$ be $k+2$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,...,k)$, $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality:*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\right| \\ &\leq \left[\frac{1}{2}v(h)+\max\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|, i=0,1,...,k-1\right]\bigvee_a^b(f) \\ &\leq v(h)\bigvee_a^b(f)\end{aligned} \tag{1.6}$$

where $v(h):=\max\{h_i|\ i=0,...,n-1\}$, $h_i:=x_{i+1}-x_i\,(i=0,1,...,k-1)$ and $\bigvee_a^b(f)$ is the total variation of f on the interval $[a,b]$.

For recent results concerning the above Ostrowski's inequality and other related results see [1]-[21].

The aim of this paper is to obtain a new generalization of Ostrowski type integral inequalities for functions of bounded variation. And we give some applications for our results.

2. Main Results

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, for all $x \in [a,b]$, we have*

$$\left|(b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right| \leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{2.1}$$

where $\lambda \in [0,1]$ and $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$.

Proof. Define the mapping $K_\lambda(x,t)$ by,

$$K_\lambda(x,t)=\begin{cases} t-\left(a+\lambda\frac{x-a}{2}\right), & a\leq t\leq x \\ t-\left(b-\lambda\frac{b-x}{2}\right) & x<t\leq b. \end{cases}$$

Integrating by parts, we get

$$\begin{aligned} &\int_a^b K_\lambda(x,t)df(t) \\ =\ & \int_a^x \left(t-\left(a+\lambda\frac{x-a}{2}\right)\right)df(t)+\int_x^b \left(t-\left(b-\lambda\frac{b-x}{2}\right)\right)df(t) \\ =\ & \left(t-a-\lambda\frac{x-a}{2}\right)f(t)\Big|_a^x-\int_a^x f(t)dt \\ & +\left(t-b+\lambda\frac{b-x}{2}\right)f(t)\Big|_x^b-\int_x^b f(t)dt \\ =\ & (x-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{x-a}{2}f(a) \\ & +\lambda\frac{b-x}{2}f(b)+(b-x)\left(1-\frac{\lambda}{2}\right)f(x)-\int_a^b f(t)dt \\ =\ & (b-a)\left(1-\frac{\lambda}{2}\right)f(x)+\lambda\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt. \end{aligned} \tag{2.2}$$

It is well known that if $g,f:[a,b]\rightarrow\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exist and

$$\left|\int_a^b g(t)df(t)\right|\leq\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{2.3}$$

On the other hand, using (2.3), we get

$$\begin{aligned}
&\left|\int_a^b K_\lambda(x,t)df(t)\right|\\
&\leq \left|\int_a^x\left(t-\left(a+\lambda\frac{x-a}{2}\right)\right)df(t)\right|+\left|\int_x^b\left(t-\left(b-\lambda\frac{b-x}{2}\right)\right)df(t)\right|\\
&\leq \sup_{t\in[a,x]}\left|t-a-\lambda\frac{x-a}{2}\right|\bigvee_a^x(f)+\sup_{t\in[x,b]}\left|t-b+\lambda\frac{b-x}{2}\right|\bigvee_x^b(f)\\
&= (x-a)\left(1-\frac{\lambda}{2}\right)\bigvee_a^x(f)+(b-x)\left(1-\frac{\lambda}{2}\right)\bigvee_x^b(f)\\
&\leq \left(1-\frac{\lambda}{2}\right)\max\{x-a,b-x\}\bigvee_a^b(f)\\
&= \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f).
\end{aligned}$$

This completes the proof. □

Remark 1. *If we choose $\lambda=0$ in Theorem 3, then the inequality (2.1) reduces the nequality (1.2).*

Corollary 1. *Under the assumption of Theorem 3 with $\lambda=1$, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{2}(b-a)f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{2}-\int_a^b f(t)dt\right|\\
&\leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f).
\end{aligned} \tag{2.4}$$

Remark 2. *If we take $x=\frac{a+b}{2}$ in Corollary 1, then we have the inequality*

$$\left|\frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)dt\right|\leq\frac{1}{4}(b-a)\bigvee_a^b(f)$$

which was given by Alomari in [3]. *The constant $\frac{1}{4}$ is the best possible.*

Corollary 2. *Under the assumption of Theorem 3 with $\lambda = \frac{2}{3}$, then we get the inequality*

(2.5)

$$\left| \frac{2}{3}(b-a) f(x) + \frac{(x-a) f(a) + (b-x) f(b)}{3} - \int_a^b f(t)dt \right|$$

$$\leq \frac{2}{3}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f).$$

Remark 3. *If we take $x = \frac{a+b}{2}$ in Corollary 2, then we have the Simpson's inequality*

$$\left| \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)dt \right| \leq \frac{1}{3}(b-a) \bigvee_a^b (f)$$

which was given by Dragomir in [7].

Corollary 3. *Under the assumption of Theorem 3. Suppose that $f \in C^1[a,b]$, then we have*

$$\left| (b-a)\left(1-\frac{\lambda}{2}\right) f(x) + \lambda \frac{(x-a) f(a) + (b-x) f(b)}{2} - \int_a^b f(t)dt \right|$$

$$\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1$$

for all $x \in [a,b]$. Here as subsequently $\|.\|_1$ is the L_1-norm

$$\|f'\|_1 := \int_a^b f'(t)dt.$$

Corollary 4. *Under the assumption of Theorem 3. Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian with the constant $L > 0$. Then*

$$\left| (b-a)\left(1-\frac{\lambda}{2}\right) f(x) + \lambda \frac{(x-a) f(a) + (b-x) f(b)}{2} - \int_a^b f(t)dt \right|$$

$$\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (b-a) L$$

for all $x \in [a,b]$.

Corollary 5. *Under the assumption of Theorem 3. Let $f : [a,b] \to \mathbb{R}$ be a monotone mapping on $[a,b]$. Then*

$$\left| (b-a)\left(1-\frac{\lambda}{2}\right) f(x) + \lambda \frac{(x-a) f(a) + (b-x) f(b)}{2} - \int_a^b f(t)dt \right|$$

$$\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] |f(b) - f(a)|$$

for all $x \in [a,b]$.

3. Application to Quadrature Formula

We now introduce the intermediate points $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ in the division $I_n : a = x_0 < x_1 < ... < x_n = b$. Let $h_i := x_{i+1} - x_i$ and $v(h) = \max\{h_i : i = 0,1,...,n-1\}$ and define the sum

(3.1)
$$A(f, I_n, \xi) \quad : \quad = \sum_{i=0}^{n} \left[\left(1 - \frac{\lambda}{2}\right) f(\xi_i) h_i + \lambda \frac{(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})}{2} \right].$$

Then the following Theorem holds:

Theorem 4. *Let f be as Theorem 3. Then*

$$\int_a^b f(t)dt = A(f, I_n, \xi) + R(f, I_n, \xi) \tag{3.2}$$

where $A(f, I_n, \xi)$ is defined as above and the remainder term $R(f, I_n, \xi)$ satisfies

$$\begin{aligned}
|R(f, I_n, \xi)| &\leq \left(1 - \frac{\lambda}{2}\right) \left[\frac{1}{2} v(h) + \max_{i \in \{0,1,...,n-1\}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b (f) \\
&\leq \left(1 - \frac{\lambda}{2}\right) v(h) \bigvee_a^b (f).
\end{aligned} \tag{3.3}$$

Proof. Application of Theorem 3 to the interval $[x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ gives

(3.4)
$$\begin{aligned}
&\left| \left(1 - \frac{\lambda}{2}\right) f(\xi_i) h_i + \lambda \frac{(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})}{2} - \int_{x_i}^{x_{i+1}} f(t)dt \right| \\
&\leq \left(1 - \frac{\lambda}{2}\right) \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} (f)
\end{aligned}$$

for all $i \in \{0,1,...,n-1\}$.

Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we have

$$\begin{aligned}
|R(f, I_n, \xi)| &\leq \left(1 - \frac{\lambda}{2}\right) \sum_{i=0}^{n} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} (f) \\
&\leq \left(1 - \frac{\lambda}{2}\right) \max_{i \in \{0,1,...,n-1\}} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \sum_{i=0}^{n} \bigvee_{x_i}^{x_{i+1}} (f) \\
&\leq \left(1 - \frac{\lambda}{2}\right) \left[\frac{1}{2} v(h) + \max_{i \in \{0,1,...,n-1\}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b (f)
\end{aligned}$$

which completes the proof of the first inequality in (3.3)

For the second inequality in (3.3), we show that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{h_i}{2} \; i \in \{0, 1, ..., n-1\}$$

and

$$\max_{i\in\{0,1,\ldots,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}v(h)$$

which completes the proof. □

Remark 4. *If we choose* $\lambda = 0$, *we get (1.3) with (1.4) and (1.5).*

Remark 5. *If we choose* $\lambda = \frac{2}{3}$ *and* $\xi_i = \frac{x_i + x_{i+1}}{2}$, *then we have*

$$\int_a^b f(t)dt = A_S(f, I_n) + R_S(f, I_n)$$

where

$$A_S(f, I_n) = \frac{1}{6}\sum_{i=0}^{n} [f(x_i) + f(x_{i+1})]\, h_i + \frac{2}{3}\sum_{i=0}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder term $R_S(f, I_n)$ *satisfies*

$$|R_S(f, I_n)| \leq \frac{1}{3}v(h) \bigvee_a^b (f)$$

which were given by Dragomir in [7].

Corollary 6. *Choosing* $\lambda = 1$ *gives*

$$\int_a^b f(t)dt = A(f, I_n, \xi) + R(f, I_n, \xi)$$

where

$$A(f, I_n, \xi) = \sum_{i=0}^{n} \left[\frac{1}{2}f(\xi_i)h_i + \frac{(\xi_i - x_i)\, f(x_i) + (x_{i+1} - \xi_i)\, f(x_{i+1})}{2}\right]$$

and the remainder term $R(f, I_n, \xi)$ *satisfies*

$$\begin{aligned} |R(f, I_n, \xi)| &\leq \frac{1}{2}\left[\frac{1}{2}v(h) + \max_{i\in\{0,1,\ldots,n-1\}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \\ &\leq \frac{1}{2}v(h) \bigvee_a^b (f). \end{aligned}$$

Particularly, if we take $\xi_i = \frac{x_i + x_{i+1}}{2}$,*then we have*

$$A(f, I_n) = \frac{1}{2}\sum_{i=0}^{n} \left[f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{f(x_i) + f(x_{i+1})}{2}\right] h_i$$

and

$$|R(f, I_n, \xi)| \leq \frac{1}{4}v(h) \bigvee_a^b (f).$$

Corollary 7. *Let $f:[a,b]\to\mathbb{R}$ be a Lipschitzian with the constant $L>0$. Then we have (3.1) and (3.2) and the remainder term satisfies*

$$\begin{aligned}|R(f,I_n,\xi)| &\leq L\left(1-\frac{\lambda}{2}\right)\left[\frac{1}{2}v(h)+\max_{i\in\{0,1,\ldots,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right](b-a)\\ &\leq L\left(1-\frac{\lambda}{2}\right)v(h)\,(b-a)\,.\end{aligned}$$

Corollary 8. *Let $f:[a,b]\to\mathbb{R}$ be a monotone mapping on $[a,b]$. Then we get (3.1) and (3.2) and the remainder term satisfies*

$$\begin{aligned}|R(f,I_n,\xi)| &\leq \left(1-\frac{\lambda}{2}\right)\left[\frac{1}{2}v(h)+\max_{i\in\{0,1,\ldots,n-1\}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]|f(b)-f(a)|\\ &\leq \left(1-\frac{\lambda}{2}\right)v(h)\,|f(b)-f(a)|\,.\end{aligned}$$

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