Received 15/05/15

ON GENERALIZED HERMITE-HADAMARD INEQUALITY FOR GENERALIZED CONVEX FUNCTION

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Abstract. In this paper, a new inequality for generalized convex functions which is related to the left side of generalized Hermite- Hadamard type inequality for generalized convex function is obtained. Some application for some generalized special means are also given.

1. Introduction

Definition 1 (Convex function). *The function $f:[a,b]\subset R\rightarrow R$, is said to be convex if the following inequality holds*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Theorem 1 (Hermite-Hadamard inequality). *Let $f:I\subseteq R\rightarrow R$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds* [4]

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [5], Sarikaya et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1. *Let $f:I^{\circ}\subset\mathbb{R}\rightarrow\mathbb{R}$ be twice differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$. If $f''\in L_1[a,b]$, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\\&=\frac{(b-a)^2}{4}\int_0^1 n(t)\left[f''(ta+(1-t)b)+f''(tb+(1-t)a)\right]dt,\end{aligned} \tag{1.2}$$

where

$$n(t):=\begin{cases}t^2 & ,t\in[0,\frac{1}{2})\\ (1-t)^2 & ,t\in[\frac{1}{2},1].\end{cases}$$

Also,one of the the main inequality in [5], pointed out as follows:

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. Genertalized Hermite-Hadamard inequality, Generalized Hölder inequality, Generalized convex functions.

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a,b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f(\frac{a+b}{2}) \right| \leq \frac{(b-a)^2}{8\,(2p+1)^{1/p}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{1/q} \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [6, 7] and so on.

Recently, the theory of Yang's fractional sets [6] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z,\ q \neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z,\ q \neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;
(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha)\, c^\alpha$;
(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2. [6] *A non-differentiable function $f : R \to R^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x) \in C_\alpha(a,b)$.

Definition 3. [6] *The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha+1)\,(f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha ... D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 4. [6] *Let $f(x) \in C_\alpha[a,b]$. Then the local fractional integral is defined by,*

$$ {}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, $$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ if $a = b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a < b$. If for any $x \in [a,b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha[a,b]$.

Definition 5 (Generalized convex function)**.** [6] *Let $f : I \subseteq R \to R^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if the following inequality*

$$ f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) $$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$, $x \ge 0$, $p > 1$;

(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$ where $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 3. *Let $f \in D_\alpha(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1, x_2 \in I$,

$$ f(x_2) - f(x_1) \ge \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}(x_2 - x_1)^\alpha. $$

Corollary 1. *Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$ f^{(2\alpha)}(x) \ge 0 \left(or\ f^{(2\alpha)}(x) \le 0 \right) $$

for all $x \in (a,b)$.

Lemma 2. [6]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x) = g(b) - g(a). $$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x). $$

Lemma 3. [6]

$$ \frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}; $$

$$ \frac{1}{\Gamma(\alpha+1)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right),\ k \in R. $$

Lemma 4 (Generalized Hölder's inequality)**.** [6] *Let $f, g \in C_\alpha[a,b]$, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\frac{1}{\Gamma(\alpha+1)}\int_a^b |f(x)g(x)|\,(dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)}\int_a^b |g(x)|^q\,(dx)^\alpha\right)^{\frac{1}{q}}.$$

In [2], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 4 (Gereneralized Hermite-Hadamard's inequality)**.** *Let $f(x) \in I_x^\alpha[a,b]$ be generalized convex function on $[a,b]$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}.$$

The interested reader is refer to [1],[2],[3],[6]-[10] for local freactional theory.

The aim of the paper is to establish some new inequality for generalized convex functions which is related to the left side of generalized Hermite- Hadamard type inequality and apply them for some generalized special means.

3. Main Results

We will start the generalized identity for local fractional integrals as follow:

Theorem 5. *Let $I \subseteq R$ be an interval, $f: I^0 \subseteq R \to R^\alpha$ (I^0 is the interior of I) such that $f \in D_{2\alpha}(I^0)$ and $f^{(2\alpha)} \in C_{2\alpha}[a,b]$ for $a, b \in I^0$ with $a<b$. Then, for all $x \in [a,b]$, we have the identity*

$$\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \tag{3.1}$$

$$= \frac{(b-a)^{2\alpha}}{2^\alpha\Gamma(1+\alpha)}\int_0^1 m(t)\left[f^{(2\alpha)}(ta+(1-t)b)+f^{(2\alpha)}(tb+(1-t)a)\right](dt)^\alpha$$

where

$$m(t)=\begin{cases} t^{2\alpha}, & t\in\left[a,\frac{1}{2}\right] \\ (1-t)^{2\alpha}, & t\in\left(\frac{1}{2},b\right]. \end{cases}$$

Proof. From definition of mapping $m(t)$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 m(t)\left[f^{(2\alpha)}(ta+(1-t)b)+f^{(2\alpha)}(tb+(1-t)a)\right](dt)^\alpha \tag{3.2}$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)\,(dt)^\alpha$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}} t^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)\,(dt)^\alpha$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}f^{(2\alpha)}(ta+(1-t)b)(dt)^{\alpha}$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}f^{(2\alpha)}(tb+(1-t)a)(dt)^{\alpha}$$
$$= K_1+K_2+K_3+K_4.$$

Using the local fractional integration by parts twice (Lemma 2), we have

$$K_1=\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}t^{2\alpha}f^{(2\alpha)}(ta+(1-t)b)(dt)^{\alpha} \tag{3.3}$$
$$= \left.\frac{t^{2\alpha}f^{(\alpha)}(ta+(1-t)b)}{(a-b)^{\alpha}}\right|_0^{\frac{1}{2}}$$
$$-\frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}t^{\alpha}f^{(\alpha)}(ta+(1-t)b)(dt)^{\alpha}$$
$$= \frac{1}{2^{2\alpha}(a-b)^{\alpha}}f^{(\alpha)}\left(\frac{a+b}{2}\right)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left.\frac{t^{\alpha}f(ta+(1-t)b)}{(a-b)^{2\alpha}}\right|_0^{\frac{1}{2}}$$
$$+\frac{1}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\Gamma(1+2\alpha)f(ta+(1-t)b)(dt)^{\alpha}$$
$$= -\frac{1}{2^{2\alpha}(b-a)^{\alpha}}f^{(\alpha)}\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha}(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)$$
$$+\frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}f(ta+(1-t)b)(dt)^{\alpha}.$$

Similarly, we have

$$K_2=\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}t^{2\alpha}f^{(2\alpha)}(tb+(1-t)a)(dt)^{\alpha} \tag{3.4}$$
$$= \frac{1}{2^{2\alpha}(b-a)^{\alpha}}f^{(\alpha)}\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha}(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)$$
$$+\frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}f(tb+(1-t)a)(dt)^{\alpha}.$$

Morover, using the local fractional integration by parts twice (Lemma 2), we have

$$
\begin{aligned}
(3.5)\ K_3 &= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha} f^{(2\alpha)}(ta+(1-t)b)\,(dt)^{\alpha} \\
&= \left.\frac{(1-t)^{2\alpha} f^{(\alpha)}(ta+(1-t)b)}{(a-b)^{\alpha}}\right|_{\frac{1}{2}}^{1} - \frac{1}{(a-b)^{\alpha}\Gamma(1+\alpha)} \\
&\quad \times \int_{\frac{1}{2}}^{1}(-1)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(1-t)^{\alpha} f^{(\alpha)}(ta+(1-t)b)\,(dt)^{\alpha} \\
&= -\frac{1}{2^{2\alpha}(a-b)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) \\
&\quad - (-1)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left.\frac{(1-t)^{\alpha} f(ta+(1-t)b)}{(a-b)^{2\alpha}}\right|_{\frac{1}{2}}^{1} \\
&\quad + \frac{1}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(-1)^{2\alpha}\Gamma(1+2\alpha) f(ta+(1-t)b)\,(dt)^{\alpha} \\
&= \frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) + \frac{(-1)^{\alpha}}{2^{\alpha}(a-b)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(a-b)^{2\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1} f(ta+(1-t)b)\,(dt)^{\alpha} \\
&= \frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1} f(ta+(1-t)b)\,(dt)^{\alpha}.
\end{aligned}
$$

Similarliy, we have

$$
\begin{aligned}
(3.6) K_4 &= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha} f^{(2\alpha)}(tb+(1-t)a)\,(dt)^{\alpha} \\
&= -\frac{1}{2^{2\alpha}(b-a)^{\alpha}} f^{(\alpha)}\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha}(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1} f(tb+(1-t)a)\,(dt)^{\alpha}.
\end{aligned}
$$

Putting equality (3.3)-(3.6) in (3.2), we obtain

$$\begin{aligned}(3.7)\quad &\frac{1}{\Gamma(1+\alpha)}\int_0^1 m(t)\left[f^{(2\alpha)}(ta+(1-t)b)+f^{(2\alpha)}(tb+(1-t)a)\right](dt)^\alpha\\ &= K_1+K_2+K_3+K_4\\ &= \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta+(1-t)b)\,(dt)^\alpha\right.\\ &\left.+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(tb+(1-t)a)\,(dt)^\alpha\right]-\frac{4^\alpha}{2^\alpha(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\\ &= \frac{2^\alpha\Gamma(1+2\alpha)}{(b-a)^{3\alpha}}\,{}_aI_b^\alpha f(t)-\frac{2^\alpha}{(b-a)^{2\alpha}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right).\end{aligned}$$

If we multiply the resulting equality (3.7) with $\frac{(b-a)^{2\alpha}}{2^\alpha}$, then we obtain the desired result. □

Remark 1. *If we assume that $\alpha=1$, then the identity (3.1) reduces the identity (1.2).*

Theorem 6. *The assumptions of Theorem 5 are satisfied. If $\left|f^{(2\alpha)}\right|$ is generalized convex on $[a,b]$, then we have the inequality*

$$\begin{aligned}(3.8)\quad &\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^{2\alpha}}{8^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right].\end{aligned}$$

Proof. Taking modulus in (3.1), we find that

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^{2\alpha}}{2^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha\right.\\ &\left.+\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|(dt)^\alpha\right].\end{aligned}$$

Since $\left|f^{(2\alpha)}\right|$ is generalized convex on $[a,b]$, then we have

$$\left|f^{(2\alpha)}(ta+(1-t)b)\right|\le t^\alpha\left|f^{(2\alpha)}(a)\right|+(1-t)^\alpha\left|f^{(2\alpha)}(b)\right|$$

and

$$\left|f^{(2\alpha)}(tb+(1-t)a)\right|\le t^\alpha\left|f^{(2\alpha)}(b)\right|+(1-t)^\alpha\left|f^{(2\alpha)}(a)\right|.$$

Then, we get

$$\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \tag{3.9}$$

$$\leq \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}t^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|\right](dt)^{\alpha}\right.$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(a)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(b)\right|\right](dt)^{\alpha}$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}t^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(b)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|\right](dt)^{\alpha}$$

$$\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}\left[t^{\alpha}\left|f^{(2\alpha)}(b)\right|+(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|\right](dt)^{\alpha}\right]$$

$$= \frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{\Gamma(1+\alpha)}\left(\int_{0}^{\frac{1}{2}}t^{2\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}t^{\alpha}(dt)^{\alpha}\right.\right.$$

$$\left.\left.+\int_{0}^{\frac{1}{2}}t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha}+\int_{\frac{1}{2}}^{1}(1-t)^{3\alpha}(dt)^{\alpha}\right)\right].$$

Usin Lemma 3, we obtain

$$\int_{0}^{\frac{1}{2}}t^{2\alpha}(dt)^{\alpha}=\int_{\frac{1}{2}}^{1}(1-t)^{3\alpha}(dt)^{\alpha}=\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{1}{16}\right)^{\alpha} \tag{3.10}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1}(1-t)^{2\alpha}t^{\alpha}(dt)^{\alpha} &= \int_{0}^{\frac{1}{2}}t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha} \\ &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^{\alpha}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{1}{16}\right)^{\alpha}.\end{aligned} \tag{3.11}$$

Substituting the equalities (3.10) and (3.11) in (3.9), we obtain desired inequality, which completes the proof. □

Remark 2. *If we assume that $\alpha = 1$, then the inequality (3.8) reduces the following inequality .*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]$$

which was proved by Sarikaya et al. in [5].

Theorem 7. *The assumptions of Theorem 5 are satisfied. If $\left|f^{(2\alpha)}\right|^q, q > 1$ is generalized convex on $[a,b]$, then we have the inequality*

$$\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \tag{3.12}$$

$$\leq \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left[\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|\right]^{\frac{1}{q}}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus in (3.1) and using generalized Hölder's inequality (Lemma 4), we have

$$\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \tag{3.13}$$

$$\leq \frac{(b-a)^{2\alpha}}{2^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha\right.$$

$$\left.+\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|(dt)^\alpha\right]$$

$$\leq \frac{(b-a)^{2\alpha}}{2^\alpha}$$

$$\times\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^\alpha\right)^{\frac{1}{p}}\right.$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(ta+(1-t)b)\right|^q (dt)^\alpha\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^\alpha\right)^{\frac{1}{p}}$$

$$\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(tb+(1-t)a)\right|^q (dt)^\alpha\right)^{\frac{1}{q}}\right].$$

Since $\left|f^{(2\alpha)}\right|^q$ is generalized convex on $[a,b]$, then we have

$$\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q \leq t^\alpha \left|f^{(2\alpha)}(a)\right|^q + (1-t)^\alpha \left|f^{(2\alpha)}(b)\right|^q$$

and

$$\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q \leq t^\alpha \left|f^{(2\alpha)}(b)\right|^q + (1-t)^\alpha \left|f^{(2\alpha)}(a)\right|^q .$$

It follows that,

$$\begin{aligned}
(3.14)\qquad & \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(ta+(1-t)b)\right|^q (dt)^\alpha \\
\leq\ & \left|f^{(2\alpha)}(a)\right|^q \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^\alpha (dt)^\alpha + \left|f^{(2\alpha)}(b)\right|^q \int_0^1 (1-t)^\alpha (dt)^\alpha \\
=\ & \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right],
\end{aligned}$$

and similarly,

$$\begin{aligned}
(3.15)\qquad & \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|f^{(2\alpha)}(tb+(1-t)a)\right|^q (dt)^\alpha \\
=\ & \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right].
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
(3.16)\qquad & \frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|^p (dt)^\alpha \\
=\ & \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2p\alpha} (dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^{2p\alpha} (dt)^\alpha \\
=\ & \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{2^{(2p+1)\alpha}} + \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{2^{(2p+1)\alpha}} \\
=\ & \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\frac{1}{4^{p\alpha}}.
\end{aligned}$$

Adding (3.14)-(3.16) in (3.13), we obtain desired result. □

Remark 3. *If we assume that $\alpha = 1$, then the inequality (3.12) reduces the inequality (1.3).*

Theorem 8. *The assumptions of Theorem 5 are satisfied. If* $\left|f^{(2\alpha)}\right|^q, q \geq 1$ *is generalized convex on* $[a,b]$, *then we have the inequality*

$$\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{2\alpha}}{4^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{2^{\alpha}}\right]^{\frac{1}{q}}. \tag{3.17}$$

Proof. Taking modulus in (3.1), we have

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^{\alpha}\right.\\ &\left.+\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right].\end{aligned} \tag{3.18}$$

Because of $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\left(\frac{1}{p}+\frac{1}{q}\right)$ can be written instead of α. Using the generalized Holder's inequality, we find that

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|(dt)^{\alpha}\right)^{\frac{1}{p}}\\ &\times\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}\right].\end{aligned}$$

If $\left|f^{(2\alpha)}\right|^q$ is generalized convex on $[a,b]$, then we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|^q (dt)^\alpha \\
\leq\ & \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}} t^{2\alpha}\left[t^\alpha\left|f^{(2\alpha)}(a)\right|^q+(1-t)^\alpha\left|f^{(2\alpha)}(b)\right|^q\right](dt)^\alpha \\
&+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1 (1-t)^{2\alpha}\left[t^\alpha\left|f^{(2\alpha)}(a)\right|^q+(1-t)^\alpha\left|f^{(2\alpha)}(b)\right|^q\right](dt)^\alpha \\
=\ & \frac{\left|f^{(2\alpha)}(a)\right|^q}{\Gamma(1+\alpha)}\left[\int_0^{\frac{1}{2}} t^{3\alpha}(dt)^\alpha+\int_{\frac{1}{2}}^1 (1-t)^{2\alpha}t^\alpha(dt)^\alpha\right] \\
&+\frac{\left|f^{(2\alpha)}(b)\right|^q}{\Gamma(1+\alpha)}\left[\int_0^{\frac{1}{2}} t^{2\alpha}(1-t)^\alpha(dt)^\alpha+\int_{\frac{1}{2}}^1 (1-t)^{3\alpha}(dt)^\alpha\right] \\
=\ & \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^\alpha\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right].
\end{aligned}$$

Similarliy, we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\left|f^{(2\alpha)}(tb+(1-t)a)\right|^q (dt)^\alpha \\
\leq\ & \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{8}\right)^\alpha\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\ {}_aI_b^\alpha f(t)-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{2\alpha}}{2^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(\frac{1}{4}\right)^\alpha\right)^{\frac{1}{p}} \\
&\times\left[2^\alpha\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{\frac{1}{q}}\left(\frac{1}{8}\right)^{\frac{\alpha}{q}}\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right]^{\frac{1}{q}}\right] \\
=\ & \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{2^\alpha}\right]^{\frac{1}{q}}.
\end{aligned}$$

Here, we used the fact that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 |m(t)|\,(dt)^\alpha = \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left(\frac{1}{4}\right)^\alpha,$$

which completes the proof. □

Remark 4. *If we assume that $\alpha = 1$, then the inequality (3.8) reduces the following inequality .*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]$$

which was proved by Sarikaya et al. in [5].

4. Applications to Some Special Means

We consider some generalized means as in [5]:

$$A(a,b) = \frac{a^\alpha + b^\alpha}{2^\alpha};$$

$$L_n(a,b) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b-a)^\alpha}\right]\right]^{\frac{1}{n}},\ n \in Z\backslash\{-1,0\},\ a,b \in R,\ a \neq b.$$

Proposition 1. *Let $a, b \in R$, $0 < a < b$, $0 \notin [a,b]$ and $n \in Z$, $|n(n-1)| \geq 3$. Then, we have the inequality*

$$\begin{aligned}&\left|\Gamma(1+2\alpha)\left[L_n(a,b)\right]^n - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right| \\ \leq\ & \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| A(a^{n-2}, b^{n-2}).\end{aligned}$$

Proof. Let us reconsider the inequality (3.8):

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{(b-a)^\alpha}\ {}_aI_b^\alpha f(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)^{2\alpha}}{8^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[\left|f^{(2\alpha)}(a)\right| + \left|f^{(2\alpha)}(b)\right|\right].\end{aligned}$$

Consider the mapping $f : (0,\infty) \to R^\alpha$, $f(x) = x^{n\alpha}$, $n \in Z\backslash\{-1,0\}$. Then, $0 < a < b$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= A^n(a,b)\ , \frac{{}_aI_b^\alpha f(t)}{(b-a)^\alpha} = \left[L_n(a,b)\right]^n, \\ \left|f^{(2\alpha)}(a)\right| &= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| a^{(n-2)\alpha}\end{aligned}$$

and

$$\left|f^{(2\alpha)}(b)\right| = \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right| b^{(n-2)\alpha}.$$

Then, we obtain

$$\begin{aligned}&\left|\Gamma(1+2\alpha)\left[L_n(a,b)\right]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\ \leq\;& \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|\left[\frac{a^{(n-2)\alpha}+b^{(n-2)\alpha}}{2^\alpha}\right]\\ =\;& \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|A(a^{n-2},b^{n-2}).\end{aligned}$$

This completes the proof. □

Proposition 2. *Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geq 3$. Then, we have the inequality*

$$\begin{aligned}&\left|\Gamma(1+2\alpha)\left[L_n(a,b)\right]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\ \leq\;& \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[A(a^{n-2},b^{n-2})\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. From Theorem 8 with $f(x)=x^{n\alpha}$, $f:(0,\infty)\to R^\alpha$ and the above equalities, we have

$$\begin{aligned}&\left|\Gamma(1+2\alpha)\left[L_n(a,b)\right]^n-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}A^n(a,b)\right|\\ \leq\;& \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[\frac{a^{(n-2)\alpha}+b^{(n-2)\alpha}}{2^\alpha}\right]^{\frac{1}{q}}.\\ =\;& \frac{(b-a)^{2\alpha}}{4^\alpha}\left(\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[A(a^{n-2},b^{n-2})\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

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