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SOME THE WEIGHTED GENERALIZATIONS THE INTEGRAL INEQUALITIES FOR CONVEX MAPPINGS

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Abstract. In this paper, an important integral identity and new Hermite-Hadamard Fejer type integral inequalities are derived. Then, we extend some estimates of the right hand and left hand side of a Hermite- Hadamard-Fejér type inequality for functions whose first derivatives absolute values are convex. In addition, new Hermite-Hadamard-type inequalities involving fractional integral are given. Some applications for special means of real numbers are also provided. The results presented here would provide extensions of those given in earlier works.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [3], [10]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f, many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([2], [3], [10], [8] and [16]) and the references included there.

In [2], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

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Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} \left(|f'(a)| + |f'(b)| \right). \tag{1.3}$$

In [1], Bullen proved the following inequality which is known in the literature as Bullen's inequality.

Theorem 2. *If f is convex and integrable, then*

$$\left(\int_{-1}^{1} f \right) - 2f(0) \leq f(-1) + f(1) - \left(\int_{-1}^{1} f \right).$$

If transformed to an arbitrary compact interval $[a, b] \subset \mathbb{R}$, $a < b$, the equivalent form of the inequality reads

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{2} \left[f\left(\frac{a+b}{2} \right) + \frac{f(a) + f(b)}{2} \right].$$

In [8], Kırmacı proved the following results connected with the left part of (1.1).

Lemma 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L(a, b)$, then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(s)\, ds - f\left(\frac{a+b}{2} \right) \\ = & \ (b-a) \left[\int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^{1} (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned}$$

Theorem 3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(s)\, ds - f\left(\frac{a+b}{2} \right) \right| \leq \frac{b-a}{8} \left(|f'(a)| + |f'(b)| \right). \tag{1.4}$$

Theorem 4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s)\, ds - f\left(\frac{a+b}{2} \right) \right| \qquad (1.5) \\ \leq & \ \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \left[\left(3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right. \\ & \left. + \left(|f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right]. \end{aligned}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [6], [7], [11]-[21]). In [5], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

Theorem 5. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\leq\frac{1}{b-a}\int_a^b f(x)w(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.6}$$

holds, where $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$.

In [11], some inequalities of Hermite-Hadamard-Fejer type for differentiable convex mappings were proved using the following lemma.

Lemma 3. *Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$, and $w:[a,b]\to[0,\infty)$ be a differentiable mapping. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\int_a^b f(x)w(x)dx=\frac{(b-a)^2}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt \tag{1.7}$$

for each $t\in[0,1]$, where

$$p(t)=\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds.$$

In this study, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard-Fejer type. The results presented here would provide extensions of those given in earlier works.

2. Main Results

Firstly, we give the following notation used to simplify the details of presentation,

$$\begin{aligned}S_{\lambda}(f,g;\alpha) \;:\; &=(1-\lambda)\left[\left(\int_a^x g(s)\,ds\right)^{\alpha}+\left(\int_x^b g(s)\,ds\right)^{\alpha}\right]f(x)\\&+\lambda\left[\left(\int_x^b g(s)\,ds\right)^{\alpha}f(b)-\left(\int_x^a g(s)\,ds\right)^{\alpha}f(a)\right]\\&-\alpha(1-\lambda)\left[\int_a^x\left(\int_a^t g(s)\,ds\right)^{\alpha-1}g(t)f(t)\,dt+\int_x^b\left(\int_t^b g(s)\,ds\right)^{\alpha-1}g(t)f(t)\,dt\right]\\&-\alpha\lambda\int_a^b\left(\int_x^t g(s)\,ds\right)^{\alpha-1}g(t)f(t)\,dt\end{aligned}$$

We will establish some new results connected with the right-hand and left hand side of (1.1) and (1.6) used the following Lemma. Some of the results mentioned will be the inequalities involving fractional integrals. Now, we give the following new Lemma for our results:

Lemma 4. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If $f',g\in L[a,b]$, then for all $x\in[a,b]$, the following*

identity holds:

$$\int_a^b P_\lambda(x,t) f'(t)\,dt = S_\lambda(f,g;\alpha) \tag{2.1}$$

where

$$P_\lambda(x,t) := \begin{cases} (1-\lambda)\left(\int_a^t g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha, & a \le t < x \\ (\lambda-1)\left(\int_t^b g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha, & x \le t \le b. \end{cases}$$

for $\lambda \in [0,1]$.

Proof. By integration by parts, we have the following identity:

$$\begin{aligned} &\int_a^b P_\lambda(x,t) f'(t)\,dt \\ = &\int_a^x \left[(1-\lambda)\left(\int_a^t g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha\right] f'(t)\,dt \\ &+\int_x^b \left[(\lambda-1)\left(\int_t^b g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha\right] f'(t)\,dt \end{aligned}$$

$$\begin{aligned} = &\left[(1-\lambda)\left(\int_a^t g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha\right] f(t)\Big|_{t=a}^{x} \\ &-\alpha\int_a^x \left[(1-\lambda)\left(\int_a^t g(s)\,ds\right)^{\alpha-1} g(t) + \lambda\left(\int_x^t g(s)\,ds\right)^{\alpha-1} g(t)\right] f(t)\,dt \\ &+\left[(\lambda-1)\left(\int_t^b g(s)\,ds\right)^\alpha + \lambda\left(\int_x^t g(s)\,ds\right)^\alpha\right] f(t)\Big|_{x}^{b} \\ &-\alpha\int_x^b \left[(1-\lambda)\left(\int_t^b g(s)\,ds\right)^{\alpha-1} g(t) + \lambda\left(\int_x^t g(s)\,ds\right)^{\alpha-1} g(t)\right] f(t)\,dt \end{aligned}$$

$$\begin{aligned}
&= (1-\lambda)\left(\int\limits_a^x g(s)\,ds\right)^{\alpha} f(x) - \lambda\left(\int\limits_x^a g(s)\,ds\right)^{\alpha} f(a) \\
&\quad -\alpha(1-\lambda)\int\limits_a^x\left(\int\limits_a^t g(s)\,ds\right)^{\alpha-1} g(t)f(t)\,dt - \alpha\lambda\int\limits_a^x\left(\int\limits_x^t g(s)\,ds\right)^{\alpha-1} g(t)f(t)\,dt \\
&\quad +\lambda\left(\int\limits_x^b g(s)\,ds\right)^{\alpha} f(b) + (1-\lambda)\left(\int\limits_x^b g(s)\,ds\right)^{\alpha} f(x) \\
&\quad -\alpha(1-\lambda)\int\limits_x^b\left(\int\limits_t^b g(s)\,ds\right)^{\alpha-1} g(t)f(t)\,dt - \alpha\lambda\int\limits_x^b\left(\int\limits_x^t g(s)\,ds\right)^{\alpha-1} g(t)f(t)\,dt
\end{aligned}$$

If necessary arrangements are made, desired result are obtained. Hence, the proof is completed. □

Remark 1. *Under the same assumptions of Lemma 4 with $\alpha = 1$; then the following identity holds:*

$$\begin{aligned}
\int\limits_a^b P_\lambda(x,t) f'(t)\,dt &= (1-\lambda)\left(\int\limits_a^b g(s)\,ds\right) f(x) + \lambda\left(\int\limits_x^b g(s)\,ds\right) f(b) \\
&\quad +\lambda\left(\int\limits_a^x g(s)\,ds\right) f(a) - \int\limits_a^b g(t)f(t)\,dt \\
&= : S_\lambda(f,g;1)
\end{aligned}$$

which is proved by Erden and Sarikaya in [4].

Remark 2. *Under the same assumptions of Lemma 4 with $\lambda = 1$; then we have*

$$\begin{aligned}
\int\limits_a^b P_1(x,t) f'(t)\,dt &= \left[\left(\int\limits_x^b g(s)\,ds\right)^{\alpha} f(b) - \left(\int\limits_x^a g(s)\,ds\right)^{\alpha} f(a)\right] \\
&\quad -\alpha\int\limits_a^b\left(\int\limits_x^t g(s)\,ds\right)^{\alpha-1} g(t)f(t)\,dt \\
&= : S_1(f,g;\alpha)
\end{aligned}$$

which is proved by Sarikaya et. al in [14].

Remark 3. *Under the same assumptions of Lemma 4 with $\lambda = 0$; then we get*

$$\int\limits_a^b P_0(x,t) f'(t)\,dt = \left[\left(\int\limits_a^x g(s)\,ds\right)^{\alpha} + \left(\int\limits_x^b g(s)\,ds\right)^{\alpha}\right] f(x)$$

$$-\alpha\left[\int_a^x\left(\int_a^t g(s)\,ds\right)^{\alpha-1} g(t)\,f(t)\,dt+\int_x^b\left(\int_t^b g(s)\,ds\right)^{\alpha-1} g(t)\,f(t)\,dt\right]$$
$$=:S_0(f,g;\alpha)$$

which is proved by Sarikaya and Erden in [12].

Corollary 1. *Let $g:[a.b]\to\mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x=\frac{a+b}{2}$ in Lemma 4. Then we have the inequality*

$$\int_a^b P_0(x,t)\,f'(t)\,dt=\left(\frac{1}{2}\int_a^b g(s)\,ds\right)^{\alpha}\left[2(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\left[f(b)-(-1)^{\alpha}f(a)\right]\right]$$

$$-\alpha(1-\lambda)\int_a^{\frac{a+b}{2}}\left(\int_a^t g(s)\,ds\right)^{\alpha-1} g(t)\,f(t)\,dt$$
$$-\alpha(1-\lambda)\int_{\frac{a+b}{2}}^b\left(\int_t^b g(s)\,ds\right)^{\alpha-1} g(t)\,f(t)\,dt-\alpha\lambda\int_a^b\left(\int_{\frac{a+b}{2}}^t g(s)\,ds\right)^{\alpha-1} g(t)\,f(t)\,dt$$
$$=:S_\lambda\left(f\left(\frac{a+b}{2}\right),g;\alpha\right).$$

Definition 1. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^0f(x)=J_{b^-}^0f(x)=f(x)$.

Corollary 2. *If we take $g(s)=1$ in Lemma 4, then we obtain*

$$\int_a^b P_\lambda(x,t)\,f'(t)\,dt \tag{2.2}$$
$$=(1-\lambda)\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)+\lambda\left[(b-x)^{\alpha}f(b)-(a-x)^{\alpha}f(a)\right]$$
$$-(1-\lambda)\Gamma(\alpha+1)\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]-\alpha\lambda\int_a^b(t-x)^{\alpha-1}f(t)\,dt$$
$$=:S_\lambda(f,1;\alpha).$$

Theorem 6. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and let $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. If $|f'|$ is convex on $[a,b]$, then,*

for all $x\in[a,b]$, *the following inequalities hold:*

$$|S_{\lambda}(f,g;\alpha)| \tag{2.3}$$

$$\begin{aligned}
&\leq \frac{(x-a)^{\alpha+1}\|g\|_{[a,x],\infty}^{\alpha}}{b-a}\left[|f'(a)|\left(\frac{(b-x)}{\alpha+1}+\frac{(1+\alpha\lambda)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.\\
&\left.+|f'(b)|\frac{(\alpha+1-\alpha\lambda)(x-a)}{(\alpha+1)(\alpha+2)}\right]\\
&+\frac{(b-x)^{\alpha+1}\|g\|_{[x,b],\infty}^{\alpha}}{b-a}\left[+|f'(b)|\left(\frac{(x-a)}{\alpha+1}+\frac{(1+\alpha\lambda)(b-x)}{(\alpha+1)(\alpha+2)}\right)\right.\\
&\left.+|f'(a)|\frac{(\alpha+1-\alpha\lambda)(b-x)}{(\alpha+1)(\alpha+2)}\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|g\|_{[a,b],\infty}^{\alpha}}{b-a}\left\{|f'(a)|\left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1}+\frac{(1+\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\right.\\
&+|f'(a)|\frac{(\alpha+1-\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+|f'(b)|\frac{(\alpha+1-\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\\
&\left.+|f'(b)|\left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1}+\frac{(1+\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\right\}
\end{aligned}$$

where $\alpha>0,\ \lambda\in[0,1]$ *and* $\|g\|_{[a,b],\infty}=\sup_{s\in[a,b]}|g(s)|$.

Proof. We take absolute of (2.1). Using bounded of the mapping g and the convexity of $|f'|$, we find that

$$\begin{aligned}
&|S_{\lambda}(f,g;\alpha)| \leq \int_a^b |P_{\lambda}(x,t)|\,|f'(t)|\,dt\\
&\leq \int_a^x\left((1-\lambda)\left|\int_a^t g(s)\,ds\right|^{\alpha}+\lambda\left|\int_x^t g(s)\,ds\right|^{\alpha}\right)\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\\
&+\int_x^b\left((1-\lambda)\left|\int_t^b g(s)\,ds\right|^{\alpha}+\lambda\left|\int_x^t g(s)\,ds\right|^{\alpha}\right)\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\\
&\leq \frac{\|g\|_{[a,x],\infty}^{\alpha}}{b-a}\left[(1-\lambda)|f'(a)|\int_a^x(t-a)^{\alpha}(b-t)\,dt+(1-\lambda)|f'(b)|\int_a^x(t-a)^{\alpha+1}\,dt\right.\\
&\left.+\lambda|f'(a)|\int_a^x(x-t)^{\alpha}(b-t)\,dt+\lambda|f'(b)|\int_a^x(x-t)^{\alpha}(t-a)\,dt\right]
\end{aligned}$$

$$+\frac{\|g\|_{[x,b],\infty}}{b-a}\left[(1-\lambda)\left|f'(a)\right|\int_x^b (b-t)^{\alpha+1}dt+(1-\lambda)\left|f'(b)\right|\int_x^b (b-t)^{\alpha}(t-a)\,dt\right.$$
$$\left.+\lambda\left|f'(a)\right|\int_x^b (t-x)^{\alpha}(b-t)\,dt+\lambda\left|f'(b)\right|\int_x^b (t-x)^{\alpha}(t-a)\,dt\right].$$

If we calculate the above eight integrals, then we obtain

$$\begin{aligned}
&\left|S_\lambda(f,g;\alpha)\right|\\
\leq\ &\frac{(x-a)^{\alpha+1}\|g\|^{\alpha}_{[a,x],\infty}}{b-a}\left[\left|f'(a)\right|\left(\frac{(b-x)}{\alpha+1}+\frac{(1+\alpha\lambda)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.\\
&\left.+\left|f'(b)\right|\frac{(\alpha+1-\alpha\lambda)(x-a)}{(\alpha+1)(\alpha+2)}\right]\\
&+\frac{(b-x)^{\alpha+1}\|g\|^{\alpha}_{[x,b],\infty}}{b-a}\left[+\left|f'(b)\right|\left(\frac{(x-a)}{\alpha+1}+\frac{(1+\alpha\lambda)(b-x)}{(\alpha+1)(\alpha+2)}\right)\right.\\
&\left.+\left|f'(a)\right|\frac{(\alpha+1-\alpha\lambda)(b-x)}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}$$

Because of $\|g\|_{[a,x],\infty},\leq\|g\|_{[a,b],\infty}$ and $\|g\|_{[x,b],\infty}\leq\|g\|_{[a,b],\infty}$, we easily deduce required inequality (2.3) which completes the proof. □

Remark 4. *Under the same assumptions of Theorem 6 with $\alpha=1$; then the following identity holds:*

$$\begin{aligned}
&\left|S_\lambda(f,g;1)\right|\\
\leq\ &\frac{\|g\|_{[a,b],\infty}}{6(b-a)}\left\{\left|f'(a)\right|(x-a)^2\left[3(b-x)+(1+\lambda)(x-a)\right]\right.\\
&+\left|f'(a)\right|(2-\lambda)(b-x)^3+\left|f'(b)\right|(2-\lambda)(x-a)^3\\
&\left.+\left|f'(b)\right|(b-x)^2\left[3(x-a)+(1+\lambda)(b-x)\right]\right\}
\end{aligned}$$

which is proved by Erden and Sarikaya in [4].

Remark 5. *Under the same assumptions of Theorem 6 with $\lambda=1$; then we have*

$$\begin{aligned}
&\left|S_1(f,g;\alpha)\right|\\
\leq\ &\frac{\|g\|^{\alpha}_{[a,b],\infty}}{b-a}\left\{\left|f'(a)\right|\left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{(\alpha+2)}\right)\right.\\
&+\left|f'(a)\right|\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\left|f'(b)\right|\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\\
&\left.\left|f'(b)\right|\left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{(\alpha+2)}\right)\right\}
\end{aligned}$$

which is proved by Sarikaya et. al in [14].

Remark 6. *Under the same assumptions of Theorem 6 with $\lambda = 0$; then we get*

$$\begin{aligned} & |S_0(f,g;\alpha)| \\ \leq\; & \frac{\|g\|_{[a,b],\infty}^{\alpha}}{b-a}\left\{ |f'(a)| \left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \right. \\ & + |f'(a)| \frac{(b-x)^{\alpha+2}}{(\alpha+2)} + |f'(b)| \frac{(x-a)^{\alpha+2}}{(\alpha+2)} \\ & \left. + |f'(b)| \left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \right\} \end{aligned}$$

which is proved by Sarikaya and Erden in [12].

Corollary 3. *Let $g : [a.b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Theorem 6. Then we have the inequality*

$$\begin{aligned} & S_\lambda\left(f\left(\frac{a+b}{2} \right), g; \alpha \right) \\ \leq\; & \frac{\|g\|_{[a,b],\infty}^{\alpha} (b-a)^{\alpha+1}}{(\alpha+1)\, 2^{\alpha+1}} \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \tag{2.4}$$

Using Theorem 6, we obtain the following inequality involving fractional integrals.

Corollary 4. *If we take $g(s) = 1$ in Theorem 6, then we obtain*

$$\begin{aligned} & S_\lambda(f,1;\alpha) \\ \leq\; & \frac{1}{b-a}\left\{ |f'(a)| \left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(1+\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \right. \\ & + |f'(a)| \frac{(\alpha+1-\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + |f'(b)| \frac{(\alpha+1-\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \\ & \left. + |f'(b)| \left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(1+\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \right\}. \end{aligned}$$

Using Theorem 6, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 5. *If we take $\alpha = 1$ and $\lambda = 1$ in (2.4), then we have the inequality*

$$\begin{aligned} & \left| \left[\frac{f(a)+f(b)}{2} \right] \int_a^b g(t)dt - \int_a^b g(t) f(t)\, dt \right| \\ \leq\; & \frac{\|g\|_{[a,b],\infty} (b-a)^2}{8} \left[|f'(a)| + |f'(b)| \right] \end{aligned} \tag{2.5}$$

*which is "**weighted trapezoid**" inequality provided that $|f'|$ is convex on $[a,b]$.*

Remark 7. *In (2.4), let $\alpha = 1$ and $\lambda = \frac{1}{2}$. Then, we have the weighted Bullen-type inequality*

$$\left| \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \int_a^b g(t)dt - \int_a^b g(t)f(t)\,dt \right| \tag{2.6}$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)^2}{8}\left[|f'(a)| + |f'(b)|\right]$$

where $|f'|$ is convex on $[a,b]$.

Remark 8. *If we take $\alpha = 1$ and $\lambda = 0$ in (2.4), we get*

$$\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)\,dt - \int_a^b g(t)f(t)\,dt \right| \tag{2.7}$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)^2}{8}\left[|f'(a)| + |f'(b)|\right]$$

*which is "**weighted midpoint**" inequality provided that $|f'|$ is convex on $[a,b]$.*

Remark 9. *In (2.6), let $g(t) = 1$. Then, we have the Bullen-type inequality*

$$\left| \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)}{8}\left[|f'(a)| + |f'(b)|\right]$$

where $|f'|$ is convex on $[a,b]$.

Remark 10. *If we choose $g(t) = 1$ in (2.5), then the inequality (2.5) reduces to (1.3).*

Remark 11. *If we choose $g(t) = 1$ in (2.7), then the inequality (2.7) reduces to (1.4).*

Theorem 7. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and let $f' \in L[a,b]$, $a,b \in I^\circ$ with $a < b$, and let $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then for all $x \in [a,b]$, the following inequality holds:*

$$|S_\lambda(f,g;\alpha)| \tag{2.8}$$

$$\leq \frac{\|g\|^\alpha_{[a,b],\infty}}{(\alpha+1)\left[(\alpha+2)(b-a)\right]^{\frac{1}{q}}}\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}\right]^{\frac{1}{p}}$$

$$\times \Big\{ |f'(a)|^q (\alpha+2)(b-x)(x-a)^{\alpha+1} + |f'(a)|^q (1+\alpha\lambda)(x-a)^{\alpha+2}$$

$$+ |f'(a)|^q (\alpha+1-\alpha\lambda)(b-x)^{\alpha+2} + |f'(b)|^q (1+\alpha\lambda)(b-x)^{\alpha+2}$$

$$+ |f'(b)|^q (\alpha+2)(x-a)(b-x)^{\alpha+1} + |f'(b)|^q (\alpha+1-\alpha\lambda)(x-a)^{\alpha+2} \Big\}^{\frac{1}{q}}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|g\|_\infty = \sup_{s\in[a,b]} |g(s)|$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} + \frac{1}{q}$ can be written instead of 1. Using Hölder's inequality, we find that

$$|S_\lambda(f,g;\alpha)| \leq \int_a^b |P_\lambda(x,t)|\,|f'(t)|\,dt \leq \left(\int_a^b |P_\lambda(x,t)|\,dt\right)^{\frac{1}{p}} \left(\int_a^b |P_\lambda(x,t)|\,|f'(t)|^q\,dt\right)^{\frac{1}{q}} \tag{2.9}$$

Using bounded of the mappings g, we calculate respectively above integrals that is in multiplication:

$$\begin{aligned} &\int_a^b |P_\lambda(x,t)|\,dt \\ \leq\ & \|g\|^\alpha_{[a,x],\infty} \int_a^x \left[(1-\lambda)(t-a)^\alpha + \lambda(x-t)^\alpha\right] dt \\ &+ \|g\|^\alpha_{[x,b],\infty} \int_x^b \left[(1-\lambda)(b-t)^\alpha + \lambda(t-x)^\alpha\right] dt \\ \leq\ & \frac{\|g\|^\alpha_{[a,b],\infty}}{\alpha+1}\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}\right] \end{aligned} \tag{2.10}$$

Using the convexity of $\left|f^{'}(t)\right|^q$ instead of that of $\left|f^{'}(t)\right|$, if the second integral is calculated as in Theorem 6, then we get

$$\begin{aligned} &\int_a^b |P_\lambda(x,t)|\,|f'(t)|^q\,dt \\ \leq\ & \frac{\|g\|^\alpha_{[a,b],\infty}}{b-a}\Bigg\{|f'(a)|^q\left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(1+\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right) \\ &+ |f'(a)|^q \frac{(\alpha+1-\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + |f'(b)|^q \frac{(\alpha+1-\alpha\lambda)(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \\ &|f'(b)|^q\left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(1+\alpha\lambda)(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\Bigg\}. \end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) in (2.9), we obtain the inequality (2.8). Hence, the proof is completed. □

Corollary 6. *Under the same assumptions of Theorem 7 with $\alpha = 1$; then the following identity holds:*

$$\begin{aligned}
&\left|S_{\lambda}\left(f,g;1\right)\right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}}{2\left[3\left(b-a\right)\right]^{\frac{1}{q}}}\left[(x-a)^{2}+(b-x)^{2}\right]^{\frac{1}{p}} \\
&\times\left\{\left|f'\left(a\right)\right|^{q}3(b-x)(x-a)^{2}+\left|f'\left(a\right)\right|^{q}\left(1+\lambda\right)\left(x-a\right)^{3}\right. \\
&+\left|f'\left(a\right)\right|^{q}\left(2-\lambda\right)\left(b-x\right)^{3}+\left|f'\left(b\right)\right|^{q}\left(1+\lambda\right)\left(b-x\right)^{3} \\
&\left.+\left|f'\left(b\right)\right|^{q}3(x-a)(b-x)^{2}+\left|f'\left(b\right)\right|^{q}\left(2-\lambda\right)\left(x-a\right)^{3}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Remark 12. *Under the same assumptions of Theorem 7 with $\lambda = 1$; then we have*

$$\begin{aligned}
&\left|S_{1}\left(f,g;\alpha\right)\right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}^{\alpha}}{\left(\alpha+1\right)\left[\left(\alpha+2\right)\left(b-a\right)\right]^{\frac{1}{q}}}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left|f'\left(a\right)\right|^{q}\left(\left(\alpha+1\right)\left(b-a\right)(x-a)^{\alpha+1}+\left(b-x\right)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\right. \\
&\left.+\left|f'\left(b\right)\right|^{q}\left(\left(\alpha+1\right)\left(b-a\right)(b-x)^{\alpha+1}+\left(x-a\right)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\right\}^{\frac{1}{q}}
\end{aligned}$$

which is proved by Sarikaya et. al in [14].

Remark 13. *Under the same assumptions of Theorem 7 with $\lambda = 0$; then we get*

$$\begin{aligned}
&\left|S_{\lambda}\left(f,g;\alpha\right)\right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}^{\alpha}}{\left(\alpha+1\right)\left[\left(\alpha+2\right)\left(b-a\right)\right]^{\frac{1}{q}}}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left|f'\left(a\right)\right|^{q}\left((b-a)(x-a)^{\alpha+1}+\left(\alpha+1\right)\left(b-x\right)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\right. \\
&\left.+\left|f'\left(b\right)\right|^{q}\left((b-a)(b-x)^{\alpha+1}+\left(\alpha+1\right)\left(x-a\right)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\right\}^{\frac{1}{q}}
\end{aligned}$$

Remark 14. *Let $g : [a.b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Theorem 7. Then we have the inequality*

$$\begin{aligned}
(2.12)\qquad & S_{\lambda}\left(f\left(\frac{a+b}{2}\right),g;\alpha\right) \\
\leq\ & \frac{\|g\|_{[a,b],\infty}^{\alpha}\left(b-a\right)^{\alpha+1}}{\left(\alpha+1\right)2^{\alpha}}\left[\frac{\left|f'\left(a\right)\right|^{q}+\left|f'\left(b\right)\right|^{q}}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Using Theorem 7, we obtain the following inequality involving fractional integrals.

Corollary 7. *If we take $g(s)=1$ in Theorem 7, then we obtain*

$$\begin{aligned}
&\left|S_{\lambda}(f,1;\alpha)\right| \\
\leq\ & \frac{\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]^{\frac{1}{p}}}{(\alpha+1)\left[(\alpha+2)(b-a)\right]^{\frac{1}{q}}} \\
&\times\Big\{\left|f'(a)\right|^{q}(\alpha+2)(b-x)(x-a)^{\alpha+1}+\left|f'(a)\right|^{q}(1+\alpha\lambda)(x-a)^{\alpha+2} \\
&+\left|f'(a)\right|^{q}(\alpha+1-\alpha\lambda)(b-x)^{\alpha+2}+\left|f'(b)\right|^{q}(1+\alpha\lambda)(b-x)^{\alpha+2} \\
&+\left|f'(b)\right|^{q}(\alpha+2)(x-a)(b-x)^{\alpha+1}+\left|f'(b)\right|^{q}(\alpha+1-\alpha\lambda)(x-a)^{\alpha+2}\Big\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 8. *Under the same assumptions of Theorem 7 with $\alpha=1$ and $\lambda=0$; then the following inequality holds:*

$$\begin{aligned}
&\left|f(x)\int_a^b g(s)ds-\int_a^b g(s)f(s)\,ds\right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}^{\alpha}}{2\left[3(b-a)\right]^{\frac{1}{q}}}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}} \\
&\times\Big\{\left|f'(a)\right|^{q}\Big((b-a)(x-a)^{2}+2(b-x)\Big[(x-a)^{2}+(b-x)^{2}\Big]\Big) \\
&+\left|f'(b)\right|^{q}\Big((b-a)(b-x)^{2}+(x-a)\Big[(x-a)^{2}+(b-x)^{2}\Big]\Big)\Big\}^{\frac{1}{q}}
\end{aligned}$$

*which is "**weighted Ostrowski**" inequality provided that $|f'|^{q}$ is convex on $[a,b]$.*

Using Theorem 7, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 9. *If we take $\alpha=1$ and $\lambda=1$ in (2.12), then we have the inequality*

$$\begin{aligned}
(2.13)\qquad &\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt-\int_a^b g(t)f(t)\,dt\right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}(b-a)^{2}}{8}\left[\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

*which is "**weighted trapezoid**" inequality provided that $|f'|^{q}$ is convex on $[a,b]$ and $f'\in L(a,b)$ where $q>1$.*

Remark 15. *In (2.12), let $\alpha = 1$ and $\lambda = \frac{1}{2}$. Then, we have the weighted Bullen-type inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt-\int_a^b g(t)f(t)\,dt\right| \tag{2.14}$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)^2}{8}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}$$

where $|f'|^q$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $q > 1$.

Remark 16. *If we take $\alpha = 1$ and $\lambda = 0$ in (2.12), we get*

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds-\int_a^b g(t)f(t)\,dt\right|$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)^2}{8}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}$$

*which is "**weighted midpoint**" inequality provided that $|f'|^q$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $q > 1$.*

Corollary 10. *In (2.14), let $g(s) = 1$. Then, we have the Bullen-type inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{\|g\|_{[a,b],\infty}(b-a)}{8}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}$$

where $|f'|^q$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $q > 1$.

In Theorem 7, we acquired an inequality by using Hölder's inequality and Lemma 1. In following theorem, we will use again Hölder's inequality and Lemma 1, but we will obtain a new inequality whose right side is independent of λ by calculating in a different way.

Theorem 8. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and let $f' \in L[a,b]$, $a, b \in I^\circ$ with $a < b$, and let $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then for all $x \in [a,b]$, the following inequality holds:*

$$|S_\lambda(f,g;\alpha)| \leq \frac{\|g\|^\alpha_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \tag{2.15}$$

$$\times\left\{(x-a)^{\alpha+1}\left[\frac{(2b-a-x)}{2}|f'(a)|^q+\frac{(x-a)}{2}|f'(b)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\alpha+1}\left[\frac{(b-x)}{2}|f'(a)|^q+\frac{(b+x-2a)}{2}|f'(b)|^q\right]^{\frac{1}{q}}\right\}$$

where $\alpha > 0$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\|g\|_\infty = \sup_{s\in[a,b]}|g(s)|$.

Proof. We take absolute value of (2.1). Using bounded of the mappings g, we find that

$$
\begin{aligned}
(2.16)\qquad & \left|S_{\lambda}\left(f,g;\alpha\right)\right| \\
\leq\ & \int_{a}^{b}\left|P_{\lambda}\left(x,t\right)\right|\left|f'\left(t\right)\right|dt \\
\leq\ & \|g\|_{[a,x],\infty}^{\alpha}\int_{a}^{x}\left[\left(1-\lambda\right)\left(t-a\right)^{\alpha}+\lambda(x-t)^{\alpha}\right]\left|f'\left(t\right)\right|dt \\
& +\|g\|_{[x,b],\infty}^{\alpha}\int_{x}^{b}\left[\left(1-\lambda\right)\left(b-t\right)^{\alpha}+\lambda(t-x)^{\alpha}\right]\left|f'\left(t\right)\right|dt \\
=\ & \|g\|_{[a,x],\infty}^{\alpha}\left[\left(1-\lambda\right)\int_{a}^{x}\left(t-a\right)^{\alpha}\left|f'\left(t\right)\right|dt+\lambda\int_{a}^{x}\left(x-t\right)^{\alpha}\left|f'\left(t\right)\right|dt\right] \\
& +\|g\|_{[x,b],\infty}^{\alpha}\left[\left(1-\lambda\right)\int_{x}^{b}\left(b-t\right)^{\alpha}\left|f'\left(t\right)\right|dt+\lambda\int_{x}^{b}\left(t-x\right)^{\alpha}\left|f'\left(t\right)\right|dt\right] \\
=\ & \|g\|_{[a,x],\infty}^{\alpha}\left[\left(1-\lambda\right)I_{1}+\lambda I_{2}\right]+\|g\|_{[x,b],\infty}^{\alpha}\left[\left(1-\lambda\right)I_{3}+\lambda I_{4}\right].
\end{aligned}
$$

Firstly, we calculate integral I_1, using Holder's inequality amd the convexity of $|f'|^q$, we find that

$$
\begin{aligned}
(2.17)\qquad & \int_{a}^{x}\left(t-a\right)^{\alpha}\left|f'\left(t\right)\right|dt \\
\leq\ & \left(\int_{a}^{x}\left(t-a\right)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{a}^{x}\left|f'\left(t\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
\leq\ & \left(\frac{\left(x-a\right)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_{a}^{x}\left[\frac{b-t}{b-a}\left|f'\left(a\right)\right|^{q}+\frac{t-a}{b-a}\left|f'\left(b\right)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{\left(x-a\right)^{\alpha+1}}{\left(\alpha p+1\right)^{\frac{1}{p}}\left(b-a\right)^{\frac{1}{q}}}\left[\frac{\left(2b-a-x\right)}{2}\left|f'\left(a\right)\right|^{q}+\frac{\left(x-a\right)}{2}\left|f'\left(b\right)\right|^{q}\right]^{\frac{1}{q}}
\end{aligned}
$$

If we calculate the other integrals as being calculated in (2.17) and substitute integrals I_1, I_2, I_3 and I_4 in (2.16), then we obtain

$$
\begin{aligned}
& \left|S_{\lambda}\left(f,g;\alpha\right)\right| \\
\leq\ & \frac{\|g\|_{[a,x],\infty}^{\alpha}\left(x-a\right)^{\alpha+1}}{\left(\alpha p+1\right)^{\frac{1}{p}}\left(b-a\right)^{\frac{1}{q}}}\left[\frac{\left(2b-a-x\right)}{2}\left|f'\left(a\right)\right|^{q}+\frac{\left(x-a\right)}{2}\left|f'\left(b\right)\right|^{q}\right]^{\frac{1}{q}} \\
& +\frac{\|g\|_{[x,b],\infty}^{\alpha}\left(b-x\right)^{\alpha+1}}{\left(\alpha p+1\right)^{\frac{1}{p}}\left(b-a\right)^{\frac{1}{q}}}\left[\frac{\left(b-x\right)}{2}\left|f'\left(a\right)\right|^{q}+\frac{\left(b+x-2a\right)}{2}\left|f'\left(b\right)\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Because of $\|g\|_{[a,x],\infty} \leq \|g\|_{[a,b],\infty}$ and $\|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty}$, we easily deduce required inequality (2.15) which completes the proof. □

Corollary 11. *Under the same assumptions of Theorem 8 with $\alpha = 1$; then the following identity holds:*

$$\begin{aligned} |S_\lambda(f,g;1)| &\leq \frac{\|g\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \\ &\times \left\{ (x-a)^2 \left[\frac{(2b-a-x)}{2} |f'(a)|^q + \frac{(x-a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ &\left. + (b-x)^2 \left[\frac{(b-x)}{2} |f'(a)|^q + \frac{(b+x-2a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 17. *Under the same assumptions of Theorem 8 with $\lambda = 1$; then we have*

$$\begin{aligned} |S_1(f,g;\alpha)| &\leq \frac{\|g\|^\alpha_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \\ &\times \left\{ (x-a)^{\alpha+1} \left[\frac{(2b-a-x)}{2} |f'(a)|^q + \frac{(x-a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ &\left. + (b-x)^{\alpha+1} \left[\frac{(b-x)}{2} |f'(a)|^q + \frac{(b+x-2a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

which is proved by Sarikaya et. al in [14].

Remark 18. *Under the same assumptions of Theorem 8 with $\lambda = 0$; then we get*

$$\begin{aligned} |S_\lambda(f,g;\alpha)| &\leq \frac{\|g\|^\alpha_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \\ &\times \left\{ (x-a)^{\alpha+1} \left[\frac{(2b-a-x)}{2} |f'(a)|^q + \frac{(x-a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ &\left. + (b-x)^{\alpha+1} \left[\frac{(b-x)}{2} |f'(a)|^q + \frac{(b+x-2a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

which is proved by Sarikaya and Erden in [12].

Remark 19. *Let $g : [a.b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Theorem 8. Then we have the inequality*

$$\begin{aligned} (2.18)\ & S_\lambda\left(f\left(\frac{a+b}{2}\right), g; \alpha\right) \\ \leq\ & \frac{\|g\|^\alpha_{[a,b],\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}2^{\alpha+1}} \left\{ \left[\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Using Theorem 8, we obtain the following inequality involving fractional integrals.

If we take $g(s) = 1$ in Theorem 8, then we obtain

$$
\begin{aligned}
|S_{\lambda}(f,1;\alpha)| &\leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \\
&\times \left\{ (x-a)^{\alpha+1}\left[\frac{(2b-a-x)}{2}\left|f'(a)\right|^{q}+\frac{(x-a)}{2}\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \right. \\
&\left. +(b-x)^{\alpha+1}\left[\frac{(b-x)}{2}\left|f'(a)\right|^{q}+\frac{(b+x-2a)}{2}\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \right\}.
\end{aligned}
$$

Using Theorem 8, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 12. *If we take $\alpha = 1$ and $\lambda = 1$ in (2.18), then we have the inequality*

$$
\begin{aligned}
&\left| \left[\frac{f(a)+f(b)}{2}\right]\int_{a}^{b} g(t)dt - \int_{a}^{b} g(t)f(t)\,dt \right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}(b-a)^{2}}{4(p+1)^{\frac{1}{p}}}\left\{ \left[\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} + \left[\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} \right\}
\end{aligned}
$$

*which is "**weighted trapezoid**" inequality provided that $|f'|^{q}$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $p > 1$.*

Remark 20. *In (2.18), let $\alpha = 1$ and $\lambda = \frac{1}{2}$. Then, we have the weighted Bullen-type inequality*

$$
\begin{aligned}
&\left| \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_{a}^{b} g(t)dt - \int_{a}^{b} g(t)f(t)\,dt \right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}(b-a)^{2}}{4(p+1)^{\frac{1}{p}}}\left\{ \left[\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} + \left[\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} \right\}
\end{aligned} \tag{2.19}
$$

where $|f'|^{q}$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $p > 1$.

Remark 21. *If we take $\alpha = 1$ and $\lambda = 0$ in (2.18), we get*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_{a}^{b} g(t)\,dt - \int_{a}^{b} g(t)f(t)\,dt \right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}(b-a)^{2}}{4(p+1)^{\frac{1}{p}}}\left\{ \left[\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} + \left[\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right]^{\frac{1}{q}} \right\}
\end{aligned} \tag{2.20}
$$

*which is "**weighted midpoint**" inequality provided that $|f'|^{q}$ is convex on $[a,b]$ and $f' \in L(a,b)$ where $p > 1$.*

Corollary 13. *In (2.19), let $g(t)=1$. Then, we have the Bullen-type inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left\{\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

where $|f'|^q$ is convex on $[a,b]$ and $f'\in L(a,b)$ where $p>1$.

Remark 22. *If we choose $g(t)=1$ in (2.20), then the inequality (2.20) reduces to (1.5).*

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