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INEQUALITIES FOR SELFADJOINT OPERATORS ON HILBERT SPACES AND PSEUDO-HILBERT SPACES

LOREDANA CIURDARIU

Abstract. In this paper are given some inequalities for selfadjoint operators on Hilbert spaces and pseudo-Hilbert spaces starting from an inequality of Kober and from a refinement of the Kittaneh-Manasrah inequality by combining inequalities for power series and inner product.

1. **Introduction**

The famous inequality of Young

$$\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}$$

where a, b are positive real numbers and $\lambda \in [0,1]$ has many improvements and refinements.

We need below the folowing one, the improvement of the inequality between arithmetic and geometric means for $n = 2$ given by H. Kober in [9]:

$$r(\sqrt{a}-\sqrt{b})^2 \leq \lambda a + (1-\lambda)b - a^{\lambda} b^{1-\lambda} \leq (1-r)\sqrt{a}-\sqrt{b})^2 \tag{1.0}$$

where a, b are positive real numbers, $\lambda \in [0,1]$ and $r = \min\{\lambda, 1-\lambda\}$.

We use also below the following result in the next section.

Proposition([10]) For $0 < a, b \leq 1$ and $\lambda \in (0,1)$ we have

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

We will take here $\lambda = \frac{1}{p}$ and replace a^{λ} by a and $b^{1-\lambda}$ by b then $1-\lambda = \frac{1}{q}$ and we obtain:

$$ab + r(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 + A(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right) \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\leq ab + (1-r)(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 + B(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right). \tag{1.1}$$

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We consider here an analytic function defined by the power series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

with real coefficients and convergent on the disk $D(0,R),\ R>0$. As in [8] the weighted version of Hölder's inequality can be stated as below:

$$|f(xy)|=\left|\sum_{n=0}^{\infty}a_nx^ny^n\right|\leq\left(\sum_{n=0}^{\infty}|a_n||x|^{pn}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}|a_n||x|^{qn}\right)^{\frac{1}{q}}$$

$$=f_A^{\frac{1}{p}}(|x|^p)f_A^{\frac{1}{q}}(|y|^q)$$

for any $x,y\in\mathbf{C}$ with $xy,\ |x|^p,|y|^q\in D(0,R)$ and $f_A(z)$ is a power series defined by $\sum_{n=0}^{\infty}|a_n|z^n$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

In the case when all coefficients of the series $f(z)$ are positive we have $f(z)=f_A(z)$.

First, it is necessary to recall that for selfadjoint operators $A,B\in B(H)$ we write $A\leq B$ (or $B\geq A$) if $<Ax,x>\leq<Bx,x>$ for every vector $x\in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(H;<.,.>)$ as in [7] and the references therein. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any $f,f\in C(Sp(A))$ and for any $\alpha,\beta\in\mathbf{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f)=\Phi(f^*)$;

(iii) $||\Phi(f)||=||f||:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$ for $t\in Sp(A.)$

Using this notation, as in [7] for example, we define

$$f(A):=\Phi(f)\quad\text{for all}\quad f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t)\geq g(t)\quad\text{for any}\quad t\in Sp(A)\quad\text{implies that}\quad f(A)\geq g(A)\tag{1.2}$$

in the operator order of $B(H)$.

We recall the definitions of pseudo-Hilbert spaces (so called Loynes Z- spaces) and of the admissible spaces in the Loynes sense and then as in [2] we use the functional calculus with functions of the class C_1 in results which will be proved below.

A locally convex space Z- is called admissible in the Loynes sense if the following conditions are satisfied:

(A.1) Z- is complete;

(A.2) there is a closed convex cone in Z, denoted Z_+, defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);

(A.3) there is an involution in Z, $z \to z^*$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$) such that $z \in Z_+$ implies $z^* = z$;

(A.4) the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);

(A.6) any monotonously decreasing sequence in Z_+ is convergent.

Let Z be an admissible space in the Loynes sense. A topological linear space $\mathcal{H}$ is called pre-Loynes Z-space if it satisfies the following properties:

(L1) $\mathcal{H}$ is endowed with an Z- valued inner product (gramian), i.e. there exists an application $(h,k) \in \mathcal{H} \times \mathcal{H} \to [h,k] \in Z$ having the properties:

(G.1) $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$;

(G.2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G.3) $[\lambda h, k] = \lambda [h,k]$;

(G.4) $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbf{C}$.

(L.2) The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $h \in \mathcal{H} \to [h,h] \in Z$ is continous.

Moreover, if $\mathcal{H}$ is a complete spaces with this topology, then $\mathcal{H}$ is called Loynes Z- space (pseudo-Hilbert space).

Let $A \in \mathcal{B}_h^*(\mathcal{H})$, where $\mathcal{H}$ is now a pseudo-Hilbert space (so called Loynes Z-spaces) and

$$m_A := \sup\{\mu : \mu[h,h] \leq [Ah,h], h \in \mathcal{H}\},$$
$$M_A := \inf\{\nu : [Ah,h] \leq \nu[h,h], h \in \mathcal{H}\}.$$

We say that the function f is in the class C_1 and we denote $f \in C_1[m_A, M_A]$ if f is positive and superior semicontinuous on $[m_A, M_A]$.

We will denote by $\overline{\mathcal{A}(A)^t}$ the strong closing of $\mathcal{A}(A)$ in $\mathcal{B}^*(\mathcal{H})$.

The mapping

$$f : C_1[m_A, M_A] \to \overline{\mathcal{A}(A)^t}, \quad f \to f(A)$$

by which to a function $f \in C_1[m_A, M_A]$ we associate the gramian self-adjoint operator denoted by $f(A)$ and defined by $f(A) = \lim_{n\to\infty} p_n(A)$ where p_n is a decreasing sequence of polynomials p_n with $f(\lambda) = \lim_{n\to\infty} p_n(\lambda)$ for any $\lambda \in [m_A, M_A]$, is called functionals calculus with functions in class C_1.

Theorem 1. *(Lemma 2.1.1*[2]*) The functional calculus with functions of the class C_1 has the following immediate properties:*

(i) the mapping $f \to f(A)$ is monotone;

(ii) $f \to f(A)$ is function of positive type and positively homogeneous;

(iii) $f \to f(A)$ is additive and multiplicative(all these three properties being inherited by passing to the limit from the functional calculus with polynomials defined at the beginning);

(iv) In addition, the functional calculus with functions of the class C_1 extends the functional calculus with continuous and positive functions on $\sigma(A)$ defined in Corollary 1.5.6,

$$f : C_+(\sigma(A)) \to \mathcal{A}(A), \quad f \to f(A)$$

if $A \in \mathcal{B}_h^*(\mathcal{H})$.

Using the definition from [7], we say that the functions $f, g : [a, b] \to \mathbf{R}$ are *syncronous* (*asyncronous*) on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for each $t, s \in [a, b]$.

Some of inequalities from this paper are continuation of some inequalities from previous papers as [3], [4], [5] and [6].

2. Some inequalities for selfadjoint operators on Hilbert spaces and on pseudo-Hilbert spaces

Some refinements of Young inequalities for power series and inner product and positive operators on Hilbert spaces and pseudo-Hilbert spaces will be presented in the following theorems.

First result was established starting from the inequality (1.0) and following the same reason as in [8] and [7].

Theorem 2. *Let* $f(z) = \sum_{n=0}^{\infty} p_n z^n$, $g(z) = \sum_{n=0}^{\infty} q_n z^n$ *be the power series with real coefficients and convergent on the open disk* $D(0, R)$, $0 < R < 1$ *(or* $R > 0$*). If* p, q *are real numbers with* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, A_1 *is a positive definite operator on the Hilbert space* $\mathcal{H}$, $A_1 \in B(H)$ *with* $Sp(A_1) \subseteq (0, R)$ *and* B *is a positive operator on the pseudo-Hilbert space* $\mathcal{K}$, $B \in \mathcal{B}_h^*(\mathcal{K})$ *with* $Sp(B) \subseteq (0, R)$, *then we have*

$$< f_A\left(A_1^{\frac{2}{p}+1}\right)x, x > [g_A\left(B^{\frac{2}{q}+1}\right)y, y] + r\{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y] +$$

$$+ < f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y] - 2 < f_A\left(A_1^{\frac{q}{2}+1}\right)x, x > [g_A\left(B^{\frac{p}{2}+1}\right)y, y]\} \leq$$

$$\leq \frac{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y]}{p} + \frac{< f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y]}{q} \leq$$

$$\leq < f_A\left(A_1^{\frac{2}{p}+1}\right)x, x > [g_A\left(B^{\frac{2}{q}+1}\right)y, y] + (1-r)\{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y] +$$

$$+ < f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y] - 2 < f_A\left(A_1^{\frac{q}{2}+1}\right)x, x > [g_A\left(B^{\frac{p}{2}+1}\right)y, y]\},$$

for each $x \in H$ *and* $y \in \mathcal{K}$, *where* $\lambda \in [0, 1]$, $r = \min\{\lambda, 1 - \lambda\}$.

Proof. As in [8], we take in inequality (1.1) $|a|^{k\frac{2}{p}}|b|^j$ instead of a and $|a|^k|b|^{j\frac{2}{q}}$ instead of b where $k, j \in \mathbf{N}$, we do not take into account the terms $A(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right)$ and $B(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right)$ (that is a form of inequality (1.0)) and we obtain:

$$|a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} + r\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) \leq$$

$$\leq \frac{|a|^{2k}|b|^{jp}}{p} + \frac{|a|^{kq}|b|^{2j}}{q} \leq$$

$$\leq |a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} + (1-r)\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right).$$

If we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over j and k from 0 to n, we have

$$\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)}+r\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp}+|a|^{qk}|b|^{2j}-2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right)\le$$

$$\le\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\frac{|a|^{2k}|b|^{jp}}{p}+\sum_{k=0}^{n}|p_j||q_k|\sum_{j=0}^{n}\frac{|a|^{kq}|b|^{2j}}{q}\le\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)}+$$

$$+(1-r)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp}+|a|^{qk}|b|^{2j}-2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right).$$

All the series whose partial sums which appear in previous inequality are convergent on the disk $D(0,R)$ therefore we can take the limit when n tends to ∞ before and obtain the inequaility

$$f_A(|a|^{\frac{2}{p}+1})g_A(|b|^{\frac{2}{q}+1})+r\{f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{p}{2}+1})\}\le$$

$$\le\frac{f_A(|a|^2)g_A(|b|^p)}{p}+\frac{f_A(|a|^q)g_A(|b|^2)}{q}\le$$

$$\le f_A(|a|^{\frac{2}{p}+1})g_A(|b|^{\frac{2}{q}+1})+(1-r)\{f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{p}{2}+1})\}.$$

Now we fix $|a|$ and apply poperty (1.2) obtaining for any $x\in\mathcal{H}$,

$$< f_A(|A_1|^{\frac{2}{p}+1})x,x> g_A(|b|^{\frac{2}{q}+1})+r\{< f_A(|A_1|^2)x,x> g_A(|b|^p)+< f_A(|A_1|^q)x,x> g_A(|b|^2)-$$

$$-2< f_A(|A_1|^{\frac{q}{2}+1})x,x> g_A(|b|^{\frac{p}{2}+1})\}\le$$

$$\le\frac{< f_A(|A_1|^2)x,x> g_A(|b|^p)}{p}+\frac{< f_A(|A_1|^q)x,x> g_A(|b|^2)}{q}\le$$

$$\le< f_A(|A_1|^{\frac{2}{p}+1})x,x> g_A(|b|^{\frac{2}{q}+1})+(1-r)\{< f_A(|A_1|^2)x,x> g_A(|b|^p)+$$

$$+< f_A(|A_1|^q)x,x> g_A(|b|^2)-2< f_A(|A_1|^{\frac{q}{2}+1})x,x> g_A(|b|^{\frac{p}{2}+1})\}.$$

Applying now for $|b|$ fixed the Theorem 1, for any $y\in\mathcal{K}$ we get the desired inequality.

∎

The second result was given starting from inequality (1.1), see [**?**] and following the same reason as in [8] and [7].

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}p_nz^n$, $g(z)=\sum_{n=0}^{\infty}q_nz^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0<R<1$. If p,q are real numbers with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and A_1 is a positive definite operator on the Hilbert space $\mathcal{H}$, $A_1\in B(H)$ with $Sp(A_1)\subseteq(0,R)$ and B is a positive operator on the pseudo-Hilbert space $\mathcal{K}$, $B\in\mathcal{B}_h^*(\mathcal{K})$ with $Sp(B)\subseteq(0,R)$ then we have*

$$< f_A\left(A_1^{\frac{2}{p}+1}\right)x,x>[g_A\left(B^{\frac{2}{q}+1}\right)y,y]+r\{< f_A\left(A_1^2\right)x,x>[g_A\left(B^p\right)y,y]+$$

$$+< f_A\left(A_1^q\right)x,x>[g_A\left(B^2\right)y,y]-2< f_A\left(A_1^{\frac{q}{2}+1}\right)x,x>[g_A\left(B^{\frac{p}{2}+1}\right)y,y]\}+A_2\left(\frac{1}{p}\right)M\le$$

$$\le\frac{< f_A\left(A_1^2\right)x,x>[g_A\left(B^p\right)y,y]}{p}+\frac{< f_A\left(A_1^q\right)x,x>[g_A\left(B^2\right)y,y]}{q}\le$$

$$\leq < f_A\left(A_1^{\frac{2}{p}+1}\right)x, x > [g_A\left(B^{\frac{2}{q}+1}\right)y, y] + (1-r)\{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y] +$$

$$+ < f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y] - 2 < f_A\left(A_1^{\frac{q}{2}+1}\right)x, x > [g_A\left(B^{\frac{p}{2}+1}\right)y, y]\} + B_2\left(\frac{1}{p}\right)M,$$

where

$$M = (2-q)^2 < S_1\left|A_1^{2+q}\right)\log^2(A_1)x, x > [g\left(B^{p+2}\right)y, y] +$$
$$+(p-2)^2 < f|A_1^{2+q})x, x > [S_2\left(B^{p+2}\right)\log^2(B)y, y] +$$
$$+2(2-q)(p-2) < S_3\left(A_1^{2+q}\right)\log(A_1)x, x > [S_4\left(B^{p+2}\right)\log(B)y, y],$$

and $S_1(a) = af_A'(a) + a^2 f_A''(a)$, $S_2(a) = ag_A'(a) + a^2 g_A''(a)$, $S_3(a) = af_A'(a)$, $S_4(a) = ag_A'(a)$ *for each* $x \in H$ *and* $y \in \mathcal{K}$, *where* $\lambda \in [0,1]$, $r = \min\{\lambda, 1-\lambda\}$.

Proof. Using the same reason as in previous theorem, we take in inequality (1.1) $|a|^{k\frac{2}{p}}|b|^j$ instead of a and $|a|^k|b|^{j\frac{2}{q}}$ instead of b where $k, j \in \mathbf{N}$ and we obtain:

$$|a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} + r\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) +$$
$$+A_2\left(\frac{1}{p}\right)|a|^{2k}|b|^{pj}|a|^{kq}|b|^{2j}\log^2\left(\frac{|a|^{2k}|b|^{pj}}{|a|^{kq}|b|^{2j}}\right) \leq \frac{|a|^{2k}|b|^{jp}}{p} + \frac{|a|^{kq}|b|^{2j}}{q} \leq$$
$$\leq |a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} + (1-r)\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) +$$
$$+B_2\left(\frac{1}{p}\right)|a|^{2k}|b|^{pj}|a|^{kq}|b|^{2j}\log^2\left(\frac{|a|^{2k}|b|^{pj}}{|a|^{kq}|b|^{2j}}\right).$$

If we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over j and k from 0 to n, we have

$$\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} + r\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) +$$
$$+A_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(q+2)}|b|^{j(p+2)}\log^2\left(|a|^{k(2-q)}|b|^{j(p-2)}\right) \leq$$
$$\leq \sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\frac{|a|^{2k}|b|^{jp}}{p} + \sum_{k=0}^{n}|p_j||q_k|\sum_{j=0}^{n}\frac{|a|^{kq}|b|^{2j}}{q} \leq \sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} +$$
$$+(1-r)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) +$$
$$+B_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(q+2)}|b|^{j(p+2)}\log^2\left(|a|^{k(2-q)}|b|^{j(p-2)}\right),$$

or by calculus,

$$\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)} +$$
$$+r\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp} + |a|^{qk}|b|^{2j} - 2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right) +$$

$$+A_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(q+2)}|b|^{j(p+2)}(k^2(2-q)^2\log^2(|a|)+j^2(p-2)^2\log^2(|b|)+$$

$$+2kj(2-q)(p-2)\log(|a|))\leq\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\frac{|a|^{2k}|b|^{jp}}{p}+\sum_{k=0}^{n}|p_j||q_k|\sum_{j=0}^{n}\frac{|a|^{kq}|b|^{2j}}{q}\leq$$

$$\leq\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(\frac{2}{p}+1)}|b|^{j(\frac{2}{q}+1)}+$$

$$+(1-r)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2k}|b|^{jp}+|a|^{qk}|b|^{2j}-2|a|^{k(1+\frac{q}{2})}|b|^{j(1+\frac{p}{2})}\right)+$$

$$+B_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{k(q+2)}|b|^{j(p+2)}(k^2(2-q)^2\log^2(|a|)+j^2(p-2)^2\log^2(|b|)$$

$$+2kj(2-q)(p-2)\log(|a|)).$$

All the series whose partial sums which appear in previous inequality are convergent on the disk $D(0,R)$ therefore we can take the limit when n tends to ∞ before and obtain the inequaility

$$f_A(|a|^{\frac{2}{p}+1})g_A(|b|^{\frac{2}{q}+1})+r\{f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{p}{2}+1})\}+$$

$$+A_2\left(\frac{1}{p}\right)\{(2-q)^2\log^2(|a|)g(|b|^{p+2})\left[|a|^{2+q}]f^{'}(|a|^{2+q})+|a|^{2(2+q)}f^{''}(|a|^{2+q})\right]+$$

$$+(p-2)^2\log^2(|b|)f(|a|^{2+q})\left[|b|^{p+2}]g^{'}(|b|^{p+2})+|b|^{2(p+2)}g^{''}(|b|^{p+2})\right]+$$

$$+2(2-q)(p-2)\log(|a|)\log(|b|)\left[|a|^{2+q}f^{'}(|a|^{2+q})|b|^{p+2}g^{'}(|b|^{p+2})\right]\}\leq$$

$$\leq\frac{f_A(|a|^2)g_A(|b|^p)}{p}+\frac{f_A(|a|^q)g_A(|b|^2)}{q}\leq$$

$$\leq f_A(|a|^{\frac{2}{p}+1})g_A(|b|^{\frac{2}{q}+1})+(1-r)\{f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{p}{2}+1})\}+$$

$$+B_2\left(\frac{1}{p}\right)\{(2-q)^2\log^2(|a|)g(|b|^{p+2})\left[|a|^{2+q}]f^{'}(|a|^{2+q})+|a|^{2(2+q)}f^{''}(|a|^{2+q})\right]+$$

$$+(p-2)^2\log^2(|b|)f(|a|^{2+q})\left[|b|^{p+2}]g^{'}(|b|^{p+2})+|b|^{2(p+2)}g^{''}(|b|^{p+2})\right]+$$

$$+2(2-q)(p-2)\log(|a|)\log(|b|)\left[|a|^{2+q}f^{'}(|a|^{2+q})|b|^{p+2}g^{'}(|b|^{p+2})\right]\}\leq$$

Now we fix $|a|$ and apply poperty (1.2) obtaining for any $x\in\mathcal{H}$,

$$<f_A(A_1^{\frac{2}{p}+1})x,x>g_A(|b|^{\frac{2}{q}+1})+r\{<f_A(A_1^2)x,x>g_A(|b|^p)+$$

$$+<f_A(A_1^q)x,x>g_A(|b|^2)-2<f_A(A_1^{\frac{q}{2}+1})x,x>g(|b|^{\frac{p}{2}+1})\}+$$

$$+A_2\left(\frac{1}{p}\right)\{(2-q)^2<S_1(A_1^{2+q})\log^2(A_1)x,x>g(|b|^{p+2})+$$

$$+(p-2)^2<f(A_1^{2+q})x,x>S_2(|b|^{p+2})\log^2(|b|)\}+$$

$$+2(2-q)(p-2)<S_3(A_1^{2+q})\log(A_1)x,x>S_4(|b|^{p+2})\log(|b|)\leq$$

$$\leq\frac{<f_A(A_1^2)x,x>g_A(|b|^p)}{p}+\frac{<f_A(A_1^q)x,x>g_A(|b|^2)}{q}\leq$$

$$\leq<f_A(A_1^{\frac{2}{p}+1})x,x>g_A(|b|^{\frac{2}{q}+1})+(1-r)\{<f_A(A_1^2)x,x>g_A(|b|^p)+$$

$$+ < f_A(A_1^q)x, x > g_A(|b|^2) - 2 < f_A(A_1^{\frac{q}{2}+1})x, x > g_A(|b|^{\frac{p}{2}+1})\}+$$
$$+B_2\left(\frac{1}{p}\right)\{(2-q)^2 < S_1(A_1^{2+q})\log^2(A_1)x, x > g(|b|^{p+2})+$$
$$+(p-2)^2 < f(A_1^{2+q})x, x > S_2(|b|^{p+2})\log^2(|b|)\}+$$
$$+2(2-q)(p-2) < S_3(A_1^{2+q})\log(A_1)x, x > S_4(|b|^{p+2})\log(|b|).$$

Applying now for $|b|$ fixed the Theorem 1, for any $y \in \mathcal{K}$ we get the desired inequality.

∎

In the last result we use the same inequality (1.1) but we will replace the variable a and b by other suitable variable and then we will follow the same steps like before.

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$, $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0 < R < 1$. If p, q are real numbers with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and A_1 is a positive definite operator on the Hilbert space $\mathcal{H}$, $A_1 \in B(H)$ with $Sp(A_1) \subseteq (0,R)$ and B is a positive operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in \mathcal{B}_h^*(\mathcal{K})$ with $Sp(B) \subseteq (0,R)$ then we have*

$$< f_A\left(A_1^2\right)x, x > [g_A\left(B^2\right)y, y] + r\{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y]+$$
$$+ < f_A\left(A_1^2\right)x, x > [g_A\left(B^q\right)y, y] - 2 < f_A\left(A_1^2\right)x, x > [g_A\left(B^{\frac{p+q}{2}}\right)y, y]\} + A_2\left(\frac{1}{p}\right)M \leq$$
$$\leq \frac{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y]}{p} + \frac{< f_A\left(A_1^2\right)x, x > [g_A\left(B^q\right)y, y]}{q} \leq$$
$$\leq < f_A\left(A_1^2\right)x, x > [g_A\left(B^2\right)y, y] + (1-r)\{< f_A\left(A_1^2\right)x, x > [g_A\left(B^p\right)y, y]+$$
$$+ < f_A\left(A_1^2\right)x, x > [g_A\left(B^q\right)y, y] - 2 < f_A\left(A_1^2\right)x, x > [g_A\left(B^{\frac{p+q}{2}}\right)y, y]\} + B_2\left(\frac{1}{p}\right)M$$

where

$$M = (p-q)^2 < f(A_1^4)x, x > [S_1(B^{p+q})\log^2(B)y, y]$$

and $S_1(a) = af_A^{'}(a) + a^2 f_A^{''}(a)$, $S_2(a) = ag_A^{'}(a) + a^2 g_A^{''}(a)$, $S_3(a) = af_A^{'}(a)$, $S_4(a) = ag_A^{'}(a)$ for each $x \in H$ and $y \in \mathcal{K}$, where $\lambda \in [0,1]$, $r = \min\{\lambda, 1-\lambda\}$.

Proof. This time we consider, in inequality (1.1) $|a|^{j\frac{2}{p}}|b|^k$ instead of a and $|a|^{j\frac{2}{q}}|b|^k$ instead of b where $k, j \in \mathbf{N}$ and we obtain:

$$|a|^{2j}|b|^{2k} + r\left(|a|^{2j}|b|^{kp} + |a|^{2j}|b|^{kq} - 2|a|^{2j}|b|^{k\frac{p+q}{2}}\right) +$$
$$+A_2\left(\frac{1}{p}\right)|a|^{4j}|b|^{k(p+q)}\log^2\left(\frac{|a|^{2j}|b|^{kp}}{|a|^{2j}|b|^{kq}}\right) \leq \frac{|a|^{2j}|b|^{kp}}{p} + \frac{|a|^{2j}|b|^{kq}}{q} \leq$$
$$\leq |a|^{2j}|b|^{2k} + (1-r)\left(|a|^{2j}|b|^{kp} + |a|^{2j}|b|^{kq} - 2|a|^{2j}|b|^{k\frac{p+q}{2}}\right) +$$
$$+B_2\left(\frac{1}{p}\right)|a|^{4j}|b|^{k(p+q)}\log^2\left(\frac{|a|^{2j}|b|^{kp}}{|a|^{2j}|b|^{kq}}\right).$$

If we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over j and k from 0 to n, we find that

$$\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{2j}|b|^{2k} + r\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2j}|b|^{kp} + |a|^{2j}|b|^{kq} - 2|a|^{2j}|b|^{k\frac{p+q}{2}}\right) +$$

$$+A_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|k^2(p-q)^2|a|^{4j}|b|^{k(p+q)}\log^2(|b|)\le$$

$$\le\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\frac{|a|^{2j}|b|^{kp}}{p}+\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\frac{|a|^{2j}|b|^{kq}}{q}\le$$

$$\le\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k||a|^{2j}|b|^{2k}+(1-r)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|\left(|a|^{2j}|b|^{kp}+|a|^{2j}|b|^{kq}-2|a|^{2j}|b|^{k\frac{p+q}{2}}\right)+$$

$$+B_2\left(\frac{1}{p}\right)\sum_{k=0}^{n}\sum_{j=0}^{n}|p_j||q_k|k^2(p-q)^2|a|^{4j}|b|^{k(p+q)}\log^2(|b|).$$

By the same reason as in previous theorem using the hypothesis we find the desired inequality. ∎

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Loredana Ciurdariu: Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No. 2, 300006-Timişoara