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INEQUALITIES OF JENSEN TYPE FOR HA-CONVEX FUNCTIONS

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Abstract. Some integral inequalities of Jensen type for HA-convex functions defined on intervals of real line are given. Applications in relation to Hermite-Hadamard inequalities and Jensen discrete inequalities are provided. Inequalities for HA-convex functions of selfadjoint operators on complex Hilbert spaces are established as well.

1. Introduction

Following [4] (see also [20]) we say that the function $f : I \subset R \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t) f(x) + t f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

The following simple but important fact is as follows:

Criterion 1. *If $f : [a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

For a convex function $h : [c,d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d h(t)\,dt \leq \frac{h(c)+h(d)}{2} \tag{1.2}$$

for any convex function $h : [c,d] \to \mathbb{R}$.

If we write the Hermite-Hadamard inequality for the convex function $g(t) = f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b}, \frac{1}{a}\right]$, then we have

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt \leq \frac{f(b)+f(a)}{2}. \tag{1.3}$$

Using the change of variable $s = \frac{1}{t}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt = \int_a^b \frac{f(s)}{s^2} ds$$

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and by (1.3) we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(s)}{s^2}ds\leq\frac{f(b)+f(a)}{2}. \tag{1.4}$$

The inequality (1.4) has been obtained in a different manner in [20] by I. Işcan.

In the recent paper [15] we also established the following inequalities for *HA*-convex functions:

Theorem 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b))\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{(L(a,b)-a)\,bf(b)+(b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}, \tag{1.5}$$

and

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2}\leq\frac{1}{b-a}\int_a^b xf(x)\,dx\leq\frac{bf(b)+af(a)}{2}. \tag{1.6}$$

We obtained in [16] the following characterization of *HA*-convex functions.

Theorem 3. *Let $f,h:[a,b]\subset(0,\infty)\to\mathbb{R}$ so that $h(t)=tf(t)$ for $t\in[a,b]$. Then f is HA-convex on the interval $[a,b]$ if and only if h is convex on $[a,b]$.*

Motivated by the above results, we establish in this paper some inequalities of Jensen type for *HA*-convex functions. Some applications for functions of selfadjoint operators on Hilbert spaces and special means are also given.

2. Jensen Type Inequalities

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ -a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega w(x)\,|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [8] the following result:

Theorem 4. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\,f\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega wd\mu=1$. Then we have the inequality:*

$$\begin{aligned}0&\leq\int_\Omega(\Phi\circ f)\,wd\mu-\Phi\left(\int_\Omega fwd\mu\right)\\&\leq\int_\Omega(\Phi'\circ f)\,fwd\mu-\int_\Omega(\Phi'\circ f)\,wd\mu\int_\Omega wfd\mu\\&\leq\frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_\Omega w\left|f-\int_\Omega fwd\mu\right|d\mu.\end{aligned} \tag{2.1}$$

If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:

$$0 \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi \circ f)\, d\mu - \Phi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\right) \tag{2.2}$$
$$\leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi' \circ f) f d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi' \circ f)\, d\mu \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu$$
$$\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \frac{1}{\mu(\Omega)} \int_{\Omega} \left| f - \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu \right| d\mu.$$

The following discrete inequality is of interest as well.

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \tag{2.3}$$
$$\leq \sum_{i=1}^{n} w_i \Phi'(x_i)\, x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i$$
$$\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|.$$

Remark 1. *We notice that the inequality between the first and the second term in (2.3) was proved in 1994 by Dragomir & Ionescu, see* [17].

On making use of (2.1) and the reverse of Schwarz's inequality [11], [12]

$$\int_{\Omega} w \left| f - \int_{\Omega} f w d\mu \right| d\mu \leq \left[\int_{\Omega} f^2 w d\mu - \left(\int_{\Omega} f w d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (M - m),$$

provided that $m \leq f \leq M$ μ-a.e. on Ω, we can state the following string of Jensen's reverse inequalities:

Lemma 1. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(x) \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega \tag{2.4}$$

and $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\, f\in L_w(\Omega,\mu)$, *where* $w\geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$, *then*

$$
\begin{aligned}
0 &\leq \int_\Omega (\Phi\circ f)\, w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \\
&\leq \int_\Omega (\Phi'\circ f)\, f w d\mu - \int_\Omega (\Phi'\circ f)\, w d\mu \int_\Omega f w d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega w f d\mu \right| w d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_\Omega f^2 w d\mu - \left(\int_\Omega f w d\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M-m).
\end{aligned} \tag{2.5}
$$

Remark 2. *We notice that the inequality between the first, second and last term from (2.5) was proved in the general case of positive linear functionals in 2001 by S. S. Dragomir in* [7].

If the differentiability condition is removed, then Φ' *can be replaced in the inequality (2.5) above with a section* φ *of the subdifferential* $\partial\Phi$. *We omit the details.*

For recent results related to Jensen's inequality, see [1]-[10], [18]-[29] and the references therein.

We have the following result concerning Jensen's type inequalities for HA-convex functions.

Theorem 5. *Let* $h: I\subset (0,\infty)\to\mathbb{R}$ *be a differentiable HA-convex function and* $[k,K]\subset \mathring{I}$. *Assume also that* $x:\Omega\to\mathbb{R}$ *is* μ*-measurable, satisfying the bounds*

$$
0 < k \leq x(t) \leq K < \infty \quad \text{for } \mu\text{-a.e. } t\in\Omega \tag{2.6}
$$

and $w\geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$.

(i) If $h\circ x,\ \frac{1}{x},\ \frac{1}{x^2},\ (h'\circ x)\, x^2,\ (h'\circ x)\, x\in L_w(\Omega,\mu)$, *then we have the following inequalities*

$$
\begin{aligned}
0 &\leq \int_\Omega (h\circ x)\, w d\mu - h\left(\frac{1}{\int_\Omega \frac{w}{x} d\mu}\right) \\
&\leq \int_\Omega (h'\circ x)\, x^2 w d\mu \int_\Omega \frac{w}{x} d\mu - \int_\Omega (h'\circ x)\, x w d\mu \\
&\leq \frac{1}{2}\left[K^2 h'(K) - k^2 h'(k)\right] \int_\Omega \left| \frac{1}{x} - \int_\Omega \frac{w}{x} d\mu \right| w d\mu \\
&\leq \frac{1}{2}\left[K^2 h'(K) - k^2 h'(k)\right] \left[\int_\Omega \frac{w}{x^2} d\mu - \left(\int_\Omega \frac{w}{x} d\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4kK}\left[K^2 h'(K) - k^2 h'(k)\right](K-k).
\end{aligned} \tag{2.7}
$$

(ii) If $h\circ x,\ (h'\circ x)\,x,\ (h\circ x)\,x, (h'\circ x)\,x^2,\ x,\ x^2\in L_w(\Omega,\mu)$ then we have

$$0\le\int_\Omega (h\circ x)\,xwd\mu-\Phi\left(\int_\Omega xwd\mu\right)\int_\Omega xwd\mu \tag{2.8}$$
$$\le\int_\Omega [h\circ x+(h'\circ x)\,x]\,xwd\mu-\int_\Omega [h\circ x+(h'\circ x)\,x]\,wd\mu\int_\Omega xwd\mu$$
$$\le\frac{1}{2}\left[h(K)+Kh'(K)-h(k)-kh'(k)\right]\int_\Omega\left|x-\int_\Omega wxd\mu\right|wd\mu$$
$$\le\frac{1}{2}\left[h(K)+Kh'(K)-h(k)-kh'(k)\right]\left[\int_\Omega x^2wd\mu-\left(\int_\Omega xwd\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le\frac{1}{4}\left[h(K)+Kh'(K)-h(k)-kh'(k)\right](K-k)\,.$$

Proof. (i) Since the function $h:[k,K]\subset\mathring{I}\to\mathbb{R}$ is differentiable HA-convex, then by Criterion 1 the function $\Phi:\left[\frac{1}{K},\frac{1}{k}\right]\to\mathbb{R},\ \Phi(s)=h\left(\frac{1}{s}\right)$ is differentiable convex and

$$\Phi'(s)=-\frac{1}{s^2}h'\left(\frac{1}{s}\right),\ \ s\in\left[\frac{1}{K},\frac{1}{k}\right].$$

Put $m:=\frac{1}{K}$ and $M:=\frac{1}{k}$ and define $f(t)=\frac{1}{x(t)},\ t\in\Omega.$

By (2.6) we then have

$$m\le f(t)\le M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega.$$

We also have

$$\Phi'(m)=\Phi'\left(\frac{1}{K}\right)=-K^2h'(K)\ \text{ and } \Phi'(M)=-k^2h'(k)\,.$$

Now if we write the inequality (2.5) for the above choices, we get the desired result (2.7).

(ii) Since the function $h:[k,K]\subset\mathring{I}\to\mathbb{R}$ is differentiable HA-convex, then by Theorem 3 we have that $\Phi:[k,K]\to\mathbb{R},\ \Phi(s)=sh(s)$ is differentiable convex and

$$\Phi'(s)=h(s)+sh'(s)\,,\ \ s\in[k,K]\,.$$

If we write the inequality (2.5) for this choice, we get the desired result (2.8). □

The following reverse of the Jensen's inequality holds [11], [12]:

Lemma 2. *Let $\Phi:I\to\mathbb{R}$ be a convex function on the interval of real numbers I and $m,M\in\mathbb{R},\ m<M$ with $[m,M]\subset\mathring{I}$. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds (2.4) and $f,\ \Phi\circ f\in L_w(\Omega,\mu)$, where $w\ge0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$, then*

$$0\le\int_\Omega(\Phi\circ f)\,wd\mu-\Phi\left(\bar{f}_{\Omega,w}\right) \tag{2.9}$$
$$\le\frac{\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)$$
$$\le\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}$$
$$\le\frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],$$

where

$$\bar{f}_{\Omega,w} := \int_\Omega w(x) f(x) d\mu(x) \in [m, M]$$

and $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We also have the inequality

$$\begin{aligned} 0 &\le \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \le \frac{1}{4}(M - m) \Psi_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \\ &\le \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{2.10}$$

provided that $\bar{f}_{\Omega,w} \in (m, M)$.

For the function $h : [k, K] \subset \mathring{I} \to \mathbb{R}$ that is assumed to be *HA*-convex, consider the function $\Phi : \left[\frac{1}{K}, \frac{1}{k}\right] \to \mathbb{R}$, $\Phi(s) = h\left(\frac{1}{s}\right)$ that, according with Criterion 1, is convex. By taking $m = \frac{1}{K}$ and $M = \frac{1}{k}$ we have

$$\Psi_\Phi(t; m, M) = \frac{h(k) - h\left(\frac{1}{t}\right)}{\frac{1}{k} - t} - \frac{h\left(\frac{1}{t}\right) - h(K)}{t - \frac{1}{K}} \text{ for } t \in \left(\frac{1}{K}, \frac{1}{k}\right).$$

For the *HA*-convex function $h : [k, K] \subset \mathring{I} \to \mathbb{R}$ we can also consider the function $\Phi : [k, K] \to \mathbb{R}$, $\Phi(s) = sh(s)$ that, by Theorem 3 is convex. We then have for $m = k$, $M = K$

$$\Psi_\Phi(t; m, M) = \frac{Kh(K) - th(t)}{K - t} - \frac{th(t) - kh(k)}{t - k} \text{ for } t \in (k, K).$$

Making use of Lemma 2 we can state the following result as well.

Theorem 6. *Let* $h : I \subset (0, \infty) \to \mathbb{R}$ *be a HA-convex function and* $[k, K] \subset \mathring{I}$. *Assume also that* $x : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfying the bounds (2.6) and* $w \ge 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$.

(i) Define the function $\Lambda_h(\cdot, k, K) : \left(\frac{1}{K}, \frac{1}{k}\right) \to \mathbb{R}$ *by*

$$\Lambda_h(t, k, K) := \frac{h(k) - h\left(\frac{1}{t}\right)}{\frac{1}{k} - t} - \frac{h\left(\frac{1}{t}\right) - h(K)}{t - \frac{1}{K}}. \tag{2.11}$$

If $h \circ x$, $\frac{1}{x} \in L_w(\Omega, \mu)$, *then*

$$\begin{aligned} 0 &\le \int_\Omega (h \circ x) w d\mu - h\left(\frac{1}{\int_\Omega \frac{w}{x} d\mu}\right) \\ &\le kK \frac{\left(\frac{1}{k} - \int_\Omega \frac{w}{x} d\mu\right)\left(\int_\Omega \frac{w}{x} d\mu - \frac{1}{K}\right)}{K - k} \sup_{t \in \left(\frac{1}{K}, \frac{1}{k}\right)} \Lambda_h(t, k, K) \\ &\le kK \left(\frac{1}{k} - \int_\Omega \frac{w}{x} d\mu\right)\left(\int_\Omega \frac{w}{x} d\mu - \frac{1}{K}\right) \frac{K^2 h'_-(K) - k^2 h'_+(k)}{K - k} \\ &\le \frac{1}{4kK}(K - k)\left[K^2 h'_-(K) - k^2 h'_+(k)\right]. \end{aligned} \tag{2.12}$$

We also have the inequality

$$0 \le \int_\Omega (h \circ x)\, w d\mu - h\left(\frac{1}{\int_\Omega \frac{w}{x} d\mu}\right) \le \frac{1}{4kK}(K-k)\,\Lambda_h\left(\int_\Omega \frac{w}{x} d\mu, k, K\right) \le \frac{1}{4kK}(K-k)\left[K^2 h'_-(K) - k^2 h'_+(k)\right], \tag{2.13}$$

provided that $\int_\Omega \frac{w}{x} d\mu \in \left(\frac{1}{K}, \frac{1}{k}\right)$.

(ii) Define the function $\Gamma_h(\cdot, k, K) : (k, K) \to \mathbb{R}$ *defined by*

$$\Gamma_h(\cdot, k, K) := \frac{Kh(K) - th(t)}{K-t} - \frac{th(t) - kh(k)}{t-k}. \tag{2.14}$$

If $(h \circ x)\, x,\ x \in L_w(\Omega, \mu)$, *then*

$$\begin{aligned} 0 &\le \int_\Omega (h \circ x)\, xwd\mu - \Phi\left(\int_\Omega xwd\mu\right)\int_\Omega xwd\mu \\ &\le \frac{\left(K - \int_\Omega xwd\mu\right)\left(\int_\Omega xwd\mu - k\right)}{K-k} \sup_{t\in(k,K)} \Gamma_h(t,k,K) \\ &\le \left(K - \int_\Omega xwd\mu\right)\left(\int_\Omega xwd\mu - k\right) \\ &\times \frac{\left[h(K) + Kh'_-(K) - h(k) - kh'_+(k)\right]}{K-k} \\ &\le \frac{1}{4}(K-k)\left[h(K) + Kh'_-(K) - h(k) - kh'_+(k)\right]. \end{aligned} \tag{2.15}$$

We also have the inequalities

$$\begin{aligned} 0 &\le \int_\Omega (h \circ x)\, xwd\mu - \Phi\left(\int_\Omega xwd\mu\right)\int_\Omega xwd\mu \\ &\le \frac{1}{4}(K-k)\,\Gamma_h\left(\int_\Omega xwd\mu, k, K\right) \\ &\le \frac{1}{4}(K-k)\left[h(K) + Kh'_-(K) - h(k) - kh'_+(k)\right], \end{aligned} \tag{2.16}$$

provided that $\int_\Omega xwd\mu \in (k, K)$.

We also have the following reverse of Jensen's inequality [11], [12]:

Lemma 3. *With the assumptions of Lemma 2, we have the inequalities*

$$\begin{aligned} 0 &\le \int_\Omega w(\Phi \circ f)\, d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\ &\le 2\max\left\{\frac{M - \bar{f}_{\Omega,w}}{M-m}, \frac{\bar{f}_{\Omega,w} - m}{M-m}\right\}\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\le 2\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{2.17}$$

Using Lemma 3 we can state the following result as well.

Theorem 7. *Let* $h : I \subset (0, \infty) \to \mathbb{R}$ *be a HA-convex function and* $[k, K] \subset \mathring{I}$. *Assume also that* $x : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfying the bounds (2.6) and* $w \ge 0$ μ*-a.e. on* Ω *with* $\int_\Omega wd\mu = 1$.

(i) If $h\circ x,\ \frac{1}{x}\in L_w(\Omega,\mu)$, *then*

$$0\leq\int_{\Omega}(h\circ x)\,wd\mu-h\left(\frac{1}{\int_{\Omega}\frac{w}{x}d\mu}\right)\leq 2kK\max\left\{\frac{\frac{1}{k}-\int_{\Omega}\frac{w}{x}d\mu}{K-k},\frac{\int_{\Omega}\frac{w}{x}d\mu-\frac{1}{K}}{K-k}\right\}\times\left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right]\leq 2\left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right]. \tag{2.18}$$

(ii) If $(h\circ x)\,x,\ x\in L_w(\Omega,\mu)$, *then*

$$0\leq\int_{\Omega}(h\circ x)\,xwd\mu-h\left(\int_{\Omega}xwd\mu\right)\int_{\Omega}xwd\mu\leq 2\max\left\{\frac{K-\int_{\Omega}xwd\mu}{K-k},\frac{\int_{\Omega}xwd\mu-k}{K-k}\right\}\times\left[\frac{kh(k)+Kh(K)}{2}-h\left(\frac{k+K}{2}\right)\frac{k+K}{2}\right]\leq 2\left[\frac{kh(k)+Kh(K)}{2}-h\left(\frac{k+K}{2}\right)\frac{k+K}{2}\right]. \tag{2.19}$$

3. Some Weighted Hermite-Hadamard Type Inequalities

Let $h:I\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function and $[a,b]\subset\mathring{I}$. Assume also that $w(t)\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt>0$. Using the results from the previous section, we can state the following weighted Hermite-Hadamard type inequalities for functions of a single real variable.

By Theorem 5 we have the following weighted inequalities:

$$0\leq\frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(t)\,dt}-h\left(\frac{\int_a^b w(t)\,dt}{\int_a^b\frac{1}{t}w(t)\,dt}\right)\leq\frac{\int_a^b h'(t)\,t^2w(t)\,dt}{\int_a^b w(t)\,dt}\cdot\frac{\int_a^b\frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}-\frac{\int_a^b h'(t)\,tw(t)\,dt}{\int_a^b w(t)\,dt}\leq\frac{1}{2}\left[b^2h'_-(b)-a^2h'_+(a)\right]\frac{\int_a^b\left|\frac{1}{t}-\frac{\int_a^b\frac{1}{s}w(s)ds}{\int_a^b w(s)ds}\right|w(t)\,dt}{\int_a^b w(t)\,dt}\leq\frac{1}{2}\left[b^2h'_-(b)-a^2h'_+(a)\right]\times\left[\frac{\int_a^b\frac{1}{t^2}w(t)\,dt}{\int_a^b w(t)\,dt}-\left(\frac{\int_a^b\frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}\right)^2\right]^{\frac{1}{2}}\leq\frac{1}{4ab}\left[b^2h'_-(b)-a^2h'_+(a)\right](b-a) \tag{3.1}$$

and

$$0 \leq \frac{\int_a^b h(t)\, tw(t)\, dt}{\int_a^b w(t)\, dt} - h\left(\frac{\int_a^b tw(t)\, dt}{\int_a^b w(t)\, dt}\right) \frac{\int_a^b tw(t)\, dt}{\int_a^b w(t)\, dt} \tag{3.2}$$
$$\leq \frac{\int_a^b [h(t) + h'(t)\, t]\, tw(t)\, dt}{\int_a^b w(t)\, dt} - \frac{\int_a^b [h(t) + h'(t)\, t]\, w(t)\, dt}{\int_a^b w(t)\, dt} \cdot \frac{\int_a^b tw(t)\, dt}{\int_a^b w(t)\, dt}$$
$$\leq \frac{1}{2}\left[h(b) + bh'_-(b) - h(a) - ah'_+(a)\right] \frac{\int_a^b \left| t - \frac{\int_a^b sw(s)ds}{\int_a^b w(s)ds} \right| w(t)\, dt}{\int_a^b w(t)\, dt}$$
$$\leq \frac{1}{2}\left[h(b) + bh'_-(b) - h(a) - ah'_+(a)\right]$$
$$\times \left[\frac{\int_a^b t^2 w(t)\, dt}{\int_a^b w(t)\, dt} - \left(\frac{\int_a^b tw(t)\, dt}{\int_a^b w(t)\, dt}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[h(b) + bh'_-(b) - h(a) - ah'_+(a)\right](b-a).$$

We define the *logarithmic mean* for the positive numbers a, b by

$$L(a,b) := \begin{cases} a \text{ if } b = a, \\ \frac{b-a}{\ln b - \ln a} \text{ if } b = a \end{cases}$$

and the *geometric mean* by $G(a,b) = \sqrt{ab}$.

If we take $w(t) = 1$ in (3.1) and (3.2), then we get

$$0 \leq \frac{1}{b-a}\int_a^b h(t)\, dt - h(L(a,b)) \tag{3.3}$$
$$\leq \frac{1}{(b-a)\, L(a,b)} \int_a^b h'(t)\, t^2 dt - \frac{1}{b-a}\int_a^b h'(t)\, tdt$$
$$\leq \frac{1}{2}\left[b^2 h'_-(b) - a^2 h'_+(a)\right] \frac{1}{b-a}\int_a^b \left|\frac{1}{t} - \frac{1}{L(a,b)}\right| dt$$
$$\leq \frac{1}{2}\frac{\left[b^2 h'_-(b) - a^2 h'_+(a)\right]}{G(a,b)\, L(a,b)}\left[L^2(a,b) - G^2(a,b)\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4G^2(a,b)}\left[b^2 h'_-(b) - a^2 h'_+(a)\right](b-a)$$

and

$$0 \leq \frac{1}{b-a}\int_a^b h(t)\, tdt - \frac{a+b}{2} h\left(\frac{a+b}{2}\right) \tag{3.4}$$
$$\leq \frac{\int_a^b [h(t) + h'(t)\, t]\, tdt}{b-a} - \frac{a+b}{2} \cdot \frac{\int_a^b [h(t) + h'(t)\, t]\, dt}{b-a}$$
$$\leq \frac{1}{8}\left[h(b) + bh'_-(b) - h(a) - ah'_+(a)\right](b-a).$$

If we take in (3.1) $w(t)=t$, then we get

$$
\begin{aligned}
0 &\leq \frac{2}{b^2-a^2}\int_a^b h(t)\,t\,dt - h\left(\frac{a+b}{2}\right) \\
&\leq \frac{2}{b^2-a^2}\left[\frac{2}{a+b}\int_a^b h'(t)\,t^3 dt - \int_a^b h'(t)\,t^2 dt\right] \\
&\leq \frac{b^2 h'_-(b) - a^2 h'_+(a)}{b^2-a^2}\int_a^b \left|\frac{1}{t}-\frac{2}{a+b}\right| t\,dt \\
&\leq \frac{1}{2}\left[b^2 h'_-(b) - a^2 h'_+(a)\right]\left(\frac{A-L}{A^2 L}\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4G^2(a,b)}\left[b^2 h'_-(b) - a^2 h'_+(a)\right](b-a).
\end{aligned} \tag{3.5}
$$

If we take in (3.2) $w(t)=\frac{1}{t}$, then we get

$$
\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b h(t)\,dt - h(L(a,b)) \\
&\leq \frac{1}{b-a}\int_a^b \left[h(t)+h'(t)\,t\right]dt - L(a,b)\frac{1}{b-a}\int_a^b \frac{h(t)+h'(t)\,t}{t}dt \\
&\leq \frac{1}{2}\left[h(b)+bh'_-(b)-h(a)-ah'_+(a)\right]\frac{1}{b-a}\int_a^b \frac{|t-L(a,b)|}{t}dt \\
&\leq \frac{1}{2}\left[h(b)+bh'_-(b)-h(a)-ah'_+(a)\right]\left(\frac{A(a,b)}{L(a,b)}-1\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4L(a,b)}\left[h(b)+bh'_-(b)-h(a)-ah'_+(a)\right](b-a).
\end{aligned} \tag{3.6}
$$

Define the function $\Lambda_h(\cdot,a,b):\left(\frac{1}{b},\frac{1}{a}\right)\to\mathbb{R}$ by

$$
\Lambda_h(t,a,b) := \frac{h(a)-h\left(\frac{1}{t}\right)}{\frac{1}{a}-t} - \frac{h\left(\frac{1}{t}\right)-h(b)}{t-\frac{1}{b}}. \tag{3.7}
$$

If we use Theorem 6, then we have

$$
\begin{aligned}
0 &\leq \frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(t)\,dt} - h\left(\frac{\int_a^b w(t)\,dt}{\int_a^b \frac{1}{t}w(t)\,dt}\right) \\
&\leq ab\frac{\left(\frac{1}{a}-\frac{\int_a^b \frac{1}{t}w(t)dt}{\int_a^b w(t)dt}\right)\left(\frac{\int_a^b \frac{1}{t}w(t)dt}{\int_a^b w(t)dt}-\frac{1}{b}\right)}{b-a}\sup_{t\in\left(\frac{1}{b},\frac{1}{a}\right)}\Lambda_h(t,a,b) \\
&\leq ab\left(\frac{1}{a}-\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}\right)\left(\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}-\frac{1}{b}\right) \\
&\quad\times\frac{b^2 h'_-(b)-a^2 h'_+(a)}{b-a} \\
&\leq \frac{1}{4ab}(b-a)\left[b^2 h'_-(b)-a^2 h'_+(a)\right].
\end{aligned} \tag{3.8}
$$

We also have the inequality

$$0\leq\frac{\int_a^b h(t)\,w(t)\,dt}{\int_a^b w(t)\,dt}-h\left(\frac{\int_a^b w(t)\,dt}{\int_a^b \frac{1}{t}w(t)\,dt}\right) \tag{3.9}$$
$$\leq\frac{1}{4ab}(b-a)\,\Lambda_h\left(\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt},a,b\right)$$
$$\leq\frac{1}{4ab}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right],$$

provided that $\frac{\int_a^b \frac{1}{t}w(t)dt}{\int_a^b w(t)dt}\in\left(\frac{1}{b},\frac{1}{a}\right)$.

If we take in (3.8) and (3.9) $w(t)=1$, then we get

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,dt-h(L(a,b)) \tag{3.10}$$
$$\leq ab\frac{\left(\frac{1}{a}-\frac{1}{L(a,b)}\right)\left(\frac{1}{L(a,b)}-\frac{1}{b}\right)}{b-a}\sup_{t\in\left(\frac{1}{b},\frac{1}{a}\right)}\Lambda_h(t,a,b)$$
$$\leq ab\left(\frac{1}{a}-\frac{1}{L(a,b)}\right)\left(\frac{1}{L(a,b)}-\frac{1}{b}\right)\frac{b^2h'_-(b)-a^2h'_+(a)}{b-a}$$
$$\leq\frac{1}{4ab}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right]$$

and

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,dt-h(L(a,b))\leq\frac{1}{4ab}(b-a)\,\Lambda_h\left(\frac{1}{L(a,b)},a,b\right) \tag{3.11}$$
$$\leq\frac{1}{4ab}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right].$$

If we take in (3.8) $w(t)=t$, then we get, by denoting $A=\frac{a+b}{2}$, that

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,tdt-A(a,b)\,h(A(a,b)) \tag{3.12}$$
$$\leq\frac{1}{4A(a,b)}(b-a)\sup_{t\in\left(\frac{1}{b},\frac{1}{a}\right)}\Lambda_h(t,a,b)$$
$$\leq\frac{1}{4A(a,b)}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right]$$
$$\leq\frac{A(a,b)}{4ab}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right]$$

and

$$0\leq\frac{1}{b-a}\int_a^b h(t)\,tdt-A(a,b)\,h(A(a,b)) \tag{3.13}$$
$$\leq\frac{A(a,b)}{4ab}(b-a)\,\Lambda_h\left(\frac{1}{A(a,b)},a,b\right)$$
$$\leq\frac{A(a,b)}{4ab}(b-a)\left[b^2h'_-(b)-a^2h'_+(a)\right].$$

Similar results may be stated by using the function

$$\Gamma_h(\cdot,a,b) := \frac{bh(b)-th(t)}{b-t} - \frac{th(t)-ah(a)}{t-a}$$

and Theorem 6. However we do not provide the details here.

If we use Theorem 7, then we get

$$\begin{aligned} 0 &\leq \frac{\int_a^b h(t)w(t)\,dt}{\int_a^b w(t)\,dt} - h\left(\frac{\int_a^b w(t)\,dt}{\int_a^b \frac{1}{t}w(t)\,dt}\right) \\ &\leq \frac{2ab}{b-a}\max\left\{\frac{1}{a} - \frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}, \frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt} - \frac{1}{b}\right\} \\ &\times\left[\frac{h(a)+h(b)}{2} - h\left(\frac{2ab}{a+b}\right)\right] \\ &\leq 2\left[\frac{h(a)+h(b)}{2} - h\left(\frac{2ab}{a+b}\right)\right] \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} 0 &\leq \frac{\int_a^b h(t)tw(t)\,dt}{\int_a^b w(t)\,dt} - h\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt}\right)\frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt} \\ &\leq \frac{2}{b-a}\max\left\{b - \frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt}, \frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt} - a\right\} \\ &\times\left[\frac{ah(a)+bh(b)}{2} - h\left(\frac{a+b}{2}\right)\frac{a+b}{2}\right] \\ &\leq 2\left[\frac{ah(a)+bh(b)}{2} - h\left(\frac{a+b}{2}\right)\frac{a+b}{2}\right]. \end{aligned} \tag{3.15}$$

If we write (3.14) for $w(t)=1$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b h(t)\,dt - h(L(a,b)) \\ &\leq 2ab\max\left\{\frac{\frac{1}{a} - \frac{1}{L(a,b)}}{b-a}, \frac{\frac{1}{L(a,b)} - \frac{1}{b}}{b-a}\right\} \\ &\times\left[\frac{h(a)+h(b)}{2} - h\left(\frac{2ab}{a+b}\right)\right] \\ &\leq 2\left[\frac{h(a)+h(b)}{2} - h\left(\frac{2ab}{a+b}\right)\right]. \end{aligned} \tag{3.16}$$

If we write (3.14) for $w(t)=t$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b h(t)\,tdt - A(a,b)\,h(A(a,b)) \\ &\leq b\left[\frac{h(a)+h(b)}{2} - h\left(\frac{2ab}{a+b}\right)\right] \end{aligned} \tag{3.17}$$

and if we take in (3.15) $w(t)=\frac{1}{t}$, then we get

$$
\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b h(t)\,dt - h(L(a,b)) \\
&\le \frac{2}{L(a,b)}\max\left\{\frac{b-L(a,b)}{b-a},\frac{L(a,b)-a}{b-a}\right\} \\
&\times\left[\frac{ah(a)+bh(b)}{2}-h\left(\frac{a+b}{2}\right)\frac{a+b}{2}\right] \\
&\le \frac{2}{L(a,b)}\left[\frac{ah(a)+bh(b)}{2}-h\left(\frac{a+b}{2}\right)\frac{a+b}{2}\right].
\end{aligned} \tag{3.18}
$$

According to Theorem 3 one can give many examples of HA-convex functions $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ by taking $f(t)=\frac{g(t)}{t}$ for $t\in[a,b]$ with $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ any convex function. Therefore $f(t)=\ln t$ is an HA-convex function.

If we write, for instance, the inequality (3.1) for $f(t)=\ln t$, then we get

$$
\begin{aligned}
0 &\le \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}-\ln\left(\frac{\int_a^b w(t)\,dt}{\int_a^b \frac{1}{t}w(t)\,dt}\right) \\
&\le \frac{\int_a^b tw(t)\,dt}{\int_a^b w(t)\,dt}\cdot\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}-1 \\
&\le \frac{1}{2}(b-a)\frac{\int_a^b\left|\frac{1}{t}-\frac{\int_a^b\frac{1}{s}w(s)ds}{\int_a^b w(s)ds}\right|w(t)\,dt}{\int_a^b w(t)\,dt} \\
&\le \frac{1}{2}(b-a)\left[\frac{\int_a^b\frac{1}{t^2}w(t)\,dt}{\int_a^b w(t)\,dt}-\left(\frac{\int_a^b\frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4ab}(b-a)^2
\end{aligned} \tag{3.19}
$$

for any $w(t)\ge 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt>0$.

If we take in the first inequality in (3.19) $w(t)=1$, $t\in[a,b]$ then we get

$$
0\le \ln I(a,b)-\ln L(a,b)\le\frac{A(a,b)}{L(a,b)}-1, \tag{3.20}
$$

where $I(a,b)$ is the *identric mean*, i.e.

$$
I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } b\ne a\\[2ex] b, & \text{if } b=a\end{cases}
$$

and $A(a,b)=\frac{a+b}{2}$ is the *arithmetic mean*.

If we write the inequality (3.14) for $f(t)=\ln t$, then we get

$$0\leq\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}-\ln\left(\frac{\int_a^b w(t)\,dt}{\int_a^b \frac{1}{t}w(t)\,dt}\right)\tag{3.21}$$
$$\leq\frac{2ab}{b-a}\max\left\{\frac{1}{a}-\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt},\frac{\int_a^b \frac{1}{t}w(t)\,dt}{\int_a^b w(t)\,dt}-\frac{1}{b}\right\}\ln\left(\frac{G(a,b)}{H(a,b)}\right)$$
$$\leq\ln\left(\frac{G(a,b)}{H(a,b)}\right)^2$$

for any $w(t)\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt>0$.

If we take in (3.21) $w(t)=1$, $t\in[a,b]$, then we get

$$\begin{aligned}0 &\leq \ln I(a,b)-\ln L(a,b)\\ &\leq \frac{2ab}{b-a}\max\left\{\frac{1}{a}-\frac{1}{L(a,b)},\frac{1}{L(a,b)}-\frac{1}{b}\right\}\ln\left(\frac{G(a,b)}{H(a,b)}\right)\\ &\leq \ln\left(\frac{G(a,b)}{H(a,b)}\right)^2,\end{aligned}\tag{3.22}$$

where $G(a,b):=\sqrt{ab}$ is the *geometric mean* and $H(a,b):=\frac{2ab}{a+b}$ is the *harmonic mean.*

4. Discrete Inequalities

Let $p=(p_1,...,p_n)$ be a probability distribution, i.e. $p_i\geq 0$, $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$. If we write the inequalities from Theorem 5 for the discrete measure we can get the following discrete inequalities.

Let $x_i\in[k,K]\subset(0,\infty)$ for $i\in\{1,...,n\}$ and $p=(p_1,...,p_n)$ be a probability distribution. If $h:[k,K]\to\mathbb{R}$ is *HA*-convex and differentiable, then by (2.7) we have

$$0\leq\sum_{i=1}^n p_i h(x_i)-h\left(\frac{1}{\sum_{i=1}^n\frac{p_i}{x_i}}\right)\tag{4.1}$$
$$\leq\sum_{i=1}^n p_i h'(x_i)x_i^2\sum_{i=1}^n\frac{p_i}{x_i}-\sum_{i=1}^n p_i h'(x_i)x_i$$
$$\leq\frac{1}{2}\left[K^2h'(K)-k^2h'(k)\right]\sum_{i=1}^n p_i\left|\frac{1}{x_i}-\sum_{j=1}^n\frac{p_j}{x_j}\right|$$
$$\leq\frac{1}{2}\left[K^2h'(K)-k^2h'(k)\right]\left[\sum_{i=1}^n\frac{p_i}{x_i^2}-\left(\sum_{i=1}^n\frac{p_i}{x_i}\right)^2\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{4kK}\left[K^2h'(K)-k^2h'(k)\right](K-k),$$

while by (2.8) we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i h(x_i) x_i - h\left(\sum_{i=1}^{n} p_i x_i\right) \sum_{i=1}^{n} p_i x_i \\
&\leq \sum_{i=1}^{n} p_i \left[h(x_i) + h'(x_i) x_i\right] x_i - \sum_{i=1}^{n} p_i \left[h(x_i) + h'(x_i) x_i\right] \sum_{i=1}^{n} p_i x_i \\
&\leq \frac{1}{2}\left[h(K) + K h'(K) - h(k) - k h'(k)\right] \sum_{i=1}^{n} p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \\
&\leq \frac{1}{2}\left[h(K) + K h'(K) - h(k) - k h'(k)\right] \left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{j=1}^{n} p_j x_j\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[h(K) + K h'(K) - h(k) - k h'(k)\right](K - k).
\end{aligned}
\tag{4.2}
$$

Consider the function $\Lambda_h(\cdot, k, K) : \left(\frac{1}{K}, \frac{1}{k}\right) \to \mathbb{R}$ defined by the equation (2.11). If $x_i \in [k, K] \subset (0, \infty)$ for $i \in \{1, ..., n\}$ and $p = (p_1, ..., p_n)$ is a probability distribution then for $h : [k, K] \to \mathbb{R}$ a HA-convex we have from (2.12)

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i h(x_i) - h\left(\frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}}\right) \\
&\leq kK \frac{\left(\frac{1}{k} - \sum_{i=1}^{n} \frac{p_i}{x_i}\right)\left(\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{K}\right)}{K - k} \sup_{t \in \left(\frac{1}{K}, \frac{1}{k}\right)} \Lambda_h(t, k, K) \\
&\leq kK \left(\frac{1}{k} - \sum_{i=1}^{n} \frac{p_i}{x_i}\right)\left(\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{K}\right) \frac{K^2 h'_-(K) - k^2 h'_+(k)}{K - k} \\
&\leq \frac{1}{4kK}(K - k)\left[K^2 h'_-(K) - k^2 h'_+(k)\right],
\end{aligned}
\tag{4.3}
$$

while from (2.13) we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i h(x_i) - h\left(\frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}}\right) \leq \frac{1}{4kK}(K - k)\Lambda_h\left(\sum_{i=1}^{n} \frac{p_i}{x_i}, k, K\right) \\
&\leq \frac{1}{4kK}(K - k)\left[K^2 h'_-(K) - k^2 h'_+(k)\right],
\end{aligned}
\tag{4.4}
$$

provided that $\sum_{i=1}^{n} \frac{p_i}{x_i} \in \left(\frac{1}{K}, \frac{1}{k}\right)$.

Consider the function $\Gamma_h(\cdot, k, K) : [k, K] \to \mathbb{R}$ defined by (2.14). Let $x_i \in [k, K] \subset (0, \infty)$ for $i \in \{1, ..., n\}$ and $p = (p_1, ..., p_n)$ be a probability distribution.

If $h:[k,K]\to\mathbb{R}$ is HA-convex then by (2.15)

$$0\le\sum_{i=1}^{n}p_ih(x_i)x_i-h\left(\sum_{i=1}^{n}p_ix_i\right)\sum_{i=1}^{n}p_ix_i \tag{4.5}$$
$$\le\frac{\left(K-\sum_{i=1}^{n}p_ix_i\right)\left(\sum_{i=1}^{n}p_ix_i-k\right)}{K-k}\sup_{t\in(k,K)}\Gamma_h(t,k,K)$$
$$\le\left(K-\sum_{i=1}^{n}p_ix_i\right)\left(\sum_{i=1}^{n}p_ix_i-k\right)$$
$$\times\frac{\left[h(K)+Kh'_-(K)-h(k)-kh'_+(k)\right]}{K-k}$$
$$\le\frac{1}{4}(K-k)\left[h(K)+Kh'_-(K)-h(k)-kh'_+(k)\right],$$

while from (2.16) we have

$$0\le\sum_{i=1}^{n}p_ih(x_i)x_i-h\left(\sum_{i=1}^{n}p_ix_i\right)\sum_{i=1}^{n}p_ix_i \tag{4.6}$$
$$\le\frac{1}{4}(K-k)\Gamma_h\left(\sum_{i=1}^{n}p_ix_i,k,K\right)$$
$$\le\frac{1}{4}(K-k)\left[h(K)+Kh'_-(K)-h(k)-kh'_+(k)\right],$$

provided that $\sum_{i=1}^{n}p_ix_i\in(k,K)$.

We also have by (2.18) that

$$0\le\sum_{i=1}^{n}p_ih(x_i)-h\left(\frac{1}{\sum_{i=1}^{n}\frac{p_i}{x_i}}\right) \tag{4.7}$$
$$\le\frac{2kK}{K-k}\max\left\{\frac{1}{k}-\sum_{i=1}^{n}\frac{p_i}{x_i},\sum_{i=1}^{n}\frac{p_i}{x_i}-\frac{1}{K}\right\}$$
$$\times\left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right]$$
$$\le2\left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right]$$

and by (2.19) that

$$0\le\sum_{i=1}^{n}p_ih(x_i)x_i-h\left(\sum_{i=1}^{n}p_ix_i\right)\sum_{i=1}^{n}p_ix_i \tag{4.8}$$
$$\le\frac{2}{K-k}\max\left\{K-\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_ix_i-k\right\}$$
$$\times\left[\frac{kh(k)+Kh(K)}{2}-h\left(\frac{k+K}{2}\right)\frac{k+K}{2}\right]$$
$$\le2\left[\frac{kh(k)+Kh(K)}{2}-h\left(\frac{k+K}{2}\right)\frac{k+K}{2}\right].$$

We consider the *weighted arithmetic mean* $A_p(\mathbf{x}) := \sum_{i=1}^n p_i x_i$, *weighted geometric mean* $G_p(\mathbf{x}) := \prod_{i=1}^n x_i^{p_i}$ and *weighted harmonic mean* $H_p(\mathbf{x}) := \frac{1}{\sum_{i=1}^n \frac{p_i}{x_i}}$ where $x_i \in (0,\infty)$ for $i \in \{1,...,n\}$ and $p = (p_1,...,p_n)$ is a probability distribution.

If we use the first inequality in (4.1) for the HA-convex function $f(t) = \ln t$ we get

$$0 \le \ln G_p(\mathbf{x}) - \ln H_p(\mathbf{x}) \le A_p(\mathbf{x}) H_p(\mathbf{x}) - 1. \tag{4.9}$$

For $[k,K] \subset (0,\infty)$ we consider the function $\ell_{k,K} : (k,K) \to \mathbb{R}$ given by

$$\ell_{k,K}(t) := \frac{\ln k + \ln t}{\frac{1}{k} - t} + \frac{\ln t + \ln(K)}{t - \frac{1}{K}}.$$

If we use the inequality (4.3) for $f(t) = \ln t$, then we can state

$$\begin{aligned} 0 &\le \ln G_p(\mathbf{x}) - \ln H_p(\mathbf{x}) \\ &\le \frac{kK}{K-k}\left(\frac{1}{k} - H_p(\mathbf{x})\right)\left(H_p(\mathbf{x}) - \frac{1}{K}\right) \sup_{t \in \left(\frac{1}{K}, \frac{1}{k}\right)} \ell_{k,K}(t) \\ &\le kK\left(\frac{1}{k} - H_p(\mathbf{x})\right)\left(H_p(\mathbf{x}) - \frac{1}{K}\right) \le \frac{1}{4kK}(K-k)^2, \end{aligned} \tag{4.10}$$

where $x_i \in [k,K] \subset (0,\infty)$ for $i \in \{1,...,n\}$ and $p = (p_1,...,p_n)$ is a probability distribution.

By (4.4) we also have

$$\begin{aligned} 0 \le \ln G_p(\mathbf{x}) - \ln H_p(\mathbf{x}) &\le \frac{1}{4kK}(K-k)\,\ell_{k,K}(H_p(\mathbf{x})) \\ &\le \frac{1}{4kK}(K-k)^2, \end{aligned} \tag{4.11}$$

provided that $H_p(\mathbf{x}) \in \left(\frac{1}{K}, \frac{1}{k}\right)$.

Consider the function $\varphi_{k,K} : (k,K) \to \mathbb{R}$ defined by

$$\varphi_{k,K}(t) := \frac{K\ln K - t\ln t}{K-t} - \frac{t\ln t - k\ln k}{t-k}.$$

If we use the inequalities (4.5) and (4.6) for the HA-convex function $f(t) = \ln t$, then, by denoting

$$S_p(\mathbf{x}) := \prod_{i=1}^n x_i^{p_i x_i}$$

we have

$$\begin{aligned} 0 &\le \ln S_p(\mathbf{x}) - \ln\left(A_p(\mathbf{x})^{A_p(\mathbf{x})}\right) \\ &\le \frac{(K - A_p(\mathbf{x}))(A_p(\mathbf{x}) - k)}{K-k} \sup_{t\in(k,K)} \varphi_{k,K}(t) \\ &\le \frac{(K - A_p(\mathbf{x}))(A_p(\mathbf{x}) - k)}{L(k,K)} \le \frac{1}{4}(K-k)(\ln K - \ln k), \end{aligned} \tag{4.12}$$

and

$$0 \le \ln S_p(\mathbf{x}) - \ln\left(A_p(\mathbf{x})^{A_p(\mathbf{x})}\right) \le \frac{1}{4}(K-k)\,\varphi_{k,K}(A_p(\mathbf{x})) \le \frac{1}{4}(K-k)(\ln K - \ln k), \tag{4.13}$$

provided that $A_p(\mathbf{x}) \in (k,K)$.

Finally, by using (4.7) and (4.8) for the function $f(t) = \ln t$ we get

$$\begin{aligned} 0 &\le \ln G_p(\mathbf{x}) - \ln H_p(\mathbf{x}) \\ &\le \frac{2kK}{K-k}\max\left\{\frac{1}{k} - H_p(\mathbf{x}), H_p(\mathbf{x}) - \frac{1}{K}\right\}\ln\left(\frac{G(k,K)}{H(k,K)}\right) \\ &\le \ln\left(\frac{G(k,K)}{H(k,K)}\right)^2 \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} 0 &\le \ln S_p(\mathbf{x}) - \ln\left(A_p(\mathbf{x})^{A_p(\mathbf{x})}\right) \\ &\le \frac{2}{K-k}\max\{K - A_p(\mathbf{x}), A_p(\mathbf{x}) - k\}\ln\left(\frac{G\left(k^k, K^K\right)}{A(k,K)^{A(k,K)}}\right) \\ &\le \ln\left(\frac{G\left(k^k, K^K\right)}{A(k,K)^{A(k,K)}}\right)^2. \end{aligned} \tag{4.15}$$

5. Applications for Functions of Selfadjoint Operators

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $\varphi : [m,M] \to [a,b]$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [18, p. 257]):

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{5.1}$$

and

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2, \tag{5.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$ for any $x \in H$.

Now, assume that $\Phi : [k,K] \subset I \to (0,\infty)$ is continuously differentiable *HA*-convex function on the interval of real numbers I, $f : [m,M] \to [k,K]$, $p : [m,M] \to (0,\infty)$ are continuous functions on $[m,M]$ and $g : [m,M] \to \mathbb{R}$ is monotonic nondecreasing on $[m,M]$.

Using the inequality (2.7) written for the Riemann-Stieltjes integral of monotonic nondecreasing integrators we can state that following inequalities

$$
\begin{aligned}
(5.3)\qquad 0 &\leq \frac{\int_m^M \Phi\left(f\left(t\right)\right)p\left(t\right)dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} - \Phi\left(\frac{\int_m^M p\left(t\right)dg\left(t\right)}{\int_m^M \frac{p(t)}{f(t)}dg\left(t\right)}\right) \\
&\leq \frac{\int_m^M \Phi'\left(f\left(t\right)\right)f^2\left(t\right)p\left(t\right)dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} \cdot \frac{\int_m^M \frac{p(t)}{f(t)}dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} \\
&- \frac{\int_m^M \Phi'\left(f\left(t\right)\right)f\left(t\right)p\left(t\right)dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} \\
&\leq \frac{1}{2}\left[K^2\Phi'\left(K\right) - k^2\Phi'\left(k\right)\right] \\
&\times \frac{\int_m^M \left|\frac{1}{f(t)} - \int_m^M \frac{p(s)}{f(s)}dg\left(s\right)\right| p\left(t\right)dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} \\
&\leq \frac{1}{2}\left[K^2\Phi'\left(K\right) - k^2\Phi'\left(k\right)\right] \\
&\times \left[\frac{\int_m^M \frac{p(t)}{f^2(t)}dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)} - \left(\frac{\int_m^M \frac{p(t)}{f(t)}dg\left(t\right)}{\int_m^M p\left(t\right)dg\left(t\right)}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4kK}\left[K^2\Phi'\left(K\right) - k^2\Phi'\left(k\right)\right]\left(K-k\right).
\end{aligned}
$$

Assume that $Sp\left(A\right)$ is included in the interval $\left[m,M\right] \subset \left(0,\infty\right)$. Now, if we apply the inequalities (5.3) for the monotonic nondecreasing function $g_x\left(\lambda\right) := \left\langle E_\lambda x, x\right\rangle$, $x \in H$, where $\left\{E_\lambda\right\}_\lambda$ is the spectral family of A, then we get

$$
\begin{aligned}
(5.4)\qquad 0 &\leq \frac{\left\langle \Phi\left(f\left(A\right)\right)p\left(A\right)x,x\right\rangle}{\left\langle p\left(A\right)x,x\right\rangle} - \Phi\left(\frac{\left\langle p\left(A\right)x,x\right\rangle}{\left\langle p\left(A\right)\left(f\left(A\right)\right)^{-1}x,x\right\rangle}\right) \\
&\leq \frac{\left\langle \Phi'\left(f\left(A\right)\right)f^2\left(A\right)p\left(A\right)x,x\right\rangle}{\left\langle p\left(A\right)x,x\right\rangle} \cdot \frac{\left\langle p\left(A\right)\left(f\left(A\right)\right)^{-1}x,x\right\rangle}{\left\langle p\left(A\right)x,x\right\rangle} \\
&- \frac{\left\langle \Phi'\left(f\left(A\right)\right)f\left(A\right)p\left(A\right)x,x\right\rangle}{\left\langle p\left(A\right)x,x\right\rangle} \\
&\leq \frac{1}{2}\left[K^2\Phi'\left(K\right) - k^2\Phi'\left(k\right)\right] \\
&\times \frac{\left\langle \left|\left(f\left(A\right)\right)^{-1} - \left\langle p\left(A\right)\left(f\left(A\right)\right)^{-1}x,x\right\rangle 1_H\right| p\left(A\right)x,x\right\rangle}{\left\langle p\left(A\right)x,x\right\rangle}
\end{aligned}
$$

$$\leq \frac{1}{2}\left[K^2\Phi'(K)-k^2\Phi'(k)\right] \times\left[\frac{\left\langle p(A)(f(A))^{-2}x,x\right\rangle}{\langle p(A)x,x\rangle}-\left(\frac{\left\langle p(A)(f(A))^{-1}x,x\right\rangle}{\langle p(A)x,x\rangle}\right)^2\right]^{1/2} \leq \frac{1}{4kK}\left[K^2\Phi'(K)-k^2\Phi'(k)\right](K-k)$$

for any $x\in H$, $x\neq 0$, where $\Phi:[k,K]\subset I\to(0,\infty)$ is continuously differentiable HA-convex function on the interval of real numbers I and $f:[m,M]\to[k,K]$, $p:[m,M]\to(0,\infty)$ are continuous functions on $[m,M]$.

In particular, if p is taken to be the constant 1, then for any $x\in H$, $\|x\|=1$, we have

$$0\leq\langle\Phi(f(A))x,x\rangle-\Phi\left(\frac{1}{\left\langle(f(A))^{-1}x,x\right\rangle}\right) \tag{5.5}$$
$$\leq\left\langle\Phi'(f(A))f^2(A)x,x\right\rangle\left\langle(f(A))^{-1}x,x\right\rangle-\langle\Phi'(f(A))f(A)x,x\rangle$$
$$\leq\frac{1}{2}\left[K^2\Phi'(K)-k^2\Phi'(k)\right]\left\langle\left|(f(A))^{-1}-\left\langle(f(A))^{-1}x,x\right\rangle 1_H\right|x,x\right\rangle$$
$$\leq\frac{1}{2}\left[K^2\Phi'(K)-k^2\Phi'(k)\right]\left[\left\langle(f(A))^{-2}x,x\right\rangle-\left\langle(f(A))^{-1}x,x\right\rangle^2\right]^{1/2}$$
$$\leq\frac{1}{4kK}\left[K^2\Phi'(K)-k^2\Phi'(k)\right](K-k).$$

Moreover, if in (5.5) we take $[m,M]=[k,K]$ and $f(t)=t$, then we have

$$0\leq\langle\Phi(A)x,x\rangle-\Phi\left(\frac{1}{\langle A^{-1}x,x\rangle}\right) \tag{5.6}$$
$$\leq\left\langle\Phi'(A)A^2x,x\right\rangle\left\langle A^{-1}x,x\right\rangle-\langle\Phi'(A)Ax,x\rangle$$
$$\leq\frac{1}{2}\left[M^2\Phi'(M)-m^2\Phi'(m)\right]\left\langle\left|(f(A))^{-1}-\left\langle(f(A))^{-1}x,x\right\rangle 1_H\right|x,x\right\rangle$$
$$\leq\frac{1}{2}\left[M^2\Phi'(M)-m^2\Phi'(m)\right]\left[\left\langle(f(A))^{-2}x,x\right\rangle-\left\langle(f(A))^{-1}x,x\right\rangle^2\right]^{1/2}$$
$$\leq\frac{1}{4mM}\left[M^2\Phi'(M)-m^2\Phi'(m)\right](M-m),$$

for any $x\in H$, $\|x\|=1$.

If we apply the inequality (5.6) for the HA-convex function $f(t) = \ln t$, then we get

$$
\begin{aligned}
(5.7)\qquad 0 &\le \langle \ln Ax, x\rangle - \ln\left(\frac{1}{\langle A^{-1}x, x\rangle}\right) \\
&\le \langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1 \\
&\le \frac{1}{2}(M-m)\left\langle \left|(\ln A)^{-1} - \left\langle (\ln A)^{-1} x, x\right\rangle 1_H\right| x, x\right\rangle \\
&\le \frac{1}{2}(M-m)\left[\left\langle (\ln A)^{-2} x, x\right\rangle - \left\langle (\ln A)^{-1} x, x\right\rangle^2\right]^{1/2} \\
&\le \frac{1}{4mM}(M-m)^2,
\end{aligned}
$$

for any $x \in H$, $\|x\| = 1$.

If we use the inequality (2.12), then we have for the operator A with $Sp(A) \subset [m, M]$ we have

$$
\begin{aligned}
(5.8)\quad 0 &\le \frac{\langle \Phi(f(A))\, p(A)\, x, x\rangle}{\langle p(A)\, x, x\rangle} - \Phi\left(\frac{\langle p(A)\, x, x\rangle}{\left\langle p(A)\,(f(A))^{-1} x, x\right\rangle}\right) \\
&\le \frac{kK}{K-k}\left(\frac{1}{k} - \frac{\left\langle p(A)\,(f(A))^{-1} x, x\right\rangle}{\langle p(A)\, x, x\rangle}\right)\left(\frac{\left\langle p(A)\,(f(A))^{-1} x, x\right\rangle}{\langle p(A)\, x, x\rangle} - \frac{1}{K}\right) \\
&\times \sup_{t\in\left(\frac{1}{K}, \frac{1}{k}\right)} \Lambda_h(t, k, K) \\
&\le kK\left(\frac{1}{k} - \frac{\left\langle p(A)\,(f(A))^{-1} x, x\right\rangle}{\langle p(A)\, x, x\rangle}\right)\left(\frac{\left\langle p(A)\,(f(A))^{-1} x, x\right\rangle}{\langle p(A)\, x, x\rangle} - \frac{1}{K}\right) \\
&\times \frac{K^2\Phi'_-(K) - k^2\Phi'_+(k)}{K-k} \\
&\le \frac{1}{4kK}(K-k)\left[K^2\Phi'_-(K) - k^2\Phi'_+(k)\right]
\end{aligned}
$$

for any $x \in H$, $x \ne 0$, where $\Phi : [k, K] \subset I \to (0, \infty)$ is a continuous HA-convex function on the interval of real numbers I and $f : [m, M] \to [k, K]$, $p : [m, M] \to (0, \infty)$ are continuous functions on $[m, M]$.

In particular, if p is taken to be the constant 1, $[m,M]=[k,K]$ and $f(t)=t$, then we have

$$
\begin{aligned}
0 &\le \langle \Phi(A)x,x\rangle - \Phi\left(\frac{1}{\langle A^{-1}x,x\rangle}\right) \\
&\le \frac{mM}{M-m}\left(\frac{1}{m}-\langle A^{-1}x,x\rangle\right)\left(\langle A^{-1}x,x\rangle-\frac{1}{M}\right) \\
&\times \sup_{t\in\left(\frac{1}{M},\frac{1}{m}\right)} \Lambda_{\Phi}(t,m,M) \\
&\le mM\left(\frac{1}{m}-\langle A^{-1}x,x\rangle\right)\left(\langle A^{-1}x,x\rangle-\frac{1}{M}\right) \\
&\times \frac{M^2\Phi'_{-}(M)-m^2\Phi'_{+}(m)}{M-m} \\
&\le \frac{1}{4mM}(M-m)\left[M^2\Phi'_{-}(M)-m^2\Phi'_{+}(m)\right]
\end{aligned} \tag{5.9}
$$

for any $x\in H$, $\|x\|=1$.

If we take in (5.9) $f(t)=\ln t$, then we get

$$
\begin{aligned}
0 &\le \langle \ln Ax,x\rangle - \ln\left(\frac{1}{\langle A^{-1}x,x\rangle}\right) \\
&\le \frac{mM}{M-m}\left(\frac{1}{m}-\langle A^{-1}x,x\rangle\right)\left(\langle A^{-1}x,x\rangle-\frac{1}{M}\right)\sup_{t\in\left(\frac{1}{M},\frac{1}{m}\right)} \ell_{m,M}(t) \\
&\le mM\left(\frac{1}{m}-\langle A^{-1}x,x\rangle\right)\left(\langle A^{-1}x,x\rangle-\frac{1}{M}\right) \\
&\le \frac{1}{4mM}(M-m)^2
\end{aligned} \tag{5.10}
$$

for any $x\in H$, $\|x\|=1$.

Using the inequality (2.18) we have

$$
\begin{aligned}
0 &\le \frac{\langle \Phi(f(A))p(A)x,x\rangle}{\langle p(A)x,x\rangle} - \Phi\left(\frac{\langle p(A)x,x\rangle}{\left\langle p(A)(f(A))^{-1}x,x\right\rangle}\right) \\
&\le \frac{2kK}{K-k} \\
&\times \max\left\{\frac{1}{k}-\frac{\left\langle p(A)(f(A))^{-1}x,x\right\rangle}{\langle p(A)x,x\rangle}, \frac{\left\langle p(A)(f(A))^{-1}x,x\right\rangle}{\langle p(A)x,x\rangle}-\frac{1}{K}\right\} \\
&\times \left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right] \\
&\le 2\left[\frac{h(k)+h(K)}{2}-h\left(\frac{2kK}{k+K}\right)\right]
\end{aligned} \tag{5.11}
$$

for any $x\in H$, $x\neq 0$, where $\Phi:[k,K]\subset I\to(0,\infty)$ is a continuous HA-convex function on the interval of real numbers I and $f:[m,M]\to[k,K]$, $p:[m,M]\to(0,\infty)$ are continuous functions on $[m,M]$.

In particular, if p is taken to be the constant 1, $[m,M]=[k,K]$ and $f(t)=t$, then we have

$$
\begin{aligned}
(5.12)\qquad 0 &\le \langle \ln Ax, x\rangle - \ln\left(\frac{1}{\langle A^{-1}x,x\rangle}\right) \\
&\le \frac{2kK}{K-k}\max\left\{\frac{1}{k}-\left\langle A^{-1}x,x\right\rangle,\left\langle A^{-1}x,x\right\rangle-\frac{1}{K}\right\}\ln\left(\frac{G(k,K)}{H(k,K)}\right) \\
&\le \ln\left(\frac{G(k,K)}{H(k,K)}\right)^2,
\end{aligned}
$$

for any $x\in H$, $\|x\|=1$.

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