

ON GENERALIZED SOME INEQUALITIES FOR CONVEX FUNCTIONS

SAMET ERDEN AND MEHMET ZEKI SARIKAYA

ABSTRACT. In this paper, a general integral identity for differentiable mapping is derived. Then, we extend some estimates of the right hand and left hand side of a Hermite- Hadamard-Fejér type inequality for functions whose first derivatives absolute values are convex. Some applications for special means of real numbers are also provided. The results presented here would provide extensions of those given in earlier works.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [2], [6]):

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f , many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([1], [2], [6], [5] and [11]) and the references included there.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$(1.2) \quad \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{b-a}{2} \int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$(1.3) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|).$$

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Theorem 2. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:

$$(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}.$$

In [5], Kırmacı proved the following results connected with the left part of (1.1).

Lemma 2. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L(a, b)$, then we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \\ = & (b-a) \left[\int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned}$$

Theorem 3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have

$$(1.5) \quad \left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [7]-[16]). In [3], Fejér gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 4. $f : [a, b] \rightarrow \mathbb{R}$, be a convex function, then the inequality

$$(1.6) \quad f\left(\frac{a+b}{2}\right) \int_a^b w(x) dx \leq \frac{1}{b-a} \int_a^b f(x) w(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b w(x) dx$$

holds, where $w : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In [7], some inequalities of Hermite-Hadamard-Fejér type for differentiable convex mappings were proved using the following lemma.

Lemma 3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \rightarrow [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:

$$(1.7) \quad \frac{f(a) + f(b)}{2} \int_a^b w(x) dx - \int_a^b f(x) w(x) dx = \frac{(b-a)^2}{2} \int_0^1 p(t) f'(ta + (1-t)b) dt$$

for each $t \in [0, 1]$, where

$$p(t) = \int_t^1 w(as + (1-s)b) ds - \int_0^t w(as + (1-s)b) ds.$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard-Fejér type. The results presented here would provide extensions of those given in earlier works.

2. MAIN RESULTS

In order to prove our main results, we need the following lemma:

Lemma 4. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$. If $f', g \in L[a, b]$, then for all $x \in [a, b]$, the following identity holds:*

$$(2.1) \quad \int_a^b P_\lambda(x, t) f'(t) dt \\ = (1 - \lambda) f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds$$

where

$$P_\lambda(x, t) := \begin{cases} (1 - \lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds & , a \leq t < x \\ (1 - \lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds & , x \leq t \leq b. \end{cases}$$

for $\lambda \in [0, 1]$.

Proof. By integration by parts, we have the following identity:

$$\begin{aligned} & \int_a^b P_\lambda(x, t) f'(t) dt \\ &= \int_a^x \left[(1 - \lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right] f'(t) dt \\ & \quad + \int_x^b \left[(1 - \lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right] f'(t) dt \\ &= \left[(1 - \lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right] f(t) \Big|_{t=a}^x - \int_a^x g(s) f(s) ds \\ & \quad + \left[(1 - \lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right] f(t) \Big|_{t=x}^b - \int_x^b g(s) f(s) ds \\ &= (1 - \lambda) f(x) \int_a^b g(s) ds + \lambda f(a) \int_a^x g(s) ds + \lambda f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds. \end{aligned}$$

This completes the proof. \square

Remark 1. Under the same assumptions of Lemma 4 with $\lambda = 1$; then the following identity holds:

$$\begin{aligned} \int_a^b P_1(x, t) f'(t) dt &= \int_a^b \left[\int_x^t g(s) ds \right] f'(t) dt \\ &= f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds \end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 2. Under the same assumptions of Lemma 4 with $\lambda = 0$; then the following identity holds:

$$\begin{aligned} \int_a^b P_0(x, t) f'(t) dt &= \int_a^x \left(\int_a^t g(s) ds \right) f'(t) dt - \int_x^b \left(\int_t^b g(s) ds \right) f'(t) dt \\ &= f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \end{aligned}$$

which is proved by Sarikaya and Erden in [8].

Remark 3. In Lemma 4, let g be symmetric to $\frac{a+b}{2}$ and let $x = \frac{a+b}{2}$. Then (2.1) can be written as

$$\begin{aligned} (2.2) \quad &\int_a^b P_\lambda \left(\frac{a+b}{2}, t \right) f'(t) dt \\ &= (1-\lambda) f \left(\frac{a+b}{2} \right) \int_a^b g(s) ds + \lambda \frac{f(a)+f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds. \end{aligned}$$

Using this Lemma we can obtain the following general integral inequalities:

Theorem 5. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then,

for all $x \in [a, b]$, the following inequalities hold:

$$\begin{aligned}
(2.3) \quad & \left| (1-\lambda) f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right. \\
& \left. + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] \right| \\
& \leq \frac{\|g\|_{[a,x],\infty}}{6(b-a)} \left[|f'(a)| (x-a)^2 ((1-\lambda)(3b-a-2x) + \lambda(3b-2a-x)) \right. \\
& \quad \left. + (2-\lambda) |f'(b)| (x-a)^3 \right] \\
& \quad + \frac{\|g\|_{[x,b],\infty}}{6(b-a)} \left[(2-\lambda) |f'(a)| (b-x)^3 \right. \\
& \quad \left. + |f'(b)| (b-x)^2 ((1-\lambda)(b-3a+2x) + \lambda(2b-3a+x)) \right] \\
& \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| (x-a)^2 ((1-\lambda)(3b-a-2x) + \lambda(3b-2a-x)) \right. \\
& \quad + |f'(a)| (2-\lambda) (b-x)^3 + |f'(b)| (2-\lambda) (x-a)^3 \\
& \quad \left. + |f'(b)| (b-x)^2 ((1-\lambda)(b-3a+2x) + \lambda(2b-3a+x)) \right\}.
\end{aligned}$$

where $\lambda \in [0, 1]$ and $\|g\|_{[a,b],\infty} = \sup_{s \in [a,b]} |g(s)|$.

Proof. We take absolute of (2.1). Using bounded of the mapping g and the convexity of $|f'|$, we find that

$$\begin{aligned}
& \left| (1-\lambda) f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] \right| \\
& \leq \int_a^b |P_\lambda(x, t)| |f'(t)| dt \\
& \leq \int_a^x \left((1-\lambda) \left| \int_a^t g(s) ds \right| + \lambda \left| \int_x^t g(s) ds \right| \right) \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| dt \\
& \quad + \int_x^b \left((1-\lambda) \left| \int_b^t g(s) ds \right| + \lambda \left| \int_x^t g(s) ds \right| \right) \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| dt
\end{aligned}$$

$$\begin{aligned}
&\leq \|g\|_{[a,x],\infty} \int_a^x [(1-\lambda)(t-a) + \lambda(x-t)] \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\
&\quad + \|g\|_{[x,b],\infty} \int_x^b [(1-\lambda)(b-t) + \lambda(t-x)] \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\
&= \frac{\|g\|_{[a,x],\infty}}{b-a} \left[(1-\lambda) |f'(a)| \int_a^x (t-a)(b-t) dt + (1-\lambda) |f'(b)| \int_a^x (t-a)^2 dt \right. \\
&\quad \left. + \lambda |f'(a)| \int_a^x (x-t)(b-t) dt + \lambda |f'(b)| \int_a^x (x-t)(t-a) dt \right] \\
&\quad + \frac{\|g\|_{[x,b],\infty}}{b-a} \left[(1-\lambda) |f'(a)| \int_x^b (b-t)^2 dt + (1-\lambda) |f'(b)| \int_x^b (b-t)(t-a) dt \right. \\
&\quad \left. + \lambda |f'(a)| \int_x^b (t-x)(b-t) dt + \lambda |f'(b)| \int_x^b (t-x)(t-a) dt \right].
\end{aligned}$$

Since

$$\begin{aligned}
\int_a^x (t-a)(b-t) dt &= \frac{(x-a)^2(3b-a-2x)}{6} \\
\int_a^x (t-a)^2 dt &= \frac{(x-a)^3}{3} \\
\int_a^x (x-t)(b-t) dt &= \frac{(x-a)^2(3b-2a-x)}{6} \\
\int_a^x (x-t)(t-a) dt &= \frac{(x-a)^3}{6}
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (b-t)^2 dt &= \frac{(b-x)^3}{3} \\
\int_x^b (b-t)(t-a) dt &= \frac{(b-x)^2(b-3a+2x)}{6} \\
\int_x^b (t-x)(b-t) dt &= \frac{(b-x)^3}{6} \\
\int_x^b (t-x)(t-a) dt &= \frac{(b-x)^2(2b-3a+x)}{6},
\end{aligned}$$

we obtain (2.3). Hence, this completes the proof. \square

Remark 4. Under the same assumptions of Theorem 5 with $\lambda = 1$; then the following inequality holds:

$$\begin{aligned}
& \left| f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{\|g\|_{[a,x],\infty}}{(b-a)} \left[\frac{|f'(a)|(x-a)^2(3b-2a-x) + |f'(b)|(x-a)^3}{6} \right] \\
& \quad + \frac{\|g\|_{[x,b],\infty}}{(b-a)} \left[\frac{|f'(a)|(b-x)^3 + |f'(b)|(b-x)^2(2b-3a+x)}{6} \right] \\
& \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| \left[(x-a)^2(3b-2a-x) + (b-x)^3 \right] \right. \\
& \quad \left. + |f'(b)| \left[(b-x)^2(2b-3a+x) + (x-a)^3 \right] \right\}
\end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 5. Under the same assumptions of Theorem 5 with $\lambda = 0$; then the following identity holds:

$$\begin{aligned}
& \left| f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{\|g\|_{[a,x],\infty}}{(b-a)} \left[\frac{|f'(a)|(x-a)^2(3b-a-2x) + 2|f'(b)|(x-a)^3}{6} \right] \\
& \quad + \frac{\|g\|_{[x,b],\infty}}{(b-a)} \left[\frac{2|f'(a)|(b-x)^3 + |f'(b)|(b-x)^2(b-3a+2x)}{6} \right] \\
& \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| \left[(x-a)^2(3b-a-2x) + 2(b-x)^3 \right] \right. \\
& \quad \left. + |f'(b)| \left[(b-x)^2(b-3a+2x) + 2(x-a)^3 \right] \right\}
\end{aligned}$$

which is proved by Sarikaya and Erden in [8].

Corollary 1. *Let $0 \leq \alpha \leq 1$ and $x = \alpha a + (1 - \alpha)b$ in Theorem 5. Then we have*

$$\begin{aligned}
(2.4) \quad & \left| (1 - \lambda) f(\alpha a + (1 - \alpha)b) \int_a^b g(s) ds + \lambda f(a) \int_a^{\alpha a + (1 - \alpha)b} g(s) ds \right. \\
& \left. + f(b) \int_{\alpha a + (1 - \alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \|g\|_{[a, \alpha a + (1 - \alpha)b], \infty} (b - a)^2 \left(|f'(b)| \frac{(2 - \lambda)(1 - \alpha)^3}{6} \right. \\
& \quad \left. + |f'(a)| \frac{(1 - \alpha)^2 [(1 - \lambda)(2\alpha + 1) + \lambda(2 + \alpha)]}{6} \right) \\
& \quad + \|g\|_{[\alpha a + (1 - \alpha)b, b], \infty} (b - a)^2 \left(|f'(a)| \frac{(2 - \lambda)\alpha^3}{6} \right. \\
& \quad \left. + |f'(b)| \frac{\alpha^2 [(1 - \lambda)(3 - 2\alpha) + \lambda(3 - \alpha)]}{6} \right) \\
& \leq \|g\|_{[a, b], \infty} (b - a)^2 \\
& \quad \times \left\{ |f'(a)| \frac{(1 - \alpha)^2 [(1 - \lambda)(2\alpha + 1) + \lambda(2 + \alpha)] + (2 - \lambda)\alpha^3}{6} \right. \\
& \quad \left. + |f'(b)| \frac{\alpha^2 [(1 - \lambda)(3 - 2\alpha) + \lambda(3 - \alpha)] + (2 - \lambda)(1 - \alpha)^3}{6} \right\}
\end{aligned}$$

for $\lambda \in [0, 1]$.

Remark 6. *If we take $\lambda = 1$ in (2.4), we get*

$$\begin{aligned}
& \left| f(a) \int_a^{\alpha a + (1 - \alpha)b} g(s) ds + f(b) \int_{\alpha a + (1 - \alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \|g\|_{[a, \alpha a + (1 - \alpha)b], \infty} (b - a)^2 \left[\frac{|f'(b)|(1 - \alpha)^3 + |f'(a)|(1 - \alpha)^2(2 + \alpha)}{6} \right] \\
& \quad + \|g\|_{[\alpha a + (1 - \alpha)b, b], \infty} (b - a)^2 \left[\frac{|f'(a)|\alpha^3 + |f'(b)|\alpha^2(3 - \alpha)}{6} \right] \\
& \leq \|g\|_{[a, b], \infty} (b - a)^2 \left(\frac{|f'(a)| [(1 - \alpha)^2(2 + \alpha) + \alpha^3] + |f'(b)| [\alpha^2(3 - \alpha) + (1 - \alpha)^3]}{6} \right)
\end{aligned}$$

which is proved by Tseng et. al in [11].

Using Theorem 5, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 2. Let $g : [a, b] \rightarrow \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $\alpha = \frac{1}{2}$ in Corollary 1. Then we have the inequalities

$$\begin{aligned}
(2.5) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds + \lambda \frac{f(a)+f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \|g\|_{[a, \frac{a+b}{2}], \infty} (b-a)^2 \left(\frac{4+\lambda}{48} |f'(a)| + \frac{2-\lambda}{48} |f'(b)| \right) \\
& \quad + \|g\|_{[\frac{a+b}{2}, b], \infty} (b-a)^2 \left(\frac{2-\lambda}{48} |f'(a)| + \frac{4+\lambda}{48} |f'(b)| \right) \\
& \leq \frac{\|g\|_{[a, b], \infty} (b-a)^2 (|f'(a)| + |f'(b)|)}{8}
\end{aligned}$$

which is "**weighted trapezoid**" inequality provided that $|f'|$ is convex on $[a, b]$.

Remark 7. If we take $\lambda = 1$ in (2.5), we obtain

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{(b-a)^2}{48} (5|f'(a)| + |f'(b)|) \|g\|_{[a, \frac{a+b}{2}], \infty} \\
& \quad + \frac{(b-a)^2}{48} (|f'(a)| + 5|f'(b)|) \|g\|_{[\frac{a+b}{2}, b], \infty} \\
& \leq \frac{(b-a)^2 (|f'(a)| + |f'(b)|)}{8} \|g\|_{[a, b], \infty}
\end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 8. If we take $\lambda = 0$ in (2.5), we get

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{(b-a)^2}{24} (2|f'(a)| + |f'(b)|) \|g\|_{[a, \frac{a+b}{2}], \infty} \\
& \quad + \frac{(b-a)^2}{24} (|f'(a)| + 2|f'(b)|) \|g\|_{[\frac{a+b}{2}, b], \infty} \\
& \leq \frac{(b-a)^2 (|f'(a)| + |f'(b)|)}{8} \|g\|_{[a, b], \infty}
\end{aligned}$$

which is "**weighted midpoint**" inequality provided that $|f'|$ is convex on $[a, b]$.

Corollary 3. *If we take $g(t) = 1$ in (2.5), we have*

$$\begin{aligned}
(2.6) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) (b-a) + \lambda \frac{f(a)+f(b)}{2} (b-a) - \int_a^b f(s) ds \right| \\
& \leq \frac{(b-a)^2}{48} ((4+\lambda)|f'(a)| + (2-\lambda)|f'(b)|) \\
& \quad + \frac{(b-a)^2}{48} ((2-\lambda)|f'(a)| + (4+\lambda)|f'(b)|) \\
& \leq \frac{(b-a)^2 (|f'(a)| + |f'(b)|)}{8}.
\end{aligned}$$

Remark 9. *If we choose $\lambda = 1$ in (2.6), then the inequality (2.6) reduces to (1.3).*

Remark 10. *If we choose $\lambda = 0$ in (2.6), then the inequality (2.6) reduces to (1.5).*

Theorem 6. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and let $f' \in L[a, b]$, $a, b \in I^\circ$ with $a < b$, and let $g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequalities hold: for $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$*

$$\begin{aligned}
(2.7) \quad & \left| (1-\lambda) f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
& \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} (b-a)^{\frac{1}{q}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\
& \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty}
\end{aligned}$$

and for $\lambda = \frac{1}{2}$

$$\begin{aligned}
& \left| \frac{f(x)}{2} \int_a^b g(s) ds + \frac{1}{2} \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{\|g\|_{[a,b],\infty} (b-a)^{\frac{1}{q}}}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|g\|_\infty = \sup_{s \in [a,b]} |g(s)|$.

Proof. We take absolute value of (2.1). Using Hölder's inequality and the convexity of $|f'(t)|^q$, we find that

$$\begin{aligned}
& \left| (1-\lambda) f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
& \leq \int_a^b |P_\lambda(x, t)| |f'(t)| dt \leq \left(\int_a^b |P_\lambda(x, t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\
& \leq \left(\int_a^x \left| (1-\lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right|^p dt + \int_x^b \left| (1-\lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_a^b \left[\frac{b-t}{b-a} |f'(a)|^q + \frac{t-a}{b-a} |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\
& = \left(\|g\|_{[a,x],\infty}^p \int_a^x [(1-\lambda)(t-a) + \lambda(x-t)]^p dt \right. \\
& \quad \left. + \|g\|_{[x,b],\infty}^p \int_x^b [(1-\lambda)(b-t) + \lambda(t-x)]^p dt \right)^{\frac{1}{p}} \\
& \quad \times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.
\end{aligned}$$

Now, we make change of variable

$$\begin{aligned}
(2.8) \quad & (1-\lambda)(t-a) + \lambda(x-t) = u & dt &= \frac{du}{1-2\lambda} \\
& (1-\lambda)(b-t) + \lambda(t-x) = v & dt &= \frac{dv}{2\lambda-1}.
\end{aligned}$$

From (2.8), it follows that

$$\begin{aligned}
& \left| (1-\lambda) f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
& \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \left[\|g\|_{[a,x],\infty}^p (x-a)^{p+1} + \|g\|_{[x,b],\infty}^p (b-x)^{p+1} \right]^{\frac{1}{p}} \\
& \quad \times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \\
& \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \|g\|_{[a,b],\infty} \left[(x-a)^{p+1} + (b-x)^{p+1} \right]^{\frac{1}{p}} \\
& \quad \times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

we obtain the inequality (2.7).

For $\lambda = \frac{1}{2}$, because of Lemma 4, and using Hölder's inequality and the convexity of $|f'(t)|^q$, we find that

$$\begin{aligned}
& \left| \frac{f(x)}{2} \int_a^b g(s) ds + \frac{1}{2} \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
& \leq \int_a^b |P_{\frac{1}{2}}(x,t)| |f'(t)| dt \leq \left(\int_a^b |P_{\frac{1}{2}}(x,t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{1}{2} \left(\|g\|_{[a,x],\infty}^p \int_a^x [(t-a) + (x-t)] dt + \|g\|_{[x,b],\infty}^p \int_x^b [(b-t) + (t-x)] dt \right)^{\frac{1}{p}} \\
& \quad \times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.
\end{aligned}$$

Hence, the proof is completed. \square

Remark 11. Under the same assumptions of Theorem 6 with $\lambda = 1$; then the following inequality holds:

$$\begin{aligned}
& \left| f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{\|g\|_{[a,b],\infty} (b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left[(x-a)^{p+1} + (b-x)^{p+1} \right]^{\frac{1}{p}} \\
& \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

which is proved by Tseng et. al in [11].

Corollary 4. Under the same assumptions of Theorem 6 with $\lambda = 0$; then the following inequality holds:

$$\begin{aligned} & \left| f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\ & \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is "weighted Ostrowski" inequality provided that $|f'|^q$ is convex on $[a, b]$.

Corollary 5. Let $0 \leq \alpha \leq 1$ and $x = \alpha a + (1-\alpha)b$ in Theorem 6. Then we have

$$\begin{aligned} (2.9) \quad & \left| (1-\lambda) f(\alpha a + (1-\alpha)b) \int_a^b g(s) ds + \lambda f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds \right. \\ & \left. + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} (b-a)^2 [(1-\alpha)^{p+1} + \alpha^{p+1}]^{\frac{1}{p}} \\ & \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

for $\lambda \in [0, 1]$.

Remark 12. If we take $\lambda = 1$ in (2.9), we get

$$\begin{aligned} & \left| f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} [(1-\alpha)^{p+1} + \alpha^{p+1}]^{\frac{1}{p}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is proved by Tseng et. al in [11].

Using Theorem 6, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 6. Let $g : [a, b] \rightarrow \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $\alpha = \frac{1}{2}$ in Corollary 5. Then we have the inequality

$$(2.10) \quad \left| (1-\lambda) f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds + \lambda \frac{f(a)+f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \frac{(b-a)^2}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty}.$$

Remark 13 (weighted trapezoid). If we take $\lambda = 1$ in (2.10), we obtain

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty}$$

which is proved by Tseng et. al in [11].

Remark 14. If we take $\lambda = 0$ in (2.10), we get

$$\left| f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty}$$

which is "weighted midpoint" inequality provided that $|f'|^q$ is convex on $[a, b]$ and $f' \in L(a, b)$ where $p > 1$.

Corollary 7. If we take $g(t) = 1$ in (2.10), we have

$$(2.11) \quad \left| (1-\lambda) f\left(\frac{a+b}{2}\right) (b-a) + \lambda \frac{f(a)+f(b)}{2} (b-a) - \int_a^b f(s) ds \right| \\ \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \frac{(b-a)^2}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

Remark 15. If we choose $\lambda = 1$ in (2.11), then the inequality (2.11) reduces to (1.4).

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BARTIN UNIVERSITY, KONURALP CAMPUS, BARTIN-TURKEY

E-mail address: erdensmt@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, KONURALP CAMPUS, DÜZCE-TURKEY

E-mail address: sarikayamz@gmail.com