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A COMPANION OF THE GENERALIZED TRAPEZOID INEQUALITY FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION AND APPLICATIONS

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Abstract. In this paper, a companion of trapezoid inequality for functions of two independent variables with bounded variation is established and some applications for general quadrature formula are given.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [19]:

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\rightarrow\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\|f'\|_{\infty}, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.*

In [15], Dragomir proved following Ostrowski type inequalities for functions of bounded variation:

Theorem 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_{a}^{b}f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_{a}^{b}(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [3], author obtained the following a companion of the generalized trapezoid inequality for mapping of bounded variation:

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Theorem 3. *Let f be as in Theorem 2. Then we have the inequality*

$$\left|(x-a)\left[f(a)+f(b)\right]+(a+b-2x)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)dt\right|$$

$$\leq \left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f)$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. Furthermore, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

2. Preliminaries and Lemmas

In 1910, Fréchet [17] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}\alpha(x_i,y_j)=\alpha(x_{i-1},y_{j-1})-\alpha(x_{i-1},y_j)-\alpha(x_i,y_{j-1})+\alpha(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}\alpha(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([13]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a \le x \le b,\ c \le y \le d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i (i = 0,1,2,...,m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j (j = 0,1,2,...,n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j),$$

$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$, or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 3. *(Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i, y_i) $(i = 0,1,2,...,m)$ be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}|\Delta_{11}f(x_i,y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f):=\bigvee_{c}^{d}\bigvee_{a}^{b}(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [18], authors proved following Lemmas for double Riemann-Stieltjes integral:

Lemma 1 (Integrating by parts)**.** *If $f\in RS(\alpha)$ on Q, then $\alpha\in RS(f)$ on Q, and we have*

$$\begin{aligned}(2.3)\qquad &\int_{c}^{d}\int_{a}^{b}f(t,s)d_td_s\alpha(t,s)+\int_{c}^{d}\int_{a}^{b}\alpha(t,s)d_td_sf(t,s)\\ =\;& f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c).\end{aligned}$$

Lemma 2. *Assume that $g\in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left|\int_{c}^{d}\int_{a}^{b}g(x,y)d_xd_y\alpha(x,y)\right|\leq\sup_{(x,y)\in Q}|g(x,y)|\bigvee_{Q}(\alpha). \tag{2.4}$$

Moreover, Jawarneh and Noorani obtained following inequalities for functions of two variables with bounded variation in [18]:

Theorem 4. *Let $f:Q\to\to\mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y)\in Q$, we have the following Ostrowski inequality*

$$\begin{aligned}(2.5)\qquad &\left|(b-a)(d-c)f(x,y)-\int_{c}^{d}\int_{a}^{b}f(t,s)dtds\right|\\ \leq\;&\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_{Q}(f)\end{aligned}$$

where $\bigvee_{Q}(f)$ denotes the total (double) variation of f on Q.

Theorem 5. *Let f be as in Theorem 4. Then we have the following trapezoid inequality*

$$\begin{aligned}(2.6)\qquad &\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}f(t,s)dsdt\right|\\ \leq\;&\frac{1}{4}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)\end{aligned}$$

The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see [6].

In [8], Budak and Sarikaya have obtained the following generalization of inequality (2.6).

Theorem 6. *Let f be as in Theorem 4. Then we have the inequality*

$$\begin{aligned} (2.7)\qquad & \left|(b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c)\right. \\ & \left.+(x-a)(d-y)f(a,d)+(x-a)(y-c)f(a,c)-\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f) \end{aligned}$$

for all $(x,y)\in Q$.

For some recent results which generalize, improve and extend the inequalities (1.2) and(2.5), see the papers ([1]-[11],[14]-[16],[20]-[22]).

The aim of this paper is to establish a companion of (2.7) for functions of two independent variables with bounded variation and apply it for quadrature formula.

3. Main Results

Theorem 7. *Let $f:Q=[a,b]\times[c,d]\to R$ be a mapping of bounded variation on Q. Then we have the inequality*

$$\begin{aligned} (3.1)\qquad & \left|(x-a)(y-c)\left[f(b,d)+f(b,c)+f(a,d)+f(a,c)\right]\right. \\ & +(x-a)(c+d-2y)\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right] \\ & +(a+b-2x)(y-c)\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right] \\ & \left.+(a+b-2x)(c+d-2y)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\left[\frac{1}{4}(d-c)+\left|y-\frac{3c+d}{4}\right|\right]\bigvee_a^b\bigvee_c^d(f) \end{aligned}$$

for all $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.

Proof. To prove, define the kernels $p(x,t)$ and $q(y,s)$ by

$$p(x,t)=\begin{cases} t-x & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ t-(a+b-x) & \text{if } t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

and

$$q(x,t)=\begin{cases} s-y & \text{if } s\in\left[c,\frac{c+d}{2}\right] \\ s-(c+d-y) & \text{if } s\in\left(\frac{c+d}{2},d\right] \end{cases}.$$

Using the mappings $p(x,t)$ and $q(y,s)$, we have

$$
\begin{aligned}
(3.2)\quad & \int_a^b \int_c^d p(x,t)q(y,s)d_s d_t f(t,s) \\
= & \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-x)(s-y)\, d_s d_t f(t,s) + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^{d} (t-x)(s-(c+d-y))\, d_s d_t f(t,s) \\
& + \int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} (t-(a+b-x))(s-y)\, d_s d_t f(t,s) \\
& + \int_{\frac{a+b}{2}}^{b} \int_{\frac{c+d}{2}}^{d} (t-(a+b-x))(s-(c+d-y))\, d_s d_t f(t,s) \\
= & K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
(3.3)\quad K_1 &= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-x)(s-y)\, d_s d_t f(t,s) \\
&= \frac{(a+b-2x)(c+d-2y)}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad + \frac{(a+b-2x)(y-c)}{2} f\left(\frac{a+b}{2}, c\right) \\
&\quad + \frac{(x-a)(c+d-2y)}{2} f\left(a, \frac{c+d}{2}\right) \\
&\quad + (x-a)(y-c) f(a,c) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(t,s) ds dt.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(3.4)\qquad K_2 &= \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}(t-x)\left(s-(c+d-y)\right)d_sd_tf(t,s)\\
&= \frac{(a+b-2x)(y-c)}{2}f\left(\frac{a+b}{2},d\right)\\
&\quad +\frac{(a+b-2x)(c+d-2y)}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad (x-a)(y-c)f(a,d)+\frac{(x-a)(c+d-2y)}{2}f\left(a,\frac{c+d}{2}\right)\\
&\quad -\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}f(t,s)dsdt,
\end{aligned}
$$

$$
\begin{aligned}
(3.5)\qquad K_3 &= \int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(t-(a+b-x)\right)(s-y)\,d_sd_tf(t,s)\\
&= \frac{(x-a)(c+d-2y)}{2}f\left(b,\frac{c+d}{2}\right)\\
&\quad +(x-a)(y-c)f(b,c)\\
&\quad +\frac{(a+b-2x)(c+d-2y)}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad \frac{(a+b-2x)(y-c)}{2}f\left(\frac{a+b}{2},c\right)-\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}f(t,s)dsdt,
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 &= \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(t-(a+b-x))\,(s-(c+d-y))\,d_sd_tf(t,s) \\
&= (x-a)\,(y-c)\,f\,(b,d)+\frac{(x-a)\,(c+d-2y)}{2}f\left(b,\frac{c+d}{2}\right) \\
&\quad+\frac{(a+b-2x)\,(y-c)}{2}f\left(\frac{a+b}{2},d\right) \\
&\quad+\frac{(a+b-2x)\,(c+d-2y)}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\quad-\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}f(t,s)dsdt.
\end{aligned}
\tag{3.6}
$$

Adding (3.3)-(3.6) in (3.2), we have

$$
\begin{aligned}
&\int_a^b\int_c^d p(x,t)q(y,s)d_sd_tf(t,s) \\
&= K_1+K_2+K_3+K_4 \\
&= (x-a)\,(y-c)\,[f(b,d)+f(b,c)+f(a,d)+f(a,c)] \\
&\quad+(x-a)\,(c+d-2y)\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right] \\
&\quad+(a+b-2x)\,(y-c)\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right] \\
&\quad+(a+b-2x)\,(c+d-2y)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_a^b\int_c^d f(t,s)dsdt.
\end{aligned}
\tag{3.7}
$$

On the other hand, taking mudulus and using Lemma 2 in (3.2), we get

$$\begin{aligned}
&\left|\int_a^b\int_c^d p(x,t)q(y,s)d_sd_tf(t,s)\right| \\
\leq\ & \left|\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-x)(s-y)\,d_sd_tf(t,s)\right|+\left|\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}(t-x)(s-(c+d-y))\,d_sd_tf(t,s)\right| \\
&+\left|\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}(t-(a+b-x))(s-y)\,d_sd_tf(t,s)\right| \\
&+\left|\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(t-(a+b-x))(s-(c+d-y))\,d_sd_tf(t,s)\right| \\
\leq\ & \max\left\{\frac{a+b}{2}-x,x-a\right\}\max\left\{\frac{c+d}{2}-y,y-x\right\}\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(f) \\
&+\max\left\{\frac{a+b}{2}-x,x-a\right\}\max\left\{\frac{c+d}{2}-y,y-x\right\}\bigvee_a^{\frac{a+b}{2}}\bigvee_{\frac{c+d}{2}}^{d}(f) \\
&+\max\left\{\frac{a+b}{2}-x,x-a\right\}\max\left\{\frac{c+d}{2}-y,y-x\right\}\bigvee_{\frac{a+b}{2}}^{b}\bigvee_c^{\frac{c+d}{2}}(f) \\
&+\max\left\{\frac{a+b}{2}-x,x-a\right\}\max\left\{\frac{c+d}{2}-y,y-x\right\}\bigvee_{\frac{a+b}{2}}^{b}\bigvee_{\frac{c+d}{2}}^{d}(f) \\
=\ & \max\left\{\frac{a+b}{2}-x,x-a\right\}\max\left\{\frac{c+d}{2}-y,y-x\right\}\bigvee_a^b\bigvee_c^d(f) \\
=\ & \left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\left[\frac{1}{4}(d-c)+\left|y-\frac{3c+d}{4}\right|\right]\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

which is desired result ☐

Remark 1. *If we take* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$ *in Theorem 7, then the inequality (3.1) reduces the inequality (2.6).*

Remark 2. *If we choose* $x=a$ *and* $y=c$ *in Theorem 7, then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_a^b\bigvee_c^d(f)$$

which was obtained by Jawarneh and Noorani in [18]*. The constant* $\frac{1}{4}$ *is the best possible. For a simple proof of sharpness of constant see* [9]*.*

Remark 3. *If we choose* $x=\frac{3a+b}{4}$ *and* $y=\frac{3c+d}{4}$ *in Theorem 7, then we get the following inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\right.\right.\\&\frac{1}{2}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]\\&\left.\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\\leq\ &\frac{1}{16}\bigvee_a^b\bigvee_c^d(f)\end{aligned}$$

which was proved by Budak and Sarikaya in [6]*. The constant* $\frac{1}{16}$ *is the best possible.*

Remark 4. *In Theorem 7, if we choose,*

a) $x=a$ *and* $y=\frac{c+d}{2}$*, then we get the following inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_a^b\bigvee_c^d(f),$$

b) $x=\frac{a+b}{2}$ *and* $y=c$*, then we get the following inequality*

$$\left|\frac{1}{2}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_Q(f)$$

which were given by Budak and Sarikaya in [10]*. The constants* $\frac{1}{4}$ *are the best possible.*

Corollary 1. *Under the assumption of Theorem 7 with* $x=a$ *and* $y=\frac{3c+d}{4}$*, we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.\\&\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\\leq\ &\frac{1}{8}\bigvee_a^b\bigvee_c^d(f).\end{aligned}\tag{3.8}$$

The constant $\frac{1}{8}$ *is the best possible.*

Proof. For sharpness of constant, assume that 3.8 holds with a constant A>0, that is,

$$\begin{aligned}(3.9)\qquad &\left|\frac{1}{2}\left[\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.\\ &\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ & A\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

Define the set E by

$$E=\left\{\left(\frac{a+b}{2},c\right),\left(\frac{a+b}{2},d\right),\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}.$$

If we choose $f:Q\to\mathbb{R}$ with

$$f(x,y)=\begin{cases}1 \text{ if } & (x,y)\in E\\ 0 \text{ if } & (x,y)\in Q\backslash E\end{cases}$$

then f is of bounded variation on Q, and

$$\frac{1}{2}\left[\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]=1,$$

$$\int_a^b\int_c^d f(t,s)dsdt=0,\ \text{and}\ \bigvee_a^b\bigvee_c^d(f)=8.$$

By (3.9), we have $1\leq 8A$, thus $A\geq\frac{1}{8}$ which implies the constant $\frac{1}{8}$ is the best possible. This completes the proof. □

Corollary 2. *Under the assumption of Theorem 7 with* $x=\frac{3a+b}{4}$ *and* $y=c$*, we have*

$$\begin{aligned}(3.10)\qquad &\left|\frac{1}{2}\left[\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right.\\ &\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ & \frac{1}{8}\bigvee_Q(f).\end{aligned}$$

The constant $\frac{1}{8}$ *is the best possible.*

Proof. The proof sharpness of constant is obtained in the same way as the above. □

4. Some Composite Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, with $h_i := x_{i+1} - x_i$, $\xi_i \in [x_i, x_{i+1}]$ and $l_j := y_{j+1} - y_j$, $\eta_j \in [y_j, y_{j+1}]$,

$$\upsilon(h) := \max\{h_i| \ i = 0, ..., n-1\},$$

and

$$\upsilon(l) := \max\{l_j| \ j = 0, ..., m-1\}.$$

Define the sum

$$\begin{aligned}
&A(f, I_n, J_m, \xi, \eta) \\
= &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\xi_i - x_i)(\eta_j - y_j)\left[f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)\right] \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\xi_i - x_i)(y_j + y_{j+1} - 2\eta_j)\left[f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right)\right] \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_i + x_{i+1} - 2\xi_i)(\eta_j - y_j)\left[f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) + f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right)\right] \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_i + x_{i+1} - 2\xi_i)(y_j + y_{j+1} - 2\eta_j)\, f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right).
\end{aligned}$$

Then the following Theorem holds.

Theorem 8. *Let f be as in Theorem 7. Then we have*

$$\int_a^b\int_c^d f(t,s)dsdt = A(f, I_n, J_m, \xi, \eta) + R(f, I_n, J_m, \xi, \eta)$$

where the remainder term $R(f, I_n, J_m, \xi, \eta)$ satisfies

$$\begin{aligned}
&|R(f, I_n, J_m, \xi, \eta)| \\
\leq &\left[\frac{1}{4}\upsilon(h) + \max_{i\in\{0,1,...,n-1\}}\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \\
&\times\left[\frac{1}{4}\upsilon(l) + \max_{j\in\{0,1,...,m-1\}}\left|\eta_j - \frac{3y_j + y_{j+1}}{4}\right|\right]\bigvee_a^b\bigvee_c^d(f) \\
\leq &\frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).
\end{aligned} \tag{4.1}$$

Proof. Appliying Theorem 7 to the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get

$$
\begin{aligned}
(4.2) \quad & \big| (\xi_i - x_i)\left(\eta_j - y_j\right)\left[f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)\right] \\
& + (\xi_i - x_i)\left(y_j + y_{j+1} - 2\eta_j\right)\left[f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right)\right] \\
& + (x_i + x_{i+1} - 2\xi_i)\left(\eta_j - y_j\right)\left[f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) + f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right)\right] \\
& + (x_i + x_{i+1} - 2\xi_i)\left(y_j + y_{j+1} - 2\eta_j\right) f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) \\
& - \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\,ds\,dt \Bigg| \\
\leq \quad & \left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]\left[\frac{1}{4}l_j + \left|\eta_j - \frac{3y_j + y_{j+1}}{4}\right|\right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f).
\end{aligned}
$$

Summing the inequality (4.2) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, then we have

$$
\begin{aligned}
& |R(f, I_n, J_m, \xi, \eta)| \\
\leq \quad & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right]\left[\frac{1}{4}l_j + \left|\eta_j - \frac{3y_j + y_{j+1}}{4}\right|\right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
\leq \quad & \max_{i \in \{0,1,\ldots,n-1\}} \left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \\
& \times \max_{j \in \{0,1,\ldots,m-1\}} \left[\frac{1}{4}l_j + \left|\eta_j - \frac{3y_j + y_{j+1}}{4}\right|\right] \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
\leq \quad & \left[\frac{1}{4}\upsilon(h) + \max_{i \in \{0,1,\ldots,n-1\}} \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \\
& \left[\frac{1}{4}\upsilon(l) + \max_{j \in \{0,1,\ldots,m-1\}} \left|\eta_j - \frac{3y_j + y_{j+1}}{4}\right|\right] \bigvee_a^b \bigvee_c^d (f)
\end{aligned}
$$

which completes the proof of the first inequality in (4.1).

For the proof of the second inequality, we show that

$$
\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right| \leq \frac{h_i}{4}
$$

and

$$
\max_{i \in \{0,1,\ldots,n-1\}} \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right| \leq \frac{1}{4}\upsilon(h).
$$

Similarly, we can observe that

$$\max_{j\in\{0,1,\dots,m-1\}}\left|\eta_j-\frac{3y_j+y_{j+1}}{4}\right|\leq\frac{1}{4}\upsilon(l).$$

This completes the proof. □

Remark 5. *If we choose $\xi_i = x_i$ and $\eta_j = y_j$ in Theorem 8, then we have the midpoint rule*

$$A_M(f,I_n,J_m)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_il_j$$

with

$$\int_a^b\int_c^d f(t,s)dsdt=A_M(f,I_n,J_m)+R_M(f,I_n,J_m)$$

and

$$|R(f,I_n,J_m,\xi,\eta)|\leq\frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f)$$

which were given by Budak and Sarikaya in [5]*.*

Remark 6. *If we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$ and $\eta_j = \frac{y_j+y_{j+1}}{2}$ in Theorem 8, then we have the trapezoid rule*

$$A_T(f,I_n,J_m)=\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_il_j$$

with

$$\int_a^b\int_c^d f(t,s)dsdt=A_T(f,I_n,J_m)+R_T(I_n,J_m,f)$$

and

$$|R_T(I_n,J_m,f)|\leq\frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f)$$

which were given by Budak and Sarikaya in [5]*.*

Corollary 3. *Under the assumption of Theorem 8 with $\xi_i = \frac{3x_i+x_{i+1}}{2}$ and $\eta_j = \frac{3y_j+y_{j+1}}{2}$, we have*

$$\begin{aligned}
&A(f,I_n,J_m,\xi,\eta)\\
=\;&\frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_{i+1},y_{j+1})+f(x_{i+1},y_j)+f(x_i,y_{j+1})+f(x_i,y_j)\right]h_il_j\\
&+\frac{1}{8}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right]h_il_j\\
&+\frac{1}{8}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right]h_il_j\\
&+\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_il_j
\end{aligned}$$

with

$$\int_a^b \int_c^d f(t,s)dsdt = A(f, I_n, J_m) + R(f, I_n, J_m)$$

where the remainder term $R(f, I_n, J_m)$ *satisfies*

$$|R(f, I_n, J_m)| \leq \frac{1}{16}\upsilon(h)\upsilon(l) \bigvee_a^b \bigvee_c^d (f).$$

One can obtain several different results according to choosing ξ_i and η_j in Theorem 8.

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