

SOME INTEGRAL INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA, SAMET ERDEN, AND HÜSEYİN BUDAK

ABSTRACT. In this paper, firstly we extend some generalization of the Hermite-Hadamard inequality and Bullen inequality to generalized convex functions. Then, we give some important integral inequalities related to these inequalities.

1. INTRODUCTION

Definition 1 (Convex function). *The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The classical Hermite-Hadamard inequality which was first published in [8] gives us an estimate of the mean value of a convex function $f : I \rightarrow \mathbb{R}$,

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

In [1], Bullen proved the following inequality which is known as Bullen's inequality for convex function.

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right].$$

An account the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [12] and [2]. Recently in [5], the author established this inequality for twice differentiable functions. In the case where f is convex then there exists an estimation better than (1.1).

In [6], Farissi gave the refinement of the inequality (1.1) as follows:

Theorem 1. *Assume that $f : I \rightarrow \mathbb{R}$ is a convex function on I . Then for all $\lambda \in [0, 1]$, we have*

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq L(\lambda) \leq \frac{f(a) + f(b)}{2},$$

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where

$$l(\lambda) := \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) := \frac{1}{2}(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda)f(b)).$$

For more information recent developments to above inequalities, please refer to [2]-[7], [9]-[11], [14] and so on.

2. PRELIMINARIES

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [15, 16] and so on.

Recently, the theory of Yang's fractional sets [15] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α -type set of element sets:

Z^α : The α -type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

Q^α : The α -type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

J^α : The α -type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z, q \neq 0\}$.

R^α : The α -type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

- (1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α ;
- (2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
- (3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
- (4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
- (5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
- (6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
- (7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2. [15] A non-differentiable function $f : R \rightarrow R^\alpha$, $x \rightarrow f(x)$ is called to be local fractional continuous at x_0 , if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b) , we denote $f(x) \in C_\alpha(a, b)$.

Definition 3. [15] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) (f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \dots$

Definition 4. [15] Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,

$${}_a I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \dots, N-1$ and $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_a I_b^\alpha f(x) = 0$ if $a = b$ and ${}_a I_b^\alpha f(x) = -{}_b I_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_a I_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha [a, b]$.

Definition 5 (Generalized convex function). [15] Let $f : I \subseteq R \rightarrow R^\alpha$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I .

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;

(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$ where $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer

function.

Theorem 2. [13] Let $f \in D_\alpha(I)$, then the following conditions are equivalent

- a) f is a generalized convex function on I
- b) $f^{(\alpha)}$ is an increasing function on I
- c) for any $x_1, x_2 \in I$,

$$f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1 + \alpha)} (x_2 - x_1)^\alpha.$$

Corollary 1. [13] Let $f \in D_{2\alpha}(a, b)$. Then f is a generalized convex function (or a generalized concave function) if and only if

$$f^{(2\alpha)}(x) \geq 0 \left(\text{or } f^{(2\alpha)}(x) \leq 0 \right)$$

for all $x \in (a, b)$.

Lemma 1. [15]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_a I_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a, b]$ and $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_a I_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_a I_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2. [15] We have

$$i) \frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\alpha)} x^{(k-1)\alpha};$$

$$ii) \frac{1}{\Gamma(\alpha + 1)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}), \quad k \in R.$$

Lemma 3 (Generalized Hölder's inequality). [15] *Let $f, g \in C_\alpha [a, b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [13], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 3 (Generalized Hermite-Hadamard inequality). *Let $f(x) \in I_x^{(\alpha)} [a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then*

$$(2.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}.$$

The aim of this paper is to extend the generalized Hermite-Hadamard inequalities and generalized Bullen inequalities to generalized convex functions.

3. MAIN RESULTS

Theorem 4 (Generalized Hermite-Hadamard-type inequality). *Let $f(x) \in I_x^{(\alpha)} [a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then*

$$(3.1) \quad f\left(\frac{a+b}{2}\right) \leq h(\lambda) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq H(\lambda) \leq \frac{f(a)+f(b)}{2^\alpha},$$

where

$$h(\lambda) := \lambda^\alpha f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)^\alpha f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$H(\lambda) := \frac{1}{2^\alpha} [f(\lambda b + (1-\lambda)a) + \lambda^\alpha f(a) + (1-\lambda)^\alpha f(b)].$$

Proof. Let f be a generalized convex. Then, applying (2.1) on the subinterval $[a, \lambda b + (1-\lambda)a]$, with $\lambda \neq 0$, we have

$$(3.2) \quad \begin{aligned} & f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) \\ & \leq \frac{1}{\lambda^\alpha (b-a)^\alpha} \int_a^{\lambda b + (1-\lambda)a} f(t) (dt)^\alpha \\ & \leq \frac{f(a) + f(\lambda b + (1-\lambda)a)}{2^\alpha}. \end{aligned}$$

Applying (2.1) again on $[\lambda b + (1-\lambda)a, b]$, with $\lambda \neq 1$, we get

$$(3.3) \quad \begin{aligned} & f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ & \leq \frac{1}{(1-\lambda)^\alpha (b-a)^\alpha} \int_{\lambda b + (1-\lambda)a}^b f(t) (dt)^\alpha \\ & \leq \frac{f(\lambda b + (1-\lambda)a) + f(b)}{2^\alpha}. \end{aligned}$$

Multiplying (3.2) by λ^α , (3.3) by $(1-\lambda)^\alpha$, and adding the resulting inequalities, we get:

$$(3.4) \quad h(\lambda) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq H(\lambda)$$

where $h(\lambda)$ and $H(\lambda)$ are defined as in Theorem 4.

Using the fact that f is a generalized convex function, we obtain

$$(3.5) \quad \begin{aligned} & f\left(\frac{a+b}{2}\right) \\ &= f\left(\lambda \frac{\lambda b + (2-\lambda)a}{2} + (1-\lambda) \frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ &\leq \lambda^\alpha f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)^\alpha f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ &\leq \frac{\lambda^\alpha}{2^\alpha} [f(\lambda b + (1-\lambda)a) + f(a)] + \frac{(1-\lambda)^\alpha}{2^\alpha} [f(b) + f(\lambda b + (1-\lambda)a)] \\ &= \frac{1}{2^\alpha} [f(\lambda b + (1-\lambda)a) + \lambda^\alpha f(a) + (1-\lambda)^\alpha f(b)] \\ &\leq \frac{f(a) + f(b)}{2^\alpha}. \end{aligned}$$

Then by (3.4) and (3.5), we get (3.1). \square

Theorem 5. Let $g(x) \in D_{2\alpha}[a, b]$ such that there exist constants $m, M \in \mathbb{R}^\alpha$ so that $m \leq g^{(2\alpha)}(x) \leq M$ for $x \in [a, b]$. Then

$$(3.6) \quad \begin{aligned} & \frac{m(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)} - \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right) \\ & \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x) - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{M}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right) - \frac{M(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)}. \end{aligned}$$

and

$$(3.7) \quad \begin{aligned} & \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right) - \frac{m(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)} \\ & \leq \frac{g(a) + g(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x) \\ & \leq \frac{M(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)} - \frac{M}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right). \end{aligned}$$

Proof. Let $f(x) = g(x) - \frac{m}{\Gamma(1+2\alpha)} x^{2\alpha}$, then $f^{(2\alpha)}(x) = g^{(2\alpha)}(x) - m \geq 0$, which shows that f is generalized convex on (a, b) . Applying inequality (2.1) for f , then

we have

$$\begin{aligned}
& g\left(\frac{a+b}{2}\right) - \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a+b}{2}\right)^{2\alpha} \\
&= f\left(\frac{a+b}{2}\right) \\
&\leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \\
&= \frac{1}{(b-a)^\alpha} \int_a^b \left[g(x) - \frac{m}{\Gamma(1+2\alpha)} x^{2\alpha} \right] (dx)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x) - \frac{1}{(b-a)^\alpha} \frac{m}{\Gamma(1+2\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} (b^{3\alpha} - a^{3\alpha}).
\end{aligned}$$

This implies that

$$\begin{aligned}
& \frac{m(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)} - \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a+b}{2}\right)^{2\alpha} \\
&\leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x) - g\left(\frac{a+b}{2}\right)
\end{aligned}$$

which proves the first inequality in (3.6). To prove the second part of (3.6), we apply the same argument for the generalized convex mapping $f(x) = \frac{M}{\Gamma(1+2\alpha)} x^{2\alpha} - g(x)$; $x \in [a, b]$.

By applying the second part of the generalized Hermite-Hadamard inequality (2.1) for the mapping $f(x) = g(x) - \frac{m}{\Gamma(1+2\alpha)} x^{2\alpha}$ as follows

$$\begin{aligned}
& \frac{g(a) + g(b)}{2^\alpha} - \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right) \\
&= \frac{f(a) + f(b)}{2^\alpha} \\
&\geq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \\
&= \frac{1}{(b-a)^\alpha} \int_a^b \left[g(x) - \frac{m}{\Gamma(1+2\alpha)} x^{2\alpha} \right] (dx)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x) - \frac{1}{(b-a)^\alpha} \frac{m}{\Gamma(1+2\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} (b^{3\alpha} - a^{3\alpha}).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
& \frac{m}{\Gamma(1+2\alpha)} \left(\frac{a^{2\alpha} + b^{2\alpha}}{2^\alpha}\right) - \frac{m(b^\alpha + a^\alpha b^\alpha + a^\alpha)}{\Gamma(1+3\alpha)} \\
&\leq \frac{g(a) + g(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha g(x)
\end{aligned}$$

which proves the rst part of (3.7). The second part is established in a similar manner; and we omit the details which completes the proof. \square

We prove the following inequality which is generalized Bullen inequality for generalized convex function.

Theorem 6 (Generalized Bullen inequality). *Let $f(x) \in I_x^{(\alpha)} [a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then we have the inequality*

$$(3.8) \quad \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \leq \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha} \right].$$

Proof. Using the Theorem 3, we find that

$$\begin{aligned} & \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x) (dx)^\alpha \\ &= \frac{2^\alpha \Gamma(1+\alpha)}{(b-a)^\alpha} \left[\frac{1}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} f(x) (dx)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b f(x) (dx)^\alpha \right] \\ &\leq \frac{f\left(\frac{a+b}{2}\right) + f(a)}{2^\alpha} + \frac{f(b) + f\left(\frac{a+b}{2}\right)}{2^\alpha} \\ &= f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha}. \end{aligned}$$

Hence, the proof is completed. \square

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_{2\alpha}(I^0)$ and $f^{(\alpha)} \in C_\alpha[a, b]$ for $a, b \in I^0$ with $a < b$. Then, for all $x \in [a, b]$, we have the following identity*

$$(3.9) \quad \begin{aligned} & \frac{1}{2^\alpha (b-a)^\alpha (\Gamma(1+\alpha))^2} \int_a^b \left(x - \frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x) (dx)^\alpha \\ &= \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha} \right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \end{aligned}$$

where

$$p(x) = \begin{cases} (a-x)^\alpha, & [a, \frac{a+b}{2}) \\ (b-x)^\alpha, & [\frac{a+b}{2}, b]. \end{cases}$$

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)} \int_a^b \left(x - \frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x) (dx)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right)^\alpha (a-x)^\alpha f^{(2\alpha)}(x) (dx)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^\alpha (b-x)^\alpha f^{(2\alpha)}(x) (dx)^\alpha \\
&= \left(x - \frac{a+b}{2}\right)^\alpha (a-x)^\alpha f^{(\alpha)}(x) \Big|_a^{\frac{a+b}{2}} \\
&\quad - \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{3a+b}{2} - 2x\right)^\alpha f^{(\alpha)}(x) (dx)^\alpha \\
&\quad + \left(x - \frac{a+b}{2}\right)^\alpha (b-x)^\alpha f^{(\alpha)}(x) \Big|_{\frac{a+b}{2}}^b \\
&\quad - \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \left(\frac{a+3b}{2} - 2x\right)^\alpha f^{(\alpha)}(x) (dx)^\alpha.
\end{aligned}$$

Using the local fractional integration by parts again, we find that

$$\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)} \int_a^b \left(x - \frac{a+b}{2}\right)^\alpha p(x) f^{(2\alpha)}(x) (dx)^\alpha \\
&= \Gamma(1+\alpha) (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \Gamma(1+\alpha) (b-a)^\alpha \frac{f(a) + f(b)}{2^\alpha} \\
&\quad - \frac{2^\alpha (\Gamma(1+\alpha))^2}{\Gamma(1+\alpha)} \int_a^b f(x) (dx)^\alpha.
\end{aligned}$$

If we divide the resulting equality with $2^\alpha \Gamma(1+\alpha) (b-a)^\alpha$, then we complete the proof. \square

Theorem 8. *Suppose that the assumptions of Theorem 7 are satisfied, then we have the following inequality*

$$\begin{aligned}
& \left| \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2^\alpha} \right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \right| \\
& \leq \frac{(b-a)^{(1+\frac{1}{p})\alpha}}{8^\alpha \Gamma(1+\alpha)} (B(p+1, p+1))^{\frac{1}{p}} \|f^{(2\alpha)}(x)\|_q
\end{aligned}$$

where, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|f^{(2\alpha)}\|_q$ is defined by

$$\|f^{(2\alpha)}\|_q = \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(2\alpha)}(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}$$

and $B(x, y)$ is defined by

$$B(x, y) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{(x-1)\alpha} (1-t)^{(y-1)\alpha} (dt)^\alpha.$$

Proof. Taking modulus in (3.9) and using generalized Hölder inequality, we have

$$\begin{aligned}
(3.10) \quad & \left| \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha} \right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \right| \\
& \leq \frac{1}{2^\alpha (b-a)^\alpha (\Gamma(1+\alpha))^2} \int_a^b \left| x - \frac{a+b}{2} \right|^\alpha |p(x)| |f^{(2\alpha)}(x)| (dx)^\alpha \\
& \leq \frac{1}{2^\alpha (b-a)^\alpha \Gamma(1+\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f^{(2\alpha)}(x)|^q (dx)^\alpha \right)^{\frac{1}{q}} \\
& \quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left| x - \frac{a+b}{2} \right|^{p\alpha} |p(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \\
& = \frac{\|f^{(2\alpha)}\|_q}{2^\alpha (b-a)^\alpha \Gamma(1+\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right)^{p\alpha} (x-a)^{p\alpha} (dx)^\alpha \right. \\
& \quad \left. + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2} \right)^{p\alpha} (b-x)^{p\alpha} (dx)^\alpha \right)^{\frac{1}{p}} \\
& = \frac{\|f^{(2\alpha)}\|_q}{2^\alpha (b-a)^\alpha \Gamma(1+\alpha)} (K_1 + K_2)^{\frac{1}{p}}.
\end{aligned}$$

For calculating integral K_1 , using changing variable with $x = (1-t)a + t\frac{a+b}{2}$, we obtain

$$\begin{aligned}
(3.11) \quad K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right)^{p\alpha} (x-a)^{p\alpha} (dx)^\alpha \\
&= \left(\frac{b-a}{2} \right)^{(2p+1)\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{p\alpha} t^{p\alpha} (dt)^\alpha \\
&= \left(\frac{b-a}{2} \right)^{(2p+1)\alpha} B(p+1, p+1).
\end{aligned}$$

Similarly, using changing variable with $x = (1-t)\frac{a+b}{2} + tb$, we have

$$(3.12) \quad \begin{aligned} K_2 &= \frac{1}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^{p\alpha} (b-x)^{p\alpha} (dx)^\alpha \\ &= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha} B(p+1, p+1) \end{aligned}$$

Putting (3.11) and (3.12) in (3.10), we obtain

$$\begin{aligned} & \left| \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha} \right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \right| \\ & \leq \frac{\|f^{(2\alpha)}\|_q}{2^\alpha (b-a)^\alpha \Gamma(1+\alpha)} \left(2^\alpha \frac{(b-a)^{(2p+1)\alpha}}{2^{(2p+1)\alpha}} B(p+1, p+1) \right)^{\frac{1}{p}} \\ & = \frac{(b-a)^{(1+\frac{1}{p})\alpha}}{8^\alpha \Gamma(1+\alpha)} (B(p+1, p+1))^{\frac{1}{p}} \|f^{(2\alpha)}\|_q \end{aligned}$$

which completes the proof. \square

Theorem 9. *The assumptions of Theorem 7 are satisfied. If the mapping*

$$\varphi(x) = \begin{cases} (a-x)^\alpha \left(x - \frac{a+b}{2}\right)^\alpha f^{(2\alpha)}(x), & [a, \frac{a+b}{2}] \\ (b-x)^\alpha \left(x - \frac{a+b}{2}\right)^\alpha f^{(2\alpha)}(x), & [\frac{a+b}{2}, b]. \end{cases}$$

is a generalized convex, then we have the inequality

$$\begin{aligned} & \frac{(b-a)^{2\alpha}}{64^\alpha (\Gamma(1+\alpha))^2} \left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right) + f^{(2\alpha)}\left(\frac{a+3b}{4}\right) \right] \\ & \leq \frac{1}{2^\alpha} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha} \right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \\ & \leq \frac{(b-a)^{2\alpha}}{128^\alpha (\Gamma(1+\alpha))^2} \left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right) + f^{(2\alpha)}\left(\frac{a+3b}{4}\right) \right]. \end{aligned}$$

Proof. Applying the first inequality (2.1) for the mapping φ , we get

$$(3.13) \quad \begin{aligned} & \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \frac{2^\alpha}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \varphi(x) (dx)^\alpha \\ & \geq \varphi\left(\frac{3a+b}{4}\right) = \frac{(b-a)^{2\alpha}}{16^\alpha} f^{(2\alpha)}\left(\frac{3a+b}{4}\right) \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} & \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \frac{2^\alpha}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \varphi(x) (dx)^\alpha \\ & \geq \varphi\left(\frac{a+3b}{4}\right) = \frac{(b-a)^{2\alpha}}{16^\alpha} f^{(2\alpha)}\left(\frac{a+3b}{4}\right). \end{aligned}$$

Applying the inequality (3.8) for the mapping φ , we have

$$\begin{aligned}
 (3.15) \quad & \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \frac{2^\alpha}{\Gamma(1+\alpha)} \int_a^{\frac{a+b}{2}} \varphi(x) (dx)^\alpha \\
 & \leq \frac{1}{2^\alpha} \left[\varphi\left(\frac{3a+b}{4}\right) + \frac{\varphi(a) + \varphi\left(\frac{a+b}{2}\right)}{2^\alpha} \right] \\
 & = \frac{(b-a)^{2\alpha}}{32^\alpha} f^{(2\alpha)}\left(\frac{3a+b}{4}\right)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.16) \quad & \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \frac{2^\alpha}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \varphi(x) (dx)^\alpha \\
 & \leq \frac{1}{2^\alpha} \left[\varphi\left(\frac{a+3b}{4}\right) + \frac{\varphi\left(\frac{a+b}{2}\right) + \varphi(b)}{2^\alpha} \right] \\
 & = \frac{(b-a)^{2\alpha}}{32^\alpha} f^{(2\alpha)}\left(\frac{a+3b}{4}\right).
 \end{aligned}$$

Adding the inequalities (3.13)-(3.16) and from Theorem 7, we write

$$\begin{aligned}
 & \frac{(b-a)^{2\alpha}}{16^\alpha} \left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right) + f^{(2\alpha)}\left(\frac{a+3b}{4}\right) \right] \\
 & \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \frac{2^\alpha}{\Gamma(1+\alpha)} \int_{\frac{a+b}{2}}^b \varphi(x) (dx)^\alpha \\
 & = 4^\alpha (\Gamma(1+\alpha))^2 \left[\frac{1}{2^\alpha} \left(f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2^\alpha} \right) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^\alpha f(x) \right] \\
 & \leq \frac{(b-a)^{2\alpha}}{32^\alpha} \left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right) + f^{(2\alpha)}\left(\frac{a+3b}{4}\right) \right].
 \end{aligned}$$

If we divide the resulting inequality with $4^\alpha (\Gamma(1+\alpha))^2$, then we complete the proof. \square

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, KONURALP CAMPUS, DÜZCE-TURKEY

E-mail address: sarikayamz@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BARTIN UNIVERSITY, BARTIN-TURKEY

E-mail address: erdensmt@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, KONURALP CAMPUS, DÜZCE-TURKEY

E-mail address: hsyn.budak@gmail.com