

**SOME GENERALIZED OSTROWSKI TYPE INEQUALITIES
INVOLVING LOCAL FRACTIONAL INTEGRALS AND
APPLICATIONS**

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ABSTRACT. In this paper, we establish the generalized Ostrowski type inequality involving local fractional integrals on fractal sets R^α ($0 < \alpha \leq 1$) of real line numbers. Some applications for special means of fractal sets R^α are also given. The results presented here would provide extensions of those given in earlier works.

1. INTRODUCTION

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [10]:

Theorem 1 (Ostrowski inequality). *Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow R$ is bounded on (a, b) , i.e. $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, we have the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the Ostrowski inequality. For more information recent development on Ostrowski inequality, please refer to [1]-[5], [7], [8] and [11]-[15].

Definition 1 (Convex function). *The function $f : [a, b] \subset R \rightarrow R$, is said to be convex if the following inequality holds*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Theorem 2 (Hermite-Hadamard inequality). *Let $f : I \subseteq R \rightarrow R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite-Hadamard inequality, holds [12]*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

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2. PRELIMINARIES

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [17, 18] and so on.

Recently, the theory of Yang's fractional sets [17] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α -type set of element sets:

Z^α : The α -type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

Q^α : The α -type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

J^α : The α -type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

R^α : The α -type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

- (1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α ;
- (2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha$;
- (3) $a^\alpha + (b^\alpha + c^\alpha) = (a + b)^\alpha + c^\alpha$;
- (4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
- (5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
- (6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
- (7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2. [17] *A non-differentiable function $f : R \rightarrow R^\alpha$, $x \rightarrow f(x)$ is called to be local fractional continuous at x_0 , if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b) , we denote $f(x) \in C_\alpha(a, b)$.

Definition 3. [17] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) (f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \dots$

Definition 4. [17] *Let $f(x) \in C_\alpha[a, b]$. Then the local fractional integral is defined by,*

$${}_a I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t) (dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \dots, N-1$ and $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_a I_b^\alpha f(x) = 0$ if $a = b$ and ${}_a I_b^\alpha f(x) = -{}_b I_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_a I_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha[a, b]$.

Definition 5 (Generalized convex function). [17] Let $f : I \subseteq R \rightarrow R^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I .

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;

(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$ where $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer

function.

Lemma 1. [17]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a, b]$, then we have

$${}_a I_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a, b]$ and $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha[a, b]$, then we have

$${}_a I_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_a I_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2. [17] We have

i) $\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha})$, $k \in R$.

Lemma 3 (Generalized Hölder's inequality). [17] Let $f, g \in C_\alpha[a, b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [9], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 3. Let $f(x) \in I_x^\alpha[a, b]$ be generalized convex function on $[a, b]$ with $a < b$. Then

$$(2.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}.$$

The interested reader is refer to [6],[9],[16]-[21] for local fractional theory.

In this paper, we establish the generalized Ostrowski type inequalities and we obtain some inequalities using generalized convex function. The results presented here would provide extensions of those given in earlier works.

3. MAIN RESULTS

Theorem 4. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a, b]$ for $a, b \in I^0$ with $a < b$. Then, for all $x \in [a, b]$, we have the identity*

$$\begin{aligned} & (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \\ &= \frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b p(x, t) f^{(\alpha)}(t) (dt)^\alpha \end{aligned}$$

where

$$p(x, t) = \begin{cases} (t - (a + h\frac{b-a}{2}))^\alpha, & t \in [a, x] \\ (t - (b - h\frac{b-a}{2}))^\alpha, & t \in (x, b] \end{cases}$$

for all $h \in [0, 1]$ and $a + h\frac{b-a}{2} \leq x \leq b - h\frac{b-a}{2}$.

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_a^b p(x, t) f^{(\alpha)}(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(t - \left(a + h\frac{b-a}{2} \right) \right)^\alpha f^{(\alpha)}(t) (dt)^\alpha \\ & \quad + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left(t - \left(b - h\frac{b-a}{2} \right) \right)^\alpha f^{(\alpha)}(t) (dt)^\alpha \\ &= \left(t - \left(a + h\frac{b-a}{2} \right) \right)^\alpha f(t) \Big|_a^x - \frac{1}{\Gamma(1+\alpha)} \int_a^x \Gamma(1+\alpha) f(t) (dt)^\alpha \\ & \quad + \left(t - \left(b - h\frac{b-a}{2} \right) \right)^\alpha f(t) \Big|_x^b - \frac{1}{\Gamma(1+\alpha)} \int_x^b \Gamma(1+\alpha) f(t) (dt)^\alpha \\ &= (b-a)^\alpha (1-h)^\alpha f(x) + \left(h\frac{b-a}{2} \right)^\alpha [f(a) + f(b)] - \Gamma(1+\alpha) {}_a I_b^\alpha f(t). \end{aligned}$$

If we divide the resulting equality with $(b-a)^\alpha$, then we complete the proof. \square

Corollary 1. *Under the same assumptions of Theorem 4 with $h = 1$, then the following equality holds:*

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b p(x, t) f^{(\alpha)}(t) (dt)^\alpha = \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t).$$

Remark 1. If we choose $h = 0$ in Theorem 4, then we have the following equality

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b p(x,t) f^{(\alpha)}(t) (dt)^\alpha = f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t)$$

which is given by Sarikaya and Budak in [16].

Theorem 5. Suppose that the assumptions of Theorem 4 are satisfied, then we have the inequality

$$(3.1) \quad \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ \leq \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[2^\alpha \left(x - \frac{a+b}{2} \right)^{2\alpha} + \frac{(b-a)^{2\alpha}}{2^\alpha} [(h-1)^{2\alpha} + h^{2\alpha}] \right]$$

for all $h \in [0, 1]$ and $a + h\frac{b-a}{2} \leq x \leq b - h\frac{b-a}{2}$.

Proof. Taking modulus in Theorem 4, we get

$$(3.2) \quad \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ \leq \frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b |p(x,t)| |f^{(\alpha)}(t)| (dt)^\alpha \\ \leq \frac{\|f^{(\alpha)}\|_\infty}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b |p(x,t)| (dt)^\alpha \\ = \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \left[\frac{1}{\Gamma(1+\alpha)} \int_a^x \left| t - \left(a + h\frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \right. \\ \left. + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left| t - \left(b - h\frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \right] \\ = \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} [K_1 + K_2].$$

Using Lemma 2, we have

$$\begin{aligned}
K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left| t - \left(a + h \frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_a^{a+h\frac{b-a}{2}} \left(a + h \frac{b-a}{2} - t \right)^\alpha (dt)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} \int_{a+h\frac{b-a}{2}}^x \left(t - \left(a + h \frac{b-a}{2} \right) \right)^\alpha (dt)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h \frac{b-a}{2} \right)^{2\alpha} + \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{2\alpha} \right]
\end{aligned}$$

and similarly

$$\begin{aligned}
K_2 &= \frac{1}{\Gamma(1+\alpha)} \int_x^b \left| t - \left(b - h \frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h \frac{b-a}{2} \right)^{2\alpha} + \left(b - h \frac{b-a}{2} - x \right)^{2\alpha} \right].
\end{aligned}$$

Substituting the calculated integrals K_1 and K_2 in (3.2), then we have

$$\begin{aligned}
&\left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\
&\leq \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h \frac{b-a}{2} \right)^{2\alpha} + \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{2\alpha} \right. \\
&\quad \left. + \left(h \frac{b-a}{2} \right)^{2\alpha} + \left(b - h \frac{b-a}{2} - x \right)^{2\alpha} \right] \\
&= \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[2^\alpha \left(h \frac{b-a}{2} \right)^{2\alpha} + 2^\alpha \left(x - \frac{a+b}{2} \right)^{2\alpha} \right. \\
&\quad \left. + \frac{(b-a)^{2\alpha}}{2^\alpha} - h^\alpha (b-a)^{2\alpha} + h^{2\alpha} (b-a)^{2\alpha} \right] \\
&= \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[2^\alpha \left(x - \frac{a+b}{2} \right)^{2\alpha} + \frac{(b-a)^{2\alpha}}{2^\alpha} [(h-1)^{2\alpha} + h^{2\alpha}] \right]
\end{aligned}$$

which completes the proof. \square

Remark 2. If we choose $h = 0$ in Theorem 5, then we have the following inequality

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \frac{\left(x - \frac{a+b}{2}\right)^{2\alpha}}{(b-a)^{2\alpha}} \right] (b-a)^\alpha \|f^{(\alpha)}\|_\infty \end{aligned}$$

which is given by Sarikaya and Budak in [16].

Remark 3. If we choose $\alpha = 1$ in Theorem 5, then we have the following inequality

$$\begin{aligned} & \left| (1-h)f(x) + h \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \\ & \leq \frac{\|f'\|_\infty}{b-a} \left[\frac{1}{4} (b-a)^2 \left[(h-1)^2 + h^2 \right] + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned}$$

which is proved by Dragomir et al. in [8].

Corollary 2. Under the same assumptions of Theorem 5 with $h = 1$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \leq \frac{(b-a)^\alpha}{2^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \|f^{(\alpha)}\|_\infty.$$

Theorem 6. Suppose that the assumptions of Theorem 4 are satisfied, then we have the inequality

$$\begin{aligned} & \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{\|f^{(\alpha)}\|_p}{(b-a)^\alpha} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \times \left[2 \left(h \frac{b-a}{2} \right)^{(q+1)} + \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{(q+1)} + \left(b - h \frac{b-a}{2} - x \right)^{(q+1)} \right]^{\frac{\alpha}{q}} \end{aligned}$$

for all $h \in [0, 1]$ and $a + h \frac{b-a}{2} \leq x \leq b - h \frac{b-a}{2}$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|f^{(\alpha)}\|_p$ is defined by

$$\|f^{(\alpha)}\|_p = \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(\alpha)}(t)|^p (dt)^\alpha \right)^{\frac{1}{p}}$$

Proof. Taking madulus in Theorem 4 and using generalized Hölder's inequality, we have

$$\begin{aligned}
& \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\
& \leq \frac{1}{\Gamma(1+\alpha)(b-a)^\alpha} \int_a^b |p(x,t)| |f^{(\alpha)}(t)| (dt)^\alpha \\
& \leq \frac{1}{(b-a)^\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(\alpha)}(t)|^p (dt)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |p(x,t)|^q (dt)^\alpha \right)^{\frac{1}{q}} \\
& \leq \frac{\|f^{(\alpha)}\|_p}{(b-a)^\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |p(x,t)|^q (dt)^\alpha \right)^{\frac{1}{q}} = \frac{\|f^{(\alpha)}\|_p}{(b-a)^\alpha} (K_3)^{\frac{1}{q}}.
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
(3.3) \quad K_3 &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left| t - \left(a + h \frac{b-a}{2} \right) \right|^{\alpha q} (dt)^\alpha \\
&+ \frac{1}{\Gamma(1+\alpha)} \int_x^b \left| t - \left(b - h \frac{b-a}{2} \right) \right|^{\alpha q} (dt)^\alpha \\
&= \frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \left[2^\alpha \left(h \frac{b-a}{2} \right)^{(q+1)\alpha} \right. \\
&\quad \left. + \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{(q+1)\alpha} + \left(b - h \frac{b-a}{2} - x \right)^{(q+1)\alpha} \right].
\end{aligned}$$

Hence, the proof is completed. \square

Remark 4. If we choose $h = 0$ in Theorem 5, then we have the following inequality

$$\begin{aligned}
& \left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\
& \leq \frac{\|f^{(\alpha)}\|_p}{(b-a)^\alpha} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \left[(x-a)^{(q+1)\alpha} + (b-x)^{(q+1)\alpha} \right]^{\frac{1}{q}}
\end{aligned}$$

which is given by Sarikaya and Budak in [16].

Corollary 3. Under the same assumptions of Theorem 5 with $h = 1$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \leq \frac{(b-a)^{\frac{\alpha}{q}}}{2^\alpha} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \|f^{(\alpha)}\|_p.$$

Theorem 7. *Suppose that the assumptions of Theorem 4 are satisfied. If $|f^{(\alpha)}|^p$ is generalized convex, then we have the inequality*

$$\begin{aligned} & \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{1}{(b-a)^{\frac{\alpha}{q}}} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \left(\left[|f^{(\alpha)}(b)|^p + |f^{(\alpha)}(a)|^p \right] \right)^{\frac{1}{p}} \\ & \quad \times \left[2 \left(h \frac{b-a}{2} \right)^{(q+1)} + \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{(q+1)} + \left(b - h \frac{b-a}{2} - x \right)^{(q+1)} \right]^{\frac{\alpha}{q}} \end{aligned}$$

for all $h \in [0, 1]$ and $a + h \frac{b-a}{2} \leq x \leq b - h \frac{b-a}{2}$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking modulus in Theorem 4 and using generalized Hölder's inequality, we have

$$\begin{aligned} (3.4) \quad & \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{1}{\Gamma(1+\alpha) (b-a)^\alpha} \int_a^b |p(x, t)| |f^{(\alpha)}(t)| (dt)^\alpha \\ & \leq \frac{1}{(b-a)^\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(\alpha)}(t)|^p (dt)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |p(x, t)|^q (dt)^\alpha \right)^{\frac{1}{q}} \end{aligned}$$

Because of the generalized convexity of $|f^{(\alpha)}|^p$, we find that

$$\begin{aligned} (3.5) \quad & \frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(\alpha)}(t)|^p (dt)^\alpha \\ & \leq \frac{|f^{(\alpha)}(b)|^p}{\Gamma(1+\alpha) (b-a)^\alpha} \int_a^b (t-a)^\alpha (dt)^\alpha + \frac{|f^{(\alpha)}(a)|^p}{\Gamma(1+\alpha) (b-a)^\alpha} \int_a^b (b-t)^\alpha (dt)^\alpha \\ & = \frac{|f^{(\alpha)}(b)|^p}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} (b-a)^{2\alpha} + \frac{|f^{(\alpha)}(a)|^p}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} (b-a)^{2\alpha} \\ & = (b-a)^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[|f^{(\alpha)}(b)|^p + |f^{(\alpha)}(a)|^p \right] \end{aligned}$$

If we substitute the inequalities (3.5) and (3.3) in (3.4), then we obtain required inequality, which completes the proof. \square

Corollary 4. *If we choose $h = 0$ in Theorem 5, then we have the following inequality*

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{1}{(b-a)^{\frac{\alpha}{q}}} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \times \left[(x-a)^{(q+1)} + (b-x)^{(q+1)} \right]^{\frac{\alpha}{q}} \left(\left[|f^{(\alpha)}(b)|^p + |f^{(\alpha)}(a)|^p \right] \right)^{\frac{1}{p}} \end{aligned}$$

which is given by Sarikaya and Budak in [16].

Corollary 5. *Under the same assumptions of Theorem 5 with $h = 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \left(\left[|f^{(\alpha)}(b)|^p + |f^{(\alpha)}(a)|^p \right] \right)^{\frac{1}{p}}. \end{aligned}$$

4. APPLICATIONS TO SOME SPECIAL MEANS

Let us recall some generalized means:

$$A(a, b) = \frac{a^\alpha + b^\alpha}{2^\alpha};$$

$$L_n(a, b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left[\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^\alpha} \right] \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad a, b \in \mathbb{R}, \quad a \neq b.$$

Now, let us reconsider the inequality (3.1):

$$\begin{aligned} & \left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(t) \right| \\ & \leq \frac{\|f^{(\alpha)}\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[2^\alpha \left(x - \frac{a+b}{2} \right)^{2\alpha} + \frac{(b-a)^{2\alpha}}{2^\alpha} [(h-1)^{2\alpha} + h^{2\alpha}] \right] \end{aligned}$$

for all $h \in [0, 1]$ and $a + h\frac{b-a}{2} \leq x \leq b - h\frac{b-a}{2}$.

Consider the mapping $f : (0, \infty) \rightarrow \mathbb{R}^\alpha$, $f(x) = x^{n\alpha}$, $n \in \mathbb{Z} \setminus \{-1, 0\}$. Then, $0 < a < b$, we have

$$\frac{f(a) + f(b)}{2^\alpha} = A(a^n, b^n) \quad \text{and} \quad \frac{1}{(b-a)^\alpha} {}_a I_b^\alpha f(t) = [L_n(a, b)]^n,$$

$$\|f^{(\alpha)}\|_\infty = \begin{cases} \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \right| b^{(n-1)\alpha}, & n > 1 \\ \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \right| a^{(n-1)\alpha}, & n \in (-\infty, 1] \setminus \{-1, 0\}, \end{cases}$$

and then, we find that

$$\begin{aligned} & |(1-h)^\alpha x^{\alpha n} + h^\alpha A(a^n, b^n) - \Gamma(1+\alpha) [L_n(a, b)]^n| \\ & \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)(b-a)^\alpha} \left[2^\alpha \left(x - \frac{a+b}{2} \right)^{2\alpha} + \frac{(b-a)^{2\alpha}}{2^\alpha} [(h-1)^{2\alpha} + h^{2\alpha}] \right] \delta_n(a, b) \end{aligned}$$

where

$$\delta_n(a, b) = \begin{cases} \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \right| b^{(n-1)\alpha}, & n > 1 \\ \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} \right| a^{(n-1)\alpha}, & n \in (-\infty, 1] \setminus \{-1, 0\}, \end{cases}$$

and $h \in [0, 1]$, $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$.

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