Received 05/05/15

FURTHER INEQUALITIES FOR LOG-CONVEX FUNCTIONS RELATED TO HERMITE-HADAMARD RESULT

S. S. DRAGOMIR[1,2]

Abstract. Some unweighted and weighted inequalities of Hermite-Hadamard type for log-convex functions defined on real intervals are given.

1. Introduction

A function $f : I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0,1]$ one has the inequality:

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \le tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let us recall the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I, $a, b \in I$ and $a < b$.

For related results, see [1]-[22], [25]-[29], [30]-[40] and [41]-[53].

Note that if we apply the above inequality for the log-convex functions $f : I \to (0,\infty)$, we have that

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \le \frac{1}{b-a}\int_a^b \ln f(x)\,dx \le \frac{\ln f(a) + \ln f(b)}{2}, \tag{1.3}$$

from which we get

$$f\left(\frac{a+b}{2}\right) \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \le \sqrt{f(a) f(b)} \tag{1.4}$$

that is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result [32].

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Integral inequalities, Log-Convex functions.

Theorem 1. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \sqrt{f(x) f(a+b-x)}dx \leq \sqrt{f(a) f(b)}. \tag{1.5}$$

The inequality between the first and second term in (1.5) may be improved as follows [32]. A different upper bound for the middle term in (1.5) can be also provided.

Theorem 2. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\, dx\right] \\ &\leq \frac{1}{b-a}\int_a^b \sqrt{f(x) f(a+b-x)}dx \\ &\leq \frac{1}{b-a}\int_a^b f(x)\, dx \leq L(f(a), f(b)), \end{aligned} \tag{1.6}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$L(p,q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p,p) := p.$$

The last inequality in (1.6) was obtained in a different context in [43].

As shown in [59], the following result also holds:

Theorem 3. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)\, dx. \tag{1.7}$$

We define the *p-logarithmic mean* as

$$L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{with } a \neq b \\ \\ a, \text{ if } a = b \end{cases}$$

for $p \neq 0, -1$ and $a, b > 0$.

In the recent work [28] we generalized the inequality (1.6) as follows:

Theorem 4. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then for any $p>0$ we have the inequality*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\le \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\
&\le \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \\
&\le \begin{cases} \left[L_{2p-1}(f(a),f(b))\right]^{1-\frac{1}{2p}}\left[L(f(a),f(b))\right]^{\frac{1}{2p}},\ p\ne\frac{1}{2}; \\ L(f(a),f(b)),\ p=\frac{1}{2}. \end{cases}
\end{aligned} \tag{1.8}$$

If $p\in\left(0,\frac{1}{2}\right)$, then we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\le \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\
&\le \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \le \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned} \tag{1.9}$$

Remark 1. *If we take in (1.8) $p=1$, then we get*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\le \left(\frac{1}{b-a}\int_a^b f(x)\,f(a+b-x)\,dx\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} \\
&\le \left[A(f(a),f(b))\right]^{\frac{1}{2}}\left[L(f(a),f(b))\right]^{\frac{1}{2}}.
\end{aligned} \tag{1.10}$$

If we take $p=\frac{1}{4}$ in (1.9), then we get

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\
&\le \left(\frac{1}{b-a}\int_a^b \sqrt[4]{f(x)\,f(a+b-x)\,dx}\right)^{2} \\
&\le \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^{2} \le \frac{1}{b-a}\int_a^b f(x)\,dx.
\end{aligned} \tag{1.11}$$

This improves the inequality (1.7).

Motivated by the above results, we establish in this paper some new unweighted and weighted inequalities for log-convex functions, some of them improving earlier results. Applications for special means are also provided.

2. New Results

The following result holds.

Theorem 5. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. Then for every $t \in [0,1]$ we have*

$$\int_a^b f(x)\,dx \geq \int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \geq \begin{cases} \frac{1}{1-2t}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du, & \text{if } t \neq \frac{1}{2} \\ (b-a)\,f\left(\frac{a+b}{2}\right), & \text{if } t = \frac{1}{2}. \end{cases} \tag{2.1}$$

Proof. The cases $t = 0, \frac{1}{2}, 1$ are obvious.

Assume that $t \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. By the log-convexity of f we have

$$\begin{aligned}[f(x)]^{1-t}[f(a+b-x)]^t &\geq f((1-t)x + t(a+b-x)) \\ &= f((1-2t)x + t(a+b))\end{aligned} \tag{2.2}$$

for any $x \in [a,b]$.

Integrating the inequality (2.2) over x on $[a,b]$ we have

$$\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \geq \int_a^b f((1-2t)x + t(a+b))\,dx.$$

Since $t \neq \frac{1}{2}$, then $u := (1-2t)x + t(a+b)$ is a change of variable with $du = (1-2t)\,dx$.

For $x = a$ we get $u = (1-t)a + tb$ and for $x = b$ we get $u = ta + (1-t)b$. Therefore

$$\int_a^b f((1-2t)x + t(a+b))\,dx = \frac{1}{1-2t}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du$$

and the second inequality in (2.1) is proved.

By the Hölder integral inequality for $p=\frac{1}{1-t}$, $q=\frac{1}{t}$ we have

$$\begin{aligned}
&\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx\\
&\le \left(\int_a^b \left([f(x)]^{1-t}\right)^{\frac{1}{1-t}}dx\right)^{1-t}\left(\int_a^b \left([f(a+b-x)]^{t}\right)^{\frac{1}{t}}dx\right)^{t}\\
&=\left(\int_a^b f(x)\,dx\right)^{1-t}\left(\int_a^b f(a+b-x)\,dx\right)^{t}\\
&=\left(\int_a^b f(x)\,dx\right)^{1-t}\left(\int_a^b f(x)\,dx\right)^{t}=\int_a^b f(x)\,dx,
\end{aligned}$$

that proves the first inequality in (2.1). □

Corollary 1. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. Then for every $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p>0$ that*

$$\begin{aligned}
&\left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}}\\
&\ge\left(\frac{1}{b-a}\int_a^b \left[f^{2p}(x)\right]^{1-t}\left[f^{2p}(a+b-x)\right]^{t}dx\right)^{\frac{1}{2p}}\\
&\ge\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\,du\right)^{\frac{1}{2p}}\\
&\ge\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u)\,f^{p}(a+b-u)\,dx\right)^{\frac{1}{2p}}\\
&\ge\exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right]\ge f\left(\frac{a+b}{2}\right).
\end{aligned}\tag{2.3}$$

Proof. Follows from Theorem 5 applied for the log-convex function f^{2p} with $p>0$ and by Theorem 4 applied for the interval $[(1-t)a+tb,ta+(1-t)b]$ when $t\in\left(0,\frac{1}{2}\right)$or $[ta+(1-t)b,(1-t)a+tb]))$ when $t\in\left(\frac{1}{2},1\right)$). □

If we take $p=1$ in (2.3), then we get

$$\begin{aligned}
\left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} &\ge\left(\frac{1}{b-a}\int_a^b \left[f^{2}(x)\right]^{1-t}\left[f^{2}(a+b-x)\right]^{t}dx\right)^{\frac{1}{2}}\\
&\ge\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2}(u)\,du\right)^{\frac{1}{2}}
\end{aligned}\tag{2.4}$$

$$\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) f(a+b-u)\, dx \right)^{\frac{1}{2}}$$
$$\geq \exp\left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\, du \right] \geq f\left(\frac{a+b}{2}\right).$$

If we take $p = \frac{1}{2}$ in (2.3), then we get

$$\frac{1}{b-a} \int_a^b f(x)\, dx \tag{2.5}$$
$$\geq \frac{1}{b-a} \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t\, dx$$
$$\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\, du$$
$$\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u) f(a+b-u)} dx$$
$$\geq \exp\left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\, du \right] \geq f\left(\frac{a+b}{2}\right).$$

Corollary 2. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. Then for every $t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ we have for $p \in \left(0, \frac{1}{2}\right)$ that*

$$\frac{1}{b-a} \int_a^b f(x)\, dx \tag{2.6}$$
$$\geq \frac{1}{b-a} \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t\, dx$$
$$\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\, du$$
$$\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\, du \right)^{\frac{1}{2p}}$$
$$\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u) f^{p}(a+b-u)\, du \right)^{\frac{1}{2p}}$$
$$\geq \exp\left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\, du \right] \geq f\left(\frac{a+b}{2}\right).$$

Follows by Theorem 5 and Theorem 4 for $p \in \left(0, \frac{1}{2}\right)$.

If we take $p = \frac{1}{4}$ in (2.6), then we get

$$\begin{aligned}
(2.7) \qquad & \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\
&\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\
&\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u)}du\right)^2 \\
&\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt[4]{f(u)f(a+b-u)}du\right)^2 \\
&\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

If we use the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du, \tag{2.8}$$

where $t \neq \frac{1}{2}$, for the log-convex function $f : [a, b] \to (0, \infty)$, $f(x) = \frac{1}{x}$, then we get

$$\frac{\ln b - \ln a}{b-a} \geq \frac{\ln(ta + (1-t)b) - \ln((1-t)a + tb)}{(ta + (1-t)b) - (((1-t)a + tb))}$$

which, in terms of logarithmic mean, is

$$L(ta + (1-t)b, (1-t)a + tb) \geq L(a, b) \tag{2.9}$$

for any $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$.

When $t = \frac{1}{2}$ the inequality (2.9) becomes $A(a, b) \geq L(a, b)$ that is also true.

For $q \neq 0, -1$ define the L_q-*logarithmic mean* as

$$L_q(a, b) := \begin{cases} \left(\frac{b^{q+1}-a^{q+1}}{(q+1)(b-a)}\right)^{\frac{1}{q}} & \text{if } b \neq a, \\ a & \text{if } b = a. \end{cases}$$

If we use the inequality (2.8) for the log-convex function $f : [a, b] \to (0, \infty)$, $f(x) = \frac{1}{x^p}$, with $p > 0$, $p \neq 1$, then we have for any $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$ that

$$\frac{1}{b-a}\int_a^b x^{-p}dx \geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} u^{-p}du,$$

i.e.

$$\frac{b^{-p+1}-a^{-p+1}}{(1-p)(b-a)} \geq \frac{(ta+(1-t)b)^{-p+1}-((1-t)a+tb)^{-p+1}}{((1-p))[(ta+(1-t)b)-(((1-t)a+tb))]},$$

which can be written as

$$L_{-p}^{-p}(a,b) \geq L_{-p}^{-p}(ta+(1-t)b,(1-t)a+tb).$$

Therefore we have

$$L_{-p}(ta+(1-t)b,(1-t)a+tb) \geq L_{-p}(a,b) \tag{2.10}$$

for any $p>0$, $p\neq 1$ and for any $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$. The case $t=\frac{1}{2}$ reduces to the known inequality $A(a,b)\geq L_{-p}(a,b)$ for any $p>0$.

3. Weighted Inequalities

We have the following generalized weighted version of the inequality (1.5).

Theorem 6. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)\,dx>0$, then*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)\,f(b)} \tag{3.1}$$

for any $p>0$.

In particular, we have

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a)\,f(b)}. \tag{3.2}$$

Proof. We know that, see [32] or [35, p. 198], if g is log-convex, then

$$g\left(\frac{a+b}{2}\right) \leq \sqrt{g(x)\,g(a+b-x)} \leq \sqrt{g(a)\,g(b)} \tag{3.3}$$

for any $x\in[a,b]$.

For any $p>0$ the function f^{2p} is log-convex and by (3.3) we have

$$f^{2p}\left(\frac{a+b}{2}\right) \leq f^p(x)\,f^p(a+b-x) \leq f^p(a)\,f^p(b) \tag{3.4}$$

for any $x\in[a,b]$.

If we multiply (3.4) by $w(x)\geq 0$ and integrate, then we get

$$\begin{aligned} f^{2p}\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx &\leq \int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx \\ &\leq f^p(a)\,f^p(b)\int_a^b w(x)\,dx \end{aligned}$$

namely

$$f^{2p}\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx} \leq f^p(a)\,f^p(b). \tag{3.5}$$

Taking the power $\frac{1}{2p}$ in (3.5) we get the desired result (3.1). □

We also have the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\leq\sqrt{f(a)f(b)}, \tag{3.6}$$

that is a weighted version of (1.5).

If we take $p=\frac{1}{4}$ in (3.1), then we get

$$f\left(\frac{a+b}{2}\right)\leq\left(\frac{\int_a^b w(x)\sqrt[4]{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right)^2\leq\sqrt{f(a)f(b)}. \tag{3.7}$$

Using Jensen's inequality for the power $p\geq 1$ ($p\in(0,1)$), namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^p\leq(\geq)\,\frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can start the following more precise result:

Corollary 3. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function and $w:[a,b]\to[0,\infty)$ be integrable and $\int_a^b w(x)\,dx>0$.*

If $p\geq 1$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq\left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \\ &\leq\left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}}\leq\sqrt{f(a)f(b)}. \end{aligned} \tag{3.8}$$

If $p\in(0,1)$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq\left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \\ &\leq\left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}}\leq\sqrt{f(a)f(b)}. \end{aligned} \tag{3.9}$$

Remark 2. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. We observe that if we take in (3.1) $w(x)=f^{-p}(a+b-x)$, $p>0$, then we get*

$$f\left(\frac{a+b}{2}\right)\leq\left(\frac{\int_a^b f^p(x)\,dx}{\int_a^b f^{-p}(x)\,dx}\right)^{\frac{1}{2p}}\leq\sqrt{f(a)f(b)} \tag{3.10}$$

for any $p>0$.

In particular, we have the inequalities

$$f\left(\frac{a+b}{2}\right)\leq\left(\frac{\int_a^b f(x)\,dx}{\int_a^b \frac{1}{f(x)}dx}\right)^{\frac{1}{2}}\leq\sqrt{f(a)f(b)}, \tag{3.11}$$

and

$$f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b \sqrt{f(x)}dx}{\int_a^b \frac{1}{\sqrt{f(x)}}dx}\leq\sqrt{f(a)f(b)}. \tag{3.12}$$

If we take in (3.10) $f(x) = x^{-1}$, $x \in [a,b] \subset (0,\infty)$, which is log-convex, then we have

$$\left(\frac{a+b}{2}\right)^{-1} \le \left(\frac{\int_a^b x^{-p}dx}{\int_a^b x^p dx}\right)^{\frac{1}{2p}} \le \left(\sqrt{f(a)f(b)}\right)^{-1} \tag{3.13}$$

Observe that

$$\frac{\int_a^b x^{-p}dx}{\int_a^b x^p dx} = \frac{\frac{1}{b-a}\int_a^b x^{-p}dx}{\frac{1}{b-a}\int_a^b x^p dx} = \frac{[L_{-p}(a,b)]^{-p}}{[L_p(a,b)]^p}$$

and by (3.13) we have

$$\left(\frac{a+b}{2}\right)^{-1} \le \left(\frac{1}{[L_p(a,b)][L_{-p}(a,b)]}\right)^{\frac{1}{2}} \le \left(\sqrt{f(a)f(b)}\right)^{-1}$$

that is equivalent to

$$G(a,b) \le \sqrt{[L_p(a,b)][L_{-p}(a,b)]} \le A(a,b).$$

This can be also written as

$$G(a,b) \le G(L_p(a,b), L_{-p}(a,b)) \le A(a,b). \tag{3.14}$$

If we take in the first inequality (3.14) $p=1$, then we get

$$G^2(a,b) \le A(a,b)L(a,b). \tag{3.15}$$

We have the following weighted version of (1.4).

Theorem 7. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)dx > 0$, then*

$$\begin{aligned} & f\left(\frac{\int_a^b w(x)xdx}{\int_a^b w(x)dx}\right) \\ & \le \exp\left(\frac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}\right) \\ & \le [f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)} \\ & \le \frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)f(a)\right]. \end{aligned} \tag{3.16}$$

Proof. Since $\ln f$ is convex, then by Jensen's inequality we have

$$\ln f\left(\frac{\int_a^b w(x)xdx}{\int_a^b w(x)dx}\right) \le \frac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}. \tag{3.17}$$

Taking the exponential in (3.17) we get the first inequality in (3.16).

Since $\ln f$ is convex, then

$$\ln f(x) = \ln f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) \le \frac{x-a}{b-a}\ln f(b) + \frac{b-x}{b-a}\ln f(a) \tag{3.18}$$

for any $x \in [a,b]$.

By taking the weighted integral mean in (3.18) we get

$$\begin{aligned}(3.19)\quad & \frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx} \\ & \leq \frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)\ln f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)\ln f(a)\right] \\ & = \ln\left([f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)}\right).\end{aligned}$$

By taking the exponential in (3.19), we get the second inequality in (3.16).

The last part of (3.16) follows by the weighted geometric mean-arithmetic mean inequality. □

Remark 3. *If we take $w(x)=1$, $x\in[a,b]$ in the first two inequalities (3.16), we recapture (1.4).*

If we take $w(x)=\frac{1}{x}$, $x\in[a,b]\subset(0,\infty)$ in (3.16), then we get

$$\begin{aligned}(3.20)\quad f(L(a,b)) & \leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(x)}{x}dx\right) \\ & \leq [f(b)]^{\frac{L(a,b)-a}{b-a}}[f(a)]^{\frac{b-L(a,b)}{b-a}} \\ & \leq \frac{(L(a,b)-a)f(b)+(b-L(a,b))f(a)}{b-a},\end{aligned}$$

where $L(a,b)$ is the logarithmic mean, i.e.

$$L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if } a\neq b \\ a & \text{if } a=b.\end{cases}$$

If we take $w(x)=\frac{1}{x^2}$, $x\in[a,b]\subset(0,\infty)$ in (3.16), then we get

$$\begin{aligned}(3.21)\quad f\left(\frac{G^2(a,b)}{L(a,b)}\right) & \leq \exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(x)}{x^2}dx\right) \\ & \leq [f(b)]^{\frac{1}{b-a}\left(\frac{G^2(a,b)}{L(a,b)}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{G^2(a,b)}{L(a,b)}\right)} \\ & \leq \frac{1}{b-a}\left[\left(\frac{G^2(a,b)}{L(a,b)}-a\right)f(b)+\left(b-\frac{G^2(a,b)}{L(a,b)}\right)f(a)\right].\end{aligned}$$

We also have the alternative result:

Theorem 8. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)\,dx>0$, then*

$$
\begin{aligned}
f\left(\frac{\int_a^b w(x)\,xdx}{\int_a^b w(x)\,dx}\right)
&\le \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\\
&\le \frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}\le\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b w(x)\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\int_a^b w(x)\,dx}\\
&\le \frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)f(a)\right].
\end{aligned}
\tag{3.22}
$$

Proof. Using Jensen's inequality for the exponential function we have

$$
\begin{aligned}
\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)&\le\frac{\int_a^b w(x)\exp(\ln f(x))\,dx}{\int_a^b w(x)\,dx}\\
&=\frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}
\end{aligned}
$$

and the second inequality in (3.22) is proved.

From (3.18) and the arithmetic mean - geometric mean inequality we have

$$
\begin{aligned}
f(x)\le[f(b)]^{\frac{x-a}{b-a}}[f(a)]^{\frac{b-x}{b-a}}&=\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}\\
&\le\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)
\end{aligned}
\tag{3.23}
$$

for any $x\in[a,b]$.

By taking the weighted integral mean in (3.23) we get

$$
\begin{aligned}
&\frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}\\
&\le\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b w(x)\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\int_a^b w(x)\,dx}\\
&\le\frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)f(a)\right]
\end{aligned}
$$

and the last part of (3.22) is proved. □

Remark 4. *If we take $w(x)=1$, $x\in[a,b]$ in (3.22), then we have*

$$f\left(\frac{a+b}{2}\right)\leq\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq L(f(a),f(b))\leq\frac{f(a)+f(b)}{2}. \tag{3.24}$$

If we take $w(x)=\frac{1}{x}$, $x\in[a,b]\subset(0,\infty)$ in (3.22), then we get

$$\begin{aligned} f(L(a,b)) &\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(x)}{x}dx\right) \\ &\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx \\ &\leq\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b\frac{1}{x}\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\ln b-\ln a} \\ &\leq\frac{(L(a,b)-a)f(b)+(b-L(a,b))f(a)}{b-a}. \end{aligned} \tag{3.25}$$

4. Inequalities for Symmetric Weights

We say that the weight $w:[a,b]\to[0,\infty)$ is *symmetric* on $[a,b]$ if

$$w(a+b-x)=w(x)\text{ for all }x\in[a,b].$$

It is well known that if $f:[a,b]\to\mathbb{R}$ is convex and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then the *Fejér inequality* holds

$$f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}\leq\frac{f(a)+f(b)}{2}. \tag{4.1}$$

If $f:[a,b]\to(0,\infty)$ is a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then by (4.1) we have

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\leq\frac{\ln f(a)+\ln f(b)}{2},$$

which is equivalent to

$$f\left(\frac{a+b}{2}\right)\leq\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\leq\sqrt{f(a)f(b)}. \tag{4.2}$$

Theorem 9. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \\ &\leq \frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\leq\frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}. \end{aligned} \tag{4.3}$$

Proof. By Jensen's integral inequality for the exponential we have

$$\exp\left(\frac{\int_a^b w(x)\ln\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right) \tag{4.4}$$
$$\le \frac{\int_a^b w(x)\exp\left(\ln\sqrt{f(x)f(a+b-x)}\right)dx}{\int_a^b w(x)\,dx}$$
$$= \frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}.$$

Observe, by the symmetry of w, that

$$\int_a^b w(x)\ln\sqrt{f(x)f(a+b-x)}dx$$
$$= \frac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx + \int_a^b w(x)\ln f(a+b-x)\,dx\right]$$
$$= \frac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx + \int_a^b w(a+b-x)\ln f(a+b-x)\,dx\right]$$
$$= \int_a^b w(x)\ln f(x)\,dx$$

since, obviously

$$\int_a^b w(a+b-x)\ln f(a+b-x)\,dx = \int_a^b w(x)\ln f(x)\,dx.$$

By (4.4) we then get the second inequality in (4.3).

By Cauchy-Buniakovski-Schwarz integral inequality we also have

$$\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx$$
$$\le \sqrt{\int_a^b w(x)f(x)\,dx}\sqrt{\int_a^b w(x)f((a+b-x))\,dx}$$
$$= \sqrt{\int_a^b w(x)f(x)\,dx}\sqrt{\int_a^b w(x)f(x)\,dx}$$
$$= \int_a^b w(x)f(x)\,dx,$$

which proves the third inequality in (4.3). □

The above inequality (4.2) may be generalized as follows by replacing f with f^{2p} for $p > 0$.

Corollary 4. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then for any $p>0$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \left(\frac{\int_a^b w(x) f^{2p}(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}}. \tag{4.5}$$

Remark 5. *We observe that for $p\geq 1$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \left(\frac{\int_a^b w(x) f^{2p}(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \tag{4.6}$$

and for $p\in(0,1)$

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \left(\frac{\int_a^b w(x) f^2(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}}. \tag{4.7}$$

Finally, we have:

Theorem 10. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then for any $p>0$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x) w(x)\,dx}{\int_a^b \frac{w(x)dx}{f^p(x)}}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}. \tag{4.8}$$

Proof. From (3.4) we have

$$f^{2p}\left(\frac{a+b}{2}\right)\frac{1}{f^{p}\left(a+b-x\right)}\leq f^{p}\left(x\right)\leq f^{p}\left(a\right)f^{p}\left(b\right)\frac{1}{f^{p}\left(a+b-x\right)} \tag{4.9}$$

for any $x\in\left[a,b\right]$.

If we multiply by $w\left(x\right)\geq 0$ and integrate on $\left[a,b\right]$, then we get

$$f^{2p}\left(\frac{a+b}{2}\right)\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(a+b-x\right)}\leq\int_{a}^{b}f^{p}\left(x\right)w\left(x\right)dx$$
$$\leq f^{p}\left(a\right)f^{p}\left(b\right)\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(a+b-x\right)}.$$

Since, by symmetry of w we have

$$\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(a+b-x\right)}=\int_{a}^{b}\frac{w\left(a+b-x\right)dx}{f^{p}\left(a+b-x\right)}=\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(x\right)},$$

which implies that

$$f^{2p}\left(\frac{a+b}{2}\right)\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(x\right)}\leq\int_{a}^{b}f^{p}\left(x\right)w\left(x\right)dx$$
$$\leq f^{p}\left(a\right)f^{p}\left(b\right)\int_{a}^{b}\frac{w\left(x\right)dx}{f^{p}\left(x\right)}.$$

and the inequality (4.8) is proved. □

Remark 6. *If we write the inequality (4.8) for the log-convex function* $f:\left[a,b\right]\subset\left(0,\infty\right)\rightarrow\left(0,\infty\right)$, $f\left(x\right)=\frac{1}{x}$, *then we have for* $p>0$

$$\left(\frac{a+b}{2}\right)^{-1}\leq\left(\frac{\int_{a}^{b}x^{-p}w\left(x\right)dx}{\int_{a}^{b}x^{p}w\left(x\right)dx}\right)^{\frac{1}{2p}}\leq\left(\sqrt{ab}\right)^{-1},$$

that is equivalent to

$$G\left(a,b\right)\leq\left(\frac{\int_{a}^{b}x^{p}w\left(x\right)dx}{\int_{a}^{b}x^{-p}w\left(x\right)dx}\right)^{\frac{1}{2p}}\leq A\left(a,b\right), \tag{4.10}$$

for any symmetric integrable weight $w:\left[a,b\right]\rightarrow\left[0,\infty\right)$.

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[5] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[6] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[7] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[8] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[9] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[10] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[11] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[12] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[13] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[14] S. S. Dragomir, Some remarks on Hadamard's inequalities for convex functions, *Extracta Math.*, **9** (2) (1994), 88-94.

[15] S. S. Dragomir, Refinements of the Hermite-Hadamard integral inequality for log-convex functions, *Austral. Math. Soc. Gaz.* **28** (2001), no. 3, 129–134.

[16] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[17] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[18] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[19] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[20] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[21] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[22] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[23] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure and Appl. Math.*, **3** (2) (2002), Art. 31.

[24] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure and Appl. Math.*, **3** (3) (2002), Art. 35.

[25] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[26] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*. Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[27] S. S. Dragomir, Some new inequalities of Hermite-Hadamard type for GA-convex functions, Preprint *RGMIA Res. Rep. Coll.* 18 (2015), Art 33. [http://rgmia.org/papers/v18/v18a33.pdf].

[28] S. S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions, Preprint *RGMIA Res. Rep. Coll.* 18 (2015), Art 42. [http://rgmia.org/papers/v18/v18a42.pdf].

[29] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[30] S. S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[31] S. S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[32] S. S. Dragomir and B. Mond, Integral inequalities of Hadamard's type for log-convex functions, *Demonstratio Math.,* **31** (2) (1998), 354-364.

[33] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[34] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[35] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[36] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[37] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[38] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[39] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[40] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[41] A. El Farissi, Simple proof and refeinment of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.

[42] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.* 24 (1906),369-390.(In Hungarian).

[43] P. M. Gill, C. E. M. Pearce and J. Pečarić, Hadamard's inequality for r-convex functions, *Journal of Mathematical Analysis and Applications.* **215** (1997), No. 2, pp. 461-470.

[44] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[45] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.

[46] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* (in press)

[47] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[48] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[49] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[50] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.

[51] M. A. Noor, K. I. Noor and M. U. Awan, Some inequalities for geometrically-arithmetically h-convex functions, *Creat. Math. Inform.* **23** (2014), No. 1, 91 - 98.

[52] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[53] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.

[54] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.

[55] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.

[56] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
[57] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
[58] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.
[59] W. T. Sulaiman, Refinements to Hadamard's inequality for log-convex functions. *Applied Mathematics*, **2011**, 2, p. 899-903.
[60] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.
[61] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.
[62] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa