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A BUZANO TYPE INEQUALITY FOR TWO HERMITIAN FORMS AND APPLICATIONS

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Abstract. A Buzano type inequality for two nonnegative Hermitian forms is obtained. Applications to inequalities for norm and numerical radius of bounded linear operators in complex Hilbert spaces are given.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. *A functional* $(\cdot,\cdot):X\times X\to\mathbb{K}$ *is said to be a Hermitian form on* X *if*

(H1) $(ax+by,z)=a(x,z)+b(y,z)$ *for* $a,b\in\mathbb{K}$ *and* $x,y,z\in X$;
(H2) $(x,y)=\overline{(y,x)}$ *for all* $x,y\in X$.

The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y,y)\geq 0$ for every $y\in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y,y)=0$, $y\in Y$ implies $y=0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0:=\{x\in X|\ (x,x)=0\},\ K:=\{x\in X|\ (x,x)<0\}$$

and, for a given $z\in X$,

$$X^{(z)}:=\{x\in X|\ (x,z)=0\}\quad\text{and}\quad L(z):=\{az|\ a\in\mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold [28].

If the functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e\in K$, then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f\in K$.

The functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ with $e\in K$ if and only if the inequality

$$|(x,y)|^2\geq(x,x)(y,y)\tag{1.1}$$

holds for all $x\in K$ and all $y\in X$.

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The functional $(\cdot,\cdot)$ is semi-definite on X if and only if the Schwarz's inequality

$$|(x,y)|^2 \le (x,x)(y,y) \tag{1.2}$$

holds for all $x,y \in X$. The case of equality holds in (1.2) for $x,y \in X$ and in (1.1), for $x \in K$, $y \in X$, respectively; if and only if there exists a scalar $a \in \mathbb{K}$ such that

$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X.

If $(\cdot,\cdot) \in \mathcal{H}(X)$, then the functional $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2 \|y\|^2 \ge |(x,y)|^2 \quad \text{or} \quad \|x\| \|y\| \ge |(x,y)| \tag{1.3}$$

for any $x,y \in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ implies that $(\cdot,\cdot)_1 + (\cdot,\cdot)_2 \in \mathcal{H}(X)$;
(ee) $\alpha \ge 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot) \in \mathcal{H}(X)$.

In 1985 the author [2] (see also [26]) established the following refinement of Schwarz inequality for inner product $\langle\cdot,\cdot\rangle$:

$$\|x\| \|y\| \ge |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| + |\langle x,e\rangle \langle e,y\rangle| \ge |\langle x,y\rangle| \tag{1.4}$$

for any $x,y,e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \ge |\langle x,e\rangle \langle e,y\rangle| - |\langle x,y\rangle|$$

and by (1.4) we get

$$\|x\| \|y\| \ge |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| + |\langle x,e\rangle \langle e,y\rangle| \ge 2|\langle x,e\rangle \langle e,y\rangle| - |\langle x,y\rangle|,$$

which implies the *Buzano inequality* [1]

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x,y\rangle|\right] \ge |\langle x,e\rangle \langle e,y\rangle| \tag{1.5}$$

that holds for any $x,y,e \in H$ with $\|e\| = 1$.

For some inequalities in inner product spaces and operators on Hilbert spaces see [3]-[26] and the references therein.

2. Vector Inequalities for Two Hermitian Forms

We can introduce on $\mathcal{H}(X)$ the following binary relation [23]:

$$(\cdot,\cdot)_2 \ge (\cdot,\cdot)_1 \quad \text{if and only if} \quad \|x\|_2 \ge \|x\|_1 \quad \text{for all} \quad x \in X. \tag{2.1}$$

We observe that the following properties hold:

(b) $(\cdot,\cdot)_2 \ge (\cdot,\cdot)_1$ for all $(\cdot,\cdot) \in \mathcal{H}(X)$;
(bb) $(\cdot,\cdot)_3 \ge (\cdot,\cdot)_2$ and $(\cdot,\cdot)_2 \ge (\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3 \ge (\cdot,\cdot)_1$;
(bbb) $(\cdot,\cdot)_2 \ge (\cdot,\cdot)_1$ and $(\cdot,\cdot)_1 \ge (\cdot,\cdot)_2$ implies that $(\cdot,\cdot)_2 = (\cdot,\cdot)_1$;

i.e., the binary relation defined by (2.1) is *an order relation* on $\mathcal{H}(X)$.

While (b) and (bb) are obvious from the definition, we should remark, for (bbb), that if $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and $(\cdot,\cdot)_1 \geq (\cdot,\cdot)_2$, then obviously $\|x\|_2 = \|x\|_1$ for all $x \in X$, which implies, by the following well known identity:

$$(x,y)_k := \frac{1}{4}\left[\|x+y\|_k^2 - \|x-y\|_k^2 + i\left(\|x+iy\|_k^2 - \|x-iy\|_k^2\right)\right] \tag{2.2}$$

with $x,y \in X$ and $k \in \{1,2\}$, that $(x,y)_2 = (x,y)_1$ for all $x,y \in X$.

The following Buzano type inequality for two Hermitian forms can be stated.

Theorem 1. *Let $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ with $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$. Then for any $x,y \in X$ we have*

$$\frac{1}{2}\left[\|x\|_2\|y\|_2 + |(x,y)_2|\right] \geq |(x,y)_1|. \tag{2.3}$$

Proof. Consider the functional $(\cdot,\cdot)_{2,1} : X \times X \to \mathbb{K}$ given by $(x,y)_{2,1} := (x,y)_2 - (x,y)_1$. Since $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ then $(x,y)_{2,1}$ is a nonnegative form and by Schwarz inequality (1.2) we have

$$\left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) \geq |(x,y)_2 - (x,y)_1|^2 \tag{2.4}$$

for any $x,y \in X$.

Using the elementary inequality for positive numbers a,b,c,d

$$(bd-ac)^2 \geq \left(b^2-a^2\right)\left(d^2-c^2\right)$$

we have

$$\left(\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1\right)^2 \geq \left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) \tag{2.5}$$

for any $x,y \in X$.

Making use of (2.4) and (2.5) we get

$$\left(\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1\right)^2 \geq |(x,y)_2 - (x,y)_1|^2 \tag{2.6}$$

for any $x,y \in X$.

Since $\|x\|_2 \geq \|x\|_1$, $\|y\|_2 \geq \|y\|_1$, then $\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1 \geq 0$ and by (2.6) we get

$$\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1 \geq |(x,y)_2 - (x,y)_1|$$

which is equivalent to

$$\|x\|_2\|y\|_2 \geq \|x\|_1\|y\|_1 + |(x,y)_2 - (x,y)_1|. \tag{2.7}$$

Using Schwarz's inequality $\|x\|_1\|y\|_1 \geq |(x,y)_1|$ and the triangle inequality for modulus

$$|(x,y)_2 - (x,y)_1| \geq |(x,y)_1| - |(x,y)_2|,$$

we have by (2.7) that

$$\|x\|_2\|y\|_2 \geq 2|(x,y)_1| - |(x,y)_2|$$

for any $x,y \in X$, which is equivalent to the desired result (2.3). □

We have the following inequalities for one operator.

Corollary 1. *Let $T : H \to H$ be a bounded linear operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. Then for any $x,y \in H$ we have*

$$\frac{1}{2}\|T\|^2\left[\|x\|\|y\| + |\langle x,y\rangle|\right] \geq |\langle Tx,Ty\rangle|. \tag{2.8}$$

Proof. We consider the Hermitian forms $(\cdot,\cdot)_1, (\cdot,\cdot)_2 : H\times H\to\mathbb{K}$ defined by

$$(x,y)_1 := \langle Tx,Ty\rangle\,,\ \ (x,y)_2 := \|T\|^2\langle x,y\rangle\,.$$

We have $\|x\|_1=\|Tx\|\le\|T\|\,\|x\|=\|x\|_2$ for any $x\in H$. By using the inequality (2.3) for these Hermitian forms we obtain the desired result (2.8). □

Corollary 2. *Let $K:H\to H$ be a bounded linear operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ such that there exists $k>0$ with the property*

$$\|Kx\|\ge k\,\|x\|\ \text{ for any } x\in H. \tag{2.9}$$

Then for any $x,y\in H$ we have

$$\frac{1}{2k^2}\left[\|Kx\|\,\|Ky\|+|\langle Kx,Ky\rangle|\right]\ge|\langle x,y\rangle|\,. \tag{2.10}$$

Follows by Theorem 1 for the Hermitian forms $(x,y)_1 := k^2\langle x,y\rangle\,,\ (x,y)_2 := \langle Kx,Ky\rangle\,.$

We say that the selfadjoint operator P is nonnegative on H and we write $P\ge0$ if $\langle Px,x\rangle\ge0$ for any $x\in H$.

We have the following Buzano type inequalities for two operators.

Corollary 3. *Let $A,B:H\to H$ be two selfadjoint operators on H with $A\ge B\ge0$. Then for any $x,y\in H$ we have*

$$\frac12\left[\langle Ax,x\rangle^{1/2}\langle Ay,y\rangle^{1/2}+|\langle Ax,y\rangle|\right]\ge|\langle Bx,y\rangle|\,. \tag{2.11}$$

Follows by Theorem 1 for the Hermitian forms $(x,y)_1 := \langle Bx,y\rangle\,,\ (x,y)_2 := \langle Ax,y\rangle\,.$

Remark 1. *If T is a bounded linear operator on H and if we take $A=\|T\|^2 1_H$ and $B=T^*T$, then $A\ge B\ge0$ and by (2.11) we recapture (2.8). If K satisfies (2.9) and if we take $A=K^*K$ and $B=k^21_H$, then $A\ge B\ge0$ and by (2.11) we recapture (2.10)*

We also remark that, if C is selfadjoint and $1_H\ge C\ge0$, then we have from (2.11) that

$$\frac12\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\ge|\langle Cx,y\rangle| \tag{2.12}$$

for any $x,y\in H$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j\perp e_k \text{ for any } j,k\in J \text{ with } j\ne k \text{ and } \|e_j\|=1 \text{ for any } j,k\in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J}|\langle x,e_j\rangle|^2\le\|x\|^2 \text{ for any } x\in H.$$

For an othonormal family $\mathcal{E}=\{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}}:H\to H$ by

$$P_{\mathcal{E}}x:=\sum_{j\in J}\langle x,e_j\rangle e_j\ ,\ x\in H. \tag{2.13}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}}x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\,,\ x, y \in H \text{ and } \langle P_{\mathcal{E}}x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2\,,\ x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$ and $\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle$, $x, y \in H$. Since $0 \le P_{\mathcal{E}} \le 1_H$, then by (2.12) applied for $C = P_{\mathcal{E}}$, we have

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \ge \left|\sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| \tag{2.14}$$

for any $x, y \in H$. This inequality becomes Buzano's inequality (1.5) if the family $\mathcal{E}$ reduces to one vector $\{e\}$.

3. Norm and Numerical Radius Inequalities

The *numerical radius* $w(T)$ of an operator T on H is given by [27, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{3.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [27, p. 9]:

Theorem 2 (Equivalent norm)**.** *For any $T \in \mathcal{B}(H)$ one has*

$$w(T) \le \|T\| \le 2w(T). \tag{3.2}$$

Utilising Buzano's inequality we obtained the following inequality for the numerical radius [12] or [13]:

Theorem 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{3.3}$$

The constant $\frac{1}{2}$ is best possible in (3.3).

For a recent survey of inequalities for numerical radius, see [21] and the references therein.

The following result holds:

Theorem 4. *Let U, V be two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and $A, B : H \to H$ be two selfadjoint operators on H with $A \ge B \ge 0$. Then we have the vector norm inequality*

$$\|V^*BUu\| \le \frac{1}{2}\left[\langle U^*AUu, u\rangle^{1/2}\|V^*AV\|^{1/2} + \|V^*AUu\|\right] \tag{3.4}$$

for any $u \in H$.

We also have the operator norm inequality

$$\|V^*BU\| \le \frac{1}{2}\left[\|U^*AU\|^{1/2}\|V^*AV\|^{1/2} + \|V^*AU\|\right]. \tag{3.5}$$

Furthermore, we have the numerical radius inequalities

$$w(V^*BU) \le \frac{1}{2}\left[\|U^*AU\|^{1/2}\|V^*AV\|^{1/2} + w(V^*AU)\right] \tag{3.6}$$

and

$$w\left(V^{*}BU\right)\leq\frac{1}{2}\left[\left\Vert \frac{U^{*}AU+V^{*}AV}{2}\right\Vert +w\left(V^{*}AU\right)\right]. \tag{3.7}$$

Proof. From (2.11) we have for $x=Uu$, $y=Vv$, with $u,v\in H$, that

$$\frac{1}{2}\left[\left\langle AUu,Uu\right\rangle^{1/2}\left\langle AVv,Vv\right\rangle^{1/2}+\left|\left\langle AUu,Vv\right\rangle\right|\right]\geq\left|\left\langle BUu,Vv\right\rangle\right|,$$

that is equivalent to

$$\frac{1}{2}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\langle V^{*}AVv,v\right\rangle^{1/2}+\left|\left\langle V^{*}AUu,v\right\rangle\right|\right]\geq\left|\left\langle V^{*}BUu,v\right\rangle\right|, \tag{3.8}$$

for any $u,v\in H$.

Taking the supremum in (3.8) over $\|v\|=1$ we have for any $u\in H$

$$\begin{aligned}\left\Vert V^{*}BUu\right\Vert &= \sup_{\|v\|=1}\left|\left\langle V^{*}BUu,v\right\rangle\right|\\ &\leq \frac{1}{2}\sup_{\|v\|=1}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\langle V^{*}AVv,v\right\rangle^{1/2}+\left|\left\langle V^{*}AUu,v\right\rangle\right|\right]\\ &\leq \frac{1}{2}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\sup_{\|v\|=1}\left\langle V^{*}AVv,v\right\rangle^{1/2}+\sup_{\|v\|=1}\left|\left\langle V^{*}AUu,v\right\rangle\right|\right]\\ &= \frac{1}{2}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\Vert V^{*}AV\right\Vert^{1/2}+\left\Vert V^{*}AUu\right\Vert\right],\end{aligned}$$

since, $V^{*}AV$ is selfadjoint and then $\sup_{\|v\|=1}\left\langle V^{*}AVv,v\right\rangle^{1/2}=\left\Vert V^{*}AV\right\Vert.$

Taking the supremum over $\|u\|=1$ in (3.4) we get

$$\begin{aligned}\left\Vert V^{*}BU\right\Vert &= \sup_{\|u\|=1}\left\Vert V^{*}BUu\right\Vert\\ &\leq \frac{1}{2}\sup_{\|u\|=1}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\Vert V^{*}AV\right\Vert^{1/2}+\left\Vert V^{*}AUu\right\Vert\right]\\ &\leq \frac{1}{2}\left[\sup_{\|u\|=1}\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\Vert V^{*}AV\right\Vert^{1/2}+\sup_{\|u\|=1}\left\Vert V^{*}AUu\right\Vert\right]\\ &= \frac{1}{2}\left[\left\Vert U^{*}AU\right\Vert^{1/2}\left\Vert V^{*}AV\right\Vert^{1/2}+\left\Vert V^{*}AU\right\Vert\right],\end{aligned}$$

that proves the inequality (3.5).

From the inequality (3.8) we have

$$\left|\left\langle V^{*}BUu,u\right\rangle\right|\leq\frac{1}{2}\left[\left\langle U^{*}AUu,u\right\rangle^{1/2}\left\langle V^{*}AVu,u\right\rangle^{1/2}+\left|\left\langle V^{*}AUu,u\right\rangle\right|\right], \tag{3.9}$$

for any $u\in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ we have

$$\begin{aligned} w(V^*BU) &= \sup_{\|u\|=1} |\langle V^*BUu, u\rangle| \\ &\le \frac{1}{2} \sup_{\|u\|=1} \left[\langle U^*AUu, u\rangle^{1/2} \langle V^*AVu, u\rangle^{1/2} + |\langle V^*AUu, u\rangle| \right] \\ &\le \frac{1}{2} \left[\sup_{\|u\|=1} \left\{ \langle U^*AUu, u\rangle^{1/2} \langle V^*AVu, u\rangle^{1/2} \right\} + \sup_{\|u\|=1} |\langle V^*AUu, u\rangle| \right] \\ &\le \frac{1}{2} \left[\sup_{\|u\|=1} \langle U^*AUu, u\rangle^{1/2} \sup_{\|u\|=1} \langle V^*AVu, u\rangle^{1/2} + \sup_{\|u\|=1} |\langle V^*AUu, u\rangle| \right] \\ &= \frac{1}{2} \left[\|U^*AU\|^{1/2} \|V^*AV\|^{1/2} + w(V^*AU) \right] \end{aligned}$$

and the inequality (3.6) is proved.

By the arithmetic mean - geometric mean inequality $\sqrt{ab} \le \frac{a+b}{2}$, $a, b \ge 0$, we have

$$\langle U^*AUu, u\rangle^{1/2} \langle V^*AVu, u\rangle^{1/2} \le \left\langle \frac{U^*AU + V^*AV}{2} u, u \right\rangle$$

for any $u \in H$.

Using (3.9) we have

$$\begin{aligned} |\langle V^*BUu, u\rangle| &\le \frac{1}{2} \left[\langle U^*AUu, u\rangle^{1/2} \langle V^*AVu, u\rangle^{1/2} + |\langle V^*AUu, u\rangle| \right] \\ &\le \frac{1}{2} \left[\left\langle \frac{U^*AU + V^*AV}{2} u, u \right\rangle + |\langle V^*AUu, u\rangle| \right] \end{aligned}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ and taking into account that $\frac{1}{2}(U^*AU + V^*AV)$ is selfadjoint and nonnegative, we deduce the desired result (3.7). □

We use the notation $|T|^2 := T^*T$ for the bounded linear operator T on the Hilbert space H.

Corollary 4. *Let $T : H \to H$ be a bounded linear operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then for any U, V two bounded linear operators on H we have*

$$\left\| V^* |T|^2 Uu \right\| \le \frac{1}{2} \|T\|^2 \left[\|Uu\| \|V\| + \|V^*Uu\| \right] \tag{3.10}$$

for any $u \in H$.

We also have the operator norm inequality

$$\left\| V^* |T|^2 U \right\| \le \frac{1}{2} \|T\|^2 \left[\|U\| \|V\| + \|V^*U\| \right]. \tag{3.11}$$

Moreover, we have the numerical radius inequalities

$$w\left(V^* |T|^2 U \right) \le \frac{1}{2} \|T\|^2 \left[\|U\| \|V\| + w(V^*U) \right] \tag{3.12}$$

and

$$w\left(V^* |T|^2 U \right) \le \frac{1}{2} \|T\|^2 \left[\left\| \frac{|U|^2 + |V|^2}{2} \right\| + w(V^*U) \right]. \tag{3.13}$$

We observe that if we take $V = U^*$ in the Corollary 4, then we get

$$\left\| U |T|^2 Uu \right\| \leq \frac{1}{2} \|T\|^2 \left[\|Uu\| \|U\| + \left\| U^2 u \right\| \right] \tag{3.14}$$

for any $u \in H$.

This implies the operator norm inequality

$$\left\| U |T|^2 U \right\| \leq \frac{1}{2} \|T\|^2 \left[\|U\|^2 + \left\| U^2 \right\| \right]. \tag{3.15}$$

Also, we have the numerical radius inequalities

$$w\left(U |T|^2 U \right) \leq \frac{1}{2} \|T\|^2 \left[\|U\|^2 + w\left(U^2\right) \right] \tag{3.16}$$

and

$$w\left(U |T|^2 U \right) \leq \frac{1}{2} \|T\|^2 \left[\left\| \frac{|U|^2 + |U^*|^2}{2} \right\| + w\left(U^2\right) \right]. \tag{3.17}$$

Since

$$\left\| \frac{|U|^2 + |U^*|^2}{2} \right\| \leq \frac{1}{2} \left[\|U^*U\| + \|UU^*\| \right] = \|U\|^2,$$

then the inequality (3.17) is better than (3.16).

Corollary 5. *Let $K : H \to H$ be a bounded linear operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ such that there exists $k > 0$ with the property (2.9). Then for any U, V two bounded linear operators on H we have*

$$\|V^*Uu\| \leq \frac{1}{2k^2} \left[\|KUu\| \|KV\| + \left\| V^* |K|^2 Uu \right\| \right] \tag{3.18}$$

for any $u \in H$.

We also have the operator norm inequality

$$\|V^*U\| \leq \frac{1}{2k^2} \left[\|KU\| \|KV\| + \left\| V^* |K|^2 U \right\| \right]. \tag{3.19}$$

Furthermore, we have the numerical radius inequalities

$$w\left(V^*U\right) \leq \frac{1}{2k^2} \left[\|KU\| \|KV\| + w\left(V^* |K|^2 U \right) \right] \tag{3.20}$$

and

$$w\left(V^*U\right) \leq \frac{1}{2k^2} \left[\left\| \frac{|KU|^2 + |KV|^2}{2} \right\| + w\left(V^* |K|^2 U \right) \right]. \tag{3.21}$$

If we take in Corollary 5 $V = U^*$, then we get

$$\left\| U^2 u \right\| \leq \frac{1}{2k^2} \left[\|KUu\| \|KU^*\| + \left\| U |K|^2 Uu \right\| \right] \tag{3.22}$$

for any $u \in H$.

We also have the operator norm inequality

$$\left\| U^2 \right\| \leq \frac{1}{2k^2} \left[\|KU\| \|KU^*\| + \left\| U |K|^2 U \right\| \right]. \tag{3.23}$$

Moreover, we have the numerical radius inequalities

$$w\left(U^2\right) \leq \frac{1}{2k^2} \left[\|KU\| \|KU^*\| + w\left(U |K|^2 U \right) \right] \tag{3.24}$$

and

$$w\left(U^2\right) \leq \frac{1}{2k^2}\left[\left\|\frac{|KU|^2+|KU^*|^2}{2}\right\| + w\left(U|K|^2 U\right)\right]. \tag{3.25}$$

Corollary 6. *Let $e \in H$, $\|e\| = 1$. Then for any U, V two bounded linear operators on H we have*

$$\frac{1}{2}\left[\langle U^*Uu,u\rangle^{1/2}\langle V^*Vv,v\rangle^{1/2} + |\langle V^*Uu,v\rangle|\right] \geq |\langle Uu,e\rangle\langle e,Vv\rangle|, \tag{3.26}$$

for any $u, v \in H$.

Proof. Consider the operator $B : H \to H$ defined by $Bx = \langle x,e\rangle e$. This is selfadjoint and $0 \leq B \leq 1_H$. Also

$$\langle V^*BUu,v\rangle = \langle V^*\langle Uu,e\rangle e,v\rangle = \langle\langle Uu,e\rangle e,Vv\rangle = \langle Uu,e\rangle\langle e,Vv\rangle$$

for any $u, v \in H$. Using (2.11) we then get (3.26). □

If we take in (3.26) $u = v$ and use the arithmetic mean - geometric mean inequality we have

$$|\langle Uu,e\rangle\langle e,Vu\rangle| \leq \frac{1}{2}\left[\left\langle\frac{U^*U+V^*V}{2}u,u\right\rangle + |\langle V^*Uu,u\rangle|\right]$$

for any $e \in H$, $\|e\| = 1$ and $u \in H$.

If we choose in this inequality $V = U^*$ and $u = e$, then we get

$$|\langle Ue,e\rangle|^2 \leq \frac{1}{2}\left[\left\langle\frac{U^*U+UU^*}{2}e,e\right\rangle + \left|\left\langle U^2e,e\right\rangle\right|\right] \tag{3.27}$$

for any $e \in H$, $\|e\| = 1$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (3.27) produces the numerical radius inequality

$$w^2(U) \leq \frac{1}{2}\left[\left\|\frac{U^*U+UU^*}{2}\right\| + w\left(U^2\right)\right]. \tag{3.28}$$

The inequality (3.28) is better than (3.3) since

$$\left\|\frac{U^*U+UU^*}{2}\right\| \leq \|U\|^2$$

for any bounded linear operator U.

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