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GENERALIZED OSTROWSKI TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. Firstly, we establish the generalized Ostrowski inequality for local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers. Secondly, we obtain some new inequalities using the generalized convex function on fractal sets R^{α}.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [12]:

Theorem 1 (Ostrowski inequality). *Let $f:[a,b]\rightarrow R$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\rightarrow R$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int\limits_{a}^{b}f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\|f'\|_{\infty}, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.* For more information recent development on Ostrowski inequality, please refer to [1]-[10],[13]-[19] *and so on.*

Definition 1 (Convex function). *The function $f:[a,b]\subset R\rightarrow R$, is said to be convex if the following inequality holds*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Theorem 2 (Hermite-Hadamard inequality). *Let $f:I\subseteq R\rightarrow R$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds* [14]

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.2}$$

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2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 21] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}:$ $p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}:$ $p,q\in Z,\ q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;
(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2. [20] *A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 3. [20] *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 4. [20] *Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 5 (Generalized convex function)**.** [20] *Let* $f : I \subseteq R \to R^{\alpha}$. *For any* $x_1, x_2 \in I$ *and* $\lambda \in [0,1]$, *if the following inequality*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda^{\alpha} f(x_1) + (1-\lambda)^{\alpha} f(x_2)$$

holds, then f *is called a generalized convex function on* I.

Here are two basic examples of generalized convex functions:
(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;
(2) $f(x) = E_{\alpha}(x^{\alpha})$, $x \in R$ where $E_{\alpha}(x^{\alpha}) = \sum\limits_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Lemma 1. [20]
(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha} f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_{\alpha}[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha} f(x) g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_a I_b^{\alpha} f^{(\alpha)}(x) g(x).$$

Lemma 2. [20]

$$\frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right), \ k \in R.$$

Lemma 3 (Generalized Hölder's inequality)**.** [20] *Let* $f, g \in C_{\alpha}[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^{\alpha} \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^{\alpha} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{\frac{1}{q}}.$$

In [11], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:
Let $f(x) \in I_x^{\alpha}[a,b]$ be generalized convex function on $[a,b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}. \tag{2.1}$$

In this paper, we establish the generalized Ostrowski inequality and we obtain some new inequalities using generalized convex function and inequality (2.1).

3. Main Results

We will start the generalized Montgomery identity for local fractional integrals as follow:

Theorem 3 (Generalized Montgomery Identity)**.** *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha [a,b]$ for $a,b \in I^0$ with $a < b$ Then. for all $x \in [a,b]$, we have the identity*

$$f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(t) = \frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b p(x,t) f^{(\alpha)}(t) (dt)^\alpha \tag{3.1}$$

where

$$p(x,t) = \begin{cases} (t-a)^\alpha, & t \in [a,x] \\ (t-b)^\alpha, & t \in (x,b]. \end{cases}$$

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)} \int_a^b p(x,t) f^{(\alpha)}(t) (dt)^\alpha \\
= & \frac{1}{\Gamma(1+\alpha)} \int_a^x (t-a)^\alpha f^{(\alpha)}(t) (dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_x^b (t-b)^\alpha f^{(\alpha)}(t) (dt)^\alpha \\
= & (t-a)^\alpha f(t)\Big|_a^x - \frac{1}{\Gamma(1+\alpha)} \int_a^x \Gamma(1+\alpha) f(t) (dt)^\alpha \\
& + (t-b)^\alpha f(t)\Big|_x^b - \frac{1}{\Gamma(1+\alpha)} \int_x^b \Gamma(1+\alpha) f(t) (dt)^\alpha \\
= & (x-a)^\alpha f(x) - (x-b)^\alpha f(x) - \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \\
= & (b-a)^\alpha f(x) - \Gamma(1+\alpha) \ {}_aI_b^\alpha f(t).
\end{aligned}$$

If we devide the rsulting equality with $(b-a)^\alpha$, then we complete the proof. □

Now, by using generalized Montgomery identity we will obtain generalized Ostrowski inequality for local fractionl integral as follow:

Theorem 4 (Generalized Ostrowski inequality)**.** *Suppose that the assumptions of Theorem 3 are satisfied, then we have the inequality*

$$\left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(t) \right| \leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] (b-a)^\alpha \left\| f^{(\alpha)} \right\|_\infty .$$

Proof. Taking modulus in Theorem 3, we get

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \tag{3.2}$$

$$\leq \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|(dt)^{\alpha}$$

$$= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{a}^{x}(t-a)^{\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{x}^{b}(b-t)^{\alpha}(dt)^{\alpha}\right]$$

$$= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\left[K_1+K_2\right].$$

Using Lemma 2, we have

$$K_1 = \frac{1}{\Gamma(1+\alpha)}\int_{a}^{x}(t-a)^{\alpha}(dt)^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(x-a)^{2\alpha}$$

and

$$K_2 = \frac{1}{\Gamma(1+\alpha)}\int_{x}^{b}(b-t)^{\alpha}(dt)^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b-x)^{2\alpha}.$$

Putting the calculated intagrals K_1 and K_2 in (3.2), then we have

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right|$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[(x-a)^{2\alpha}+(b-x)^{2\alpha}\right]$$

$$= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[(x-a)^{2}+(b-x)^{2}\right]^{\alpha}$$

$$= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{(b-a)^{2\alpha}}{2^{\alpha}}+2^{\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}\right]$$

$$= 2^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{1}{4^{\alpha}}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right](b-a)^{\alpha}\left\|f^{\alpha}\right\|_{\infty}$$

which completes the proof. ☐

Theorem 5. *Suppose that the assumptions of Theorem 3 are satisfied, then we have the inequality*

$$\begin{aligned} &\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \\ \leq\ & \frac{1}{(b-a)^{\alpha}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]^{\frac{1}{p}}\left\|f^{(\alpha)}\right\|_{q}\end{aligned}$$

where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\left\|f^{(\alpha)}\right\|_{q}$ *is defined by*

$$\left\|f^{(\alpha)}\right\|_{q}=\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(\alpha)}(t)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}.$$

Proof. Taking modulus in Theorem 3 and generalized Hölder's inequality (Lemma 3), we obtain

$$\begin{aligned} &\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right| \\ \leq\ & \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha} \\ \leq\ & \frac{1}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}|p(x,t)|^{p}(dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(\alpha)}(t)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}} \\ =\ & \frac{\left\|f^{(\alpha)}\right\|_{q}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{x}(x-a)^{p\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(b-x)^{p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}} \\ =\ & \frac{\left\|f^{(\alpha)}\right\|_{q}}{(b-a)^{\alpha}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]\right)^{\frac{1}{p}}\end{aligned}$$

which completes the proof. ☐

Theorem 6. *Suppose that the assumptions of Theorem 3 are satisfied, then we have the equality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(u)-f(x)=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_{0}^{1}q(t)f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \tag{3.3}$$

where

$$q(t)=\begin{cases} t^{\alpha}, & t\in\left[0,\frac{b-x}{b-a}\right] \\ (t-1)^{\alpha}, & t\in\left(\frac{b-x}{b-a},1\right]. \end{cases}$$

Proof. Using the local fractional integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 q(t)f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{b-x}{b-a}} t^{\alpha}f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^1 (t-1)^{\alpha}f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \\
&= \left.\frac{t^{\alpha}f(ta+(1-t)b)}{(a-b)^{\alpha}}\right|_0^{\frac{b-x}{b-a}}-\frac{1}{\Gamma(1+\alpha)(a-b)^{\alpha}}\int_0^{\frac{b-x}{b-a}}\Gamma(1+\alpha)f(ta+(1-t)b)(dt)^{\alpha} \\
&\quad+\left.\frac{(t-1)^{\alpha}f(ta+(1-t)b)}{(a-b)^{\alpha}}\right|_{\frac{b-x}{b-a}}^1-\frac{1}{\Gamma(1+\alpha)(a-b)^{\alpha}}\int_{\frac{b-x}{b-a}}^1\Gamma(1+\alpha)f(ta+(1-t)b)(dt) \\
&= \frac{(b-x)^{\alpha}}{(b-a)^{\alpha}(a-b)^{\alpha}}f(x)-\frac{(a-x)^{\alpha}}{(b-a)^{\alpha}(a-b)^{\alpha}}f(x)-\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)(a-b)^{\alpha}}\int_0^1 f(ta+(1-t)b)(dt).
\end{aligned}
$$

Changing variables with $u=ta+(1-t)b$ and $du=(a-b)dt$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 q(t)f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha} \\
&= -\frac{f(x)}{(b-a)^{\alpha}}+\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)(a-b)^{2\alpha}}\int_a^b f(u)(du)^{\alpha} \\
&= -\frac{f(x)}{(b-a)^{\alpha}}+\frac{\Gamma(1+\alpha)}{(b-a)^{2\alpha}}\ {}_aI_b^{\alpha}f(u).
\end{aligned}
$$

Multiplying the last equality with $(b-a)^{\alpha}$, we obtain the desired result. □

Theorem 7. *The assumptions of Theorem 6 are satisfied. If $|f^{\alpha}|^q$ is generalized convex on $[a,b]$, then we have the inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(u)-f(x)\right| \qquad (3.4)\\
&\leq \frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}(b-a)^{\frac{\alpha}{p}}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}} \\
&\quad\times\left[(b-x)^{\frac{p+1}{p}\alpha}\left[\frac{|f^{\alpha}(x)|^q+|f^{\alpha}(b)|^q}{2^{\alpha}}\right]^{\frac{1}{q}}+(x-a)^{\frac{p+1}{p}\alpha}\left[\frac{|f^{\alpha}(a)|^q+|f^{\alpha}(x)|^q}{2^{\alpha}}\right]^{\frac{1}{q}}\right]
\end{aligned}
$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking madulus in Theorem 6 and using generalized Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(u)-f(x)\right| \qquad (3.5)\\
&\leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_{0}^{1}|q(t)|\left|f^{(\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\\
&= (b-a)^{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}|t|^{\alpha}\left|f^{(\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\right.\\
&\qquad\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{1}|t-1|^{\alpha}\left|f^{(\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\right]\\
&\leq (b-a)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}|t|^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\right.\\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&\qquad+\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{1}|t-1|^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\qquad\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{1}\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}|t|^{\alpha p}(dt)^{\alpha} &= \frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}t^{\alpha p}(dt)^{\alpha} \qquad (3.6)\\
&= \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left(\frac{b-x}{b-a}\right)^{(p+1)\alpha}
\end{aligned}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{1}|t-1|^{\alpha p}(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha p}(dt)^{\alpha} \tag{3.7}$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{x-a}{b-a}}u^{\alpha p}(du)^{\alpha}$$

$$= \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left(\frac{x-a}{b-a}\right)^{(p+1)\alpha}.$$

Since $|f^{\alpha}|^{q}$ is generalized convex on $[a,b]$, by generalized Hermite-Hadamard inequality (Theorem 2), we have

$$\int_{0}^{\frac{b-x}{b-a}}\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha} = \frac{1}{(b-a)^{\alpha}}\int_{x}^{b}\left|f^{(\alpha)}(u)\right|^{q}(du)^{\alpha} \tag{3.8}$$

$$\leq \frac{|f^{\alpha}(x)|^{q}+|f^{\alpha}(b)|^{q}}{2^{\alpha}},$$

and

$$\int_{\frac{b-x}{b-a}}^{1}\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha} = \frac{1}{(b-a)^{\alpha}}\int_{a}^{x}\left|f^{(\alpha)}(u)\right|^{q}(du)^{\alpha} \tag{3.9}$$

$$\leq \frac{|f^{\alpha}(a)|^{q}+|f^{\alpha}(x)|^{q}}{2^{\alpha}}.$$

Putting the equalities (3.6)-(3.7) and inequalities (3.8)-(3.9) in (3.5), we obtain

$$\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(u)-f(x)\right|$$

$$\leq (b-a)^{\alpha}\left[\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}\alpha}\frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}}\left[\frac{\left|f^{(\alpha)}(x)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{2^{\alpha}}\right]^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}\alpha}\frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}}\left[\frac{\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}(x)\right|^{q}}{2^{\alpha}}\right]^{\frac{1}{q}}\right]$$

$$= \frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}(b-a)^{\frac{\alpha}{p}}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}$$

$$\times\left[(b-x)^{\frac{p+1}{p}\alpha}\left[\frac{\left|f^{(\alpha)}(x)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{2^{\alpha}}\right]^{\frac{1}{q}}+(x-a)^{\frac{p+1}{p}\alpha}\left[\frac{\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}(x)\right|^{q}}{2^{\alpha}}\right]^{\frac{1}{q}}\right]$$

which completes the proof. □

Theorem 8. *The assumptions of Theorem 6 are satisfied. If* $|f^{\alpha}|^{q}$ *is generalized convex on* $[a,b]$*, then we have the inequality*

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(u)-f(x)\right|\\ \leq\ & \frac{1}{(b-a)^{\frac{\alpha}{p}}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\\ &\times\left[(b-x)^{(p+1)\alpha}+(x-a)^{(p+1)\alpha}\right]^{\frac{1}{p}}\left[|f^{\alpha}(a)|^{q}+|f^{\alpha}(b)|^{q}\right]^{\frac{1}{q}}\end{aligned}$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking madulus in Theorem 6 and using generalized Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(u)-f(x)\right|\\ \leq\ & \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_0^1|q(t)|\left|f^{(\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\\ \leq\ & (b-a)^{\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|q(t)|^{p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}.\end{aligned}$$

Using generalized convexity of $|f^{\alpha}|^{q}$, we obtain

$$\begin{aligned}&\left|\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(u)-f(x)\right|\\ \leq\ & (b-a)^{\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{b-x}{b-a}}|t|^{\alpha p}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^1|t-1|^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\ &\times\left(\frac{\left|f^{(\alpha)}(a)\right|^{q}}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(b)\right|^{q}}{\Gamma(1+\alpha)}\int_0^1(1-t)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\ =\ & (b-a)^{\alpha}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[\left(\frac{b-x}{b-a}\right)^{(p+1)\alpha}+\left(\frac{x-a}{b-a}\right)^{(p+1)\alpha}\right]\right)^{\frac{1}{p}}\\ &\times\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[|f^{\alpha}(a)|^{q}+|f^{\alpha}(b)|^{q}\right]\right)^{\frac{1}{p}}.\end{aligned}$$

This completes the proof. □

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