Received 16/05/15

FEJÉR TYPE INEQUALITIES FOR HARMONICALLY-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, a new weighted identity involving harmonically symmetric functions and differentiable functions is established. By using the notion of harmonic symmetricity, harmonic convexity, analysis and some auxiliary results, some new Fejér type integral inequalities are presented for the class of harmonically convex functions. Applications of our results to special means of positive real numbers are given as well.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)t) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1.1)$$

holds if and only if f is convex and are know as Hermite-Hadamard inequality after the name of C. Hermite and J. Hadamard (see [18]). The inequalities in (1.1) hold in reversed direction if f is a concave function. A vast literature have been produced by a number of mathematicians for convex functions but (1.1) is considered to be the most famous inequality for convex mappings due to its usefulness and many applications in various branches of pure and applied mathematics.

The definition of classical or usual convex functions has been generalized in a variety of ways and as a consequence many researchers have established a number of Hermite-Hadamard type inequalities by using different generalizations of the classical convexity, see for instance [2]-[23] and the references mentioned in these papers.

One of the generalizations of classical convexity is the harmonic convexity stated in the definition below.

Date: November 15, 2012.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26A51, 26E60, 41A55.

Key words and phrases. Hermite-Hadamard's inequality, Fejér's inequality, convex function, harmonically-convex function, Hölder's inequality, power mean inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Definition 1. [10] *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{1.2}$$

for all $x,\ y\in I$ and $t\in[0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

The following result explains how the usual convexity and the harmonic convexity are connected.

Proposition 1. [10] *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval and $f:I\to\mathbb{R}$ is function, then:*

- *if $I\subset(0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
- *if $I\subset(0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
- *if $I\subset(-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.*
- *if $I\subset(-\infty,0)$ and f is convex and nonincreasing function then f is harmonically convex.*

Most recently, İşcan [10], has proved the following results for harmonically convex functions.

Theorem 1. [10] *Let $I\subset\mathbb{R}\backslash\{0\}$ be a harmonically convex function and $a,\ b\in I$ with $a<b$. If $f\in L([a,b])$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

Theorem 2. [10] *Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,\ b\in I^\circ$ with $a<b$, and $f'\in L([a,b])$. If $\left|f'\right|^q$ is harmonically convex $[a,b]$ for $q\geq1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\&\leq\frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q+\lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}},\end{aligned}$$

where

$$\begin{aligned}\lambda_1&=\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),\\ \lambda_2&=-\frac{1}{b(b-a)}+\frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),\\ \lambda_2&=\frac{1}{a(b-a)}-\frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)\\&=\lambda_1-\lambda_2.\end{aligned}$$

Theorem 3. [10] *Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, a, $b\in I^{\circ}$ with $a<b$, and $f^{'}\in L([a,b])$. If $\left|f^{'}\right|^{q}$ is harmonically convex $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\mu_1\left|f^{'}(a)\right|^{q}+\mu_1\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}},$$

where

$$\mu_1=\frac{a^{2-2q}+b^{1-2q}\left[(b-a)(1-2q)-a\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2=\frac{b^{2-2q}-a^{1-2q}\left[(b-a)(1-2q)+b\right]}{2(b-a)^2(1-q)(1-2q)}.$$

Some applications of the above results can also be found in [10].

Chen and Wu [3] established the following Fejér type inequality for harmonically convex functions which provides a weighted generalization of the result given in Theorem 1.

Theorem 4. [3] *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a harmonically convex function and a, $b\in I$ with $a<b$. If $f\in L([a,b])$, them one has be continuous*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx\leq\int_a^b\frac{f(x)g(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx, \tag{1.3}$$

$g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and satisfies

$$g\left(\frac{ab}{x}\right)=g\left(\frac{ab}{a+b-x}\right).$$

The main purpose of the present paper is to introduce a new notion of harmonically symmetric functions and to establish an identity involving a harmonically symmetric function and a differentiable function. We will prove some Fejér type inequalities by using this identity related with the first part of the inequality given above by (1.3). Some applications of our results to special means of positive real numbers will also be provided in Section 3. We believe that our findings are novel, new and better than those already exist and will open new ways for further research in this filed.

2. Main Results

Throughout this section we take $L(t)=\frac{2ab}{(1-t)a+(1+t)b}$ and $U(t)=\frac{2ab}{(1+t)a+(1-t)b}$. The Beta function, the Gamma function and the integral from of the hypergeometric function are defined as follows to be used in the sequel of the paper

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\alpha>0,\beta>0,$$
$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\alpha>0$$

and

$$ {}_2F_1(\alpha,\beta;\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt $$

for $|z|<1, \gamma>\beta>0$.

The notion of harmonically symmetric functions is given in following definition.

Definition 2. *A function $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if*

$$ g(x)=g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right) $$

holds for all $x\in[a,b]$.

Now we prove a weighted integral identity which will play an important role to establish our main results.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on I° and a, $b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$. If $f'\in L([a,b])$, then the following equality holds*

$$ f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx=\left(\frac{b-a}{2ab}\right) \times\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x^2}dx\right)\left[(L(t))^2f'(L(t))-(U(t))^2f'(U(t))\right]dt. \quad (2.1) $$

Proof. Let

$$ I_1=\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x^2}dx\right)(U(t))^2f'(U(t))\,dt $$

and

$$ I_2=\int_0^1\left(\int_a^{L(t)}\frac{g(x)}{x^2}dx\right)(L(t))^2f'(L(t))\,dt. $$

Since $g:[a,b]\to[0,\infty)$ is harmonically symmetric to $\frac{2ab}{a+b}$, then $g(U(t))=g(L(t))$ for all $t\in[0,1]$ and

$$ \int_a^{L(t)}\frac{g(x)}{x^2}dx=\int_{U(t)}^b\frac{g(x)}{x^2}dx. $$

Hence, we have

$$I_1 = \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x^2} dx \right) (U(t))^2 f'(U(t))\, dt$$
$$= \frac{2ab}{b-a} \int_0^1 \left(\int_{U(t)}^{b} \frac{g(x)}{x^2} dx \right) d\left[f(U(t)) \right]$$
$$= \frac{2ab}{b-a} \left(\int_{U(t)}^{b} \frac{g(x)}{x^2} dx \right) f(U(t)) \Big|_0^1 + \int_0^1 g(U(t))\, f(U(t))\, dt$$
$$= -\frac{2ab}{b-a} f\left(\frac{2ab}{a+b} \right) \int_a^b \frac{g(x)}{x^2} dx - \int_0^1 g(U(t))\, f(U(t))\, dt$$
$$= -\frac{2ab}{b-a} f\left(\frac{2ab}{a+b} \right) \int_a^b \frac{g(x)}{x^2} dx + \frac{2ab}{b-a} \int_{\frac{2ab}{a+b}}^{b} \frac{g(x) f(x)}{x^2} dx. \tag{2.2}$$

Analogously, we have

$$I_2 = \frac{2ab}{b-a} f\left(\frac{2ab}{a+b} \right) \int_a^b \frac{g(x)}{x^2} dx - \frac{2ab}{b-a} \int_a^{\frac{2ab}{a+b}} \frac{g(x) f(x)}{x^2} dx. \tag{2.3}$$

Adding (2.2) and (2.3) and multiplying the result by $\frac{b-a}{2ab}$, we get the required identity. This completes the proof of the Lemma. □

Lemma 2. *For $v > u > 0$, we have*

$$\int_0^1 (1-t) \left[\frac{2uv}{(1+t)u + (1-t)v} \right]^2 dt = \left(\frac{2uv}{v-u} \right)^2 \lambda_1(u,v),$$
$$\int_0^1 (1-t)^2 \left[\frac{2uv}{(1+t)u + (1-t)v} \right]^2 dt = \left(\frac{2uv}{v-u} \right)^2 \lambda_2(u,v),$$
$$\int_0^1 \left(1-t^2\right) \left[\frac{2uv}{(1+t)u + (1-t)v} \right]^2 dt = \left(\frac{2uv}{v-u} \right)^2 \lambda_3(u,v),$$

where

$$\lambda_1(u,v) \triangleq \frac{u-v}{u+v} - \ln\left(\frac{2u}{u+v} \right),$$
$$\lambda_2(u,v) \triangleq \frac{4u^2 - (u+v)^2}{u^2 - v^2} - \frac{4u}{u-v} \ln\left(\frac{2u}{u+v} \right)$$

and

$$\lambda_3(u,v) \triangleq \frac{4v^2 - (u+v)^2}{u^2 - v^2} + \frac{2(u+v)}{u-v} \ln\left(\frac{2u}{u+v} \right).$$

Proof. The proof follows from a straightforward computation. □

Lemma 3. *For $v > u > 0$ and $p > 1$, we have*

$$\int_0^1 (1+t) \left[\frac{2uv}{(1+t)u + (1-t)v} \right]^{2p} dt = \left(\frac{2uv}{v+u} \right)^{2p} \zeta_1(u,v;p),$$
$$\int_0^1 (1-t) \left[\frac{2uv}{(1+t)u + (1-t)v} \right]^{2p} dt = \left(\frac{2uv}{v+u} \right)^{2p} \zeta_2(u,v;p),$$

$$\int_0^1 \left[\frac{2ab}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \varsigma(u,v;p)$$

where

$$\zeta_1(u,v;p) \triangleq \frac{2^{1-2p}u\left(\frac{u}{u+v}\right)^{-2p}[(1-2p)(u-v)-v]}{(v-u)^2(p-1)(2p-1)} - \frac{(u+v)[(1-2p)(u-v)-2v]}{2(v-u)^2(p-1)(2p-1)},$$

$$\zeta_2(u,v;p) \triangleq \frac{4^{1-p}u^2\left(\frac{u}{u+v}\right)^{-2p}+(u+v)[(2p-1)(u-v)-2u]}{2(v-u)^2(p-1)(2p-1)}$$

and

$$\varsigma(u,v;p) \triangleq \frac{(u+b)^{-2p+1}-(2u)^{-2p+1}}{(2p-1)(u-v)(u+v)^{-2p}}.$$

Proof. The proof follows from a straightforward computation. □

Now we present new Fejér type inequalities for harmonically-convex functions, which provide weighted generalization of some of the results established in recent literature.

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L([a,b])$. If $\left|f'\right|^q$ is harmonically-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)\,g(x)}{x^2}dx\right| \leq \left(\frac{1}{2}\right)^{1/q} \|g\|_\infty \left\{[\lambda_1(a,b)]^{1-1/q}\left[\lambda_2(a,b)\left|f'(a)\right|^q+\lambda_3(a,b)\left|f'(b)\right|^q\right]^{1/q} + [\lambda_1(b,a)]^{1-1/q}\left[\lambda_3(b,a)\left|f'(a)\right|^q+\lambda_2(b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (2.4)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ and $\lambda_1(\cdot,\cdot)$, $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ are defined in Lemma 2.

Proof. From Lemma 1, we get

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)\,g(x)}{x^2}dx\right| \leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_\infty \times\left\{\left(\int_0^1 (1-t)(U(t))^2 dt\right)^{1-1/q}\left(\int_0^1 (1-t)(U(t))^2\left|f'(U(t))\right|^q dt\right)^{1/q} + \left(\int_0^1 (1-t)(L(t))^2 dt\right)^{1-1/q}\left(\int_0^1 (1-t)(L(t))^2\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \quad (2.5)$$

By the harmonic-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 2, we have

$$\int_0^1 (1-t)(U(t))^2 \left|f'(U(t))\right|^q dt = \int_0^1 (1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2$$

$$\times\left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt \leq \frac{1}{2}\left|f'(a)\right|^q \int_0^1 (1-t)^2\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 dt$$

$$+\frac{1}{2}\left|f'(b)\right|^q \int_0^1 \left(1-t^2\right)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 dt$$

$$=\frac{1}{2}\left(\frac{2ab}{b-a}\right)^2\left\{\lambda_2(a,b)\left|f'(a)\right|^q+\lambda_3(a,b)\left|f'(b)\right|^q\right\} \tag{2.6}$$

and

$$\int_0^1 (1-t)(L(t))^2 \left|f'(L(t))\right|^q dt = \int_0^1 (1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2$$

$$\times\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt \leq \frac{1}{2}\left|f'(a)\right|^q \int_0^1 \left(1-t^2\right)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt$$

$$+\frac{1}{2}\left|f'(b)\right|^q \int_0^1 (1-t)^2\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt$$

$$=\frac{1}{2}\left(\frac{2ab}{b-a}\right)^2\left\{\lambda_3(b,a)\left|f'(a)\right|^q+\lambda_2(b,a)\left|f'(b)\right|^q\right\}. \tag{2.7}$$

Moreover,

$$\int_0^1 (1-t)(U(t))^2 dt = \int_0^1 (1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 dt$$

$$=\left(\frac{2ab}{b-a}\right)^2 \lambda_1(a,b) \tag{2.8}$$

and

$$\int_0^1 (1-t)(L(t))^2 dt = \int_0^1 (1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt$$

$$=\left(\frac{2ab}{b-a}\right)^2 \lambda_1(b,a) \tag{2.9}$$

A combination of (2.5), (2.6), (2.7), (2.8) and (2.9) gives the required result. This completes the proof of the theorem. □

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied. If $q=1$, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$

$$\leq \left(\frac{1}{2}\right)\|g\|_\infty \left\{\left[\lambda_2(a,b)+\lambda_3(b,a)\right]\left|f'(a)\right| + \left[\lambda_3(a,b)+\lambda_2(b,a)\right]\left|f'(b)\right|\right\}, \tag{2.10}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ *and* $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ *are defined in Lemma 2.*

Corollary 2. *If* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$ *in Theorem 5, then*

$$\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \left(\frac{1}{2}\right)^{1/q}\left(\frac{ab}{b-a}\right)$$
$$\times\left\{[\lambda_1(a,b)]^{1-1/q}\left[\lambda_2(a,b)\left|f'(a)\right|^q + \lambda_3(a,b)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+[\lambda_1(b,a)]^{1-1/q}\left[\lambda_3(b,a)\left|f'(a)\right|^q + \lambda_2(b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (2.11)$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ *and* $\lambda_1(\cdot,\cdot)$, $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ *are defined in Lemma 2.*

Corollary 3. *If* $q = 1$ *in Corollary 2, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)$$
$$\times\left\{[\lambda_2(a,b)+\lambda_3(b,a)]\left|f'(a)\right| + [\lambda_3(a,b)+\lambda_2(b,a)]\left|f'(b)\right|\right\}, \quad (2.12)$$

where $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ *are defined in Lemma 2.*

Theorem 6. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a < b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f' \in L([a,b])$. *If* $\left|f'\right|^q$ *is harmonically-convex on* $[a,b]$ *for* $q > 1$, *then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\le \|g\|_\infty\left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left[\zeta_2(a,b;q)\left|f'(a)\right|^q + \zeta_1(a,b;q)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\zeta_1(b,a;q)\left|f'(a)\right|^q + \zeta_2(b,a;q)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (2.13)$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Proof. From Lemma 1 and Hölder's inequality, we have

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\le \left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left(\int_0^1 (1-t)^{q/(q-1)}\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 (U(t))^{2q}\left|f'(U(t))\right|^q\right)^{1/q} + \left(\int_0^1 (L(t))^{2q}\left|f'(L(t))\right|^q\right)^{1/q}\right\}. \quad (2.14)$$

Since $\left|f^{'}\right|^{q}$ is harmonically-convex on $[a,b]$, we obtain

$$\begin{aligned}\int_0^1 [U(t)]^{2q}\left|f^{'}(U(t))\right|^{q} &= \int_0^1 \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}\left|f^{'}\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^{q}dt \\ &\le \frac{1}{2}\left|f^{'}(a)\right|^{q}\int_0^1 (1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt \\ &\quad + \frac{1}{2}\left|f^{'}(b)\right|^{q}\int_0^1 (1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt \qquad (2.15)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 [L(t)]^{2q}\left|f^{'}(L(t))\right|^{q} &= \int_0^1 \left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}\left|f^{'}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^{q}dt \\ &\le \frac{1}{2}\left|f^{'}(a)\right|^{q}\int_0^1 (1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt \\ &\quad + \frac{1}{2}\left|f^{'}(b)\right|^{q}\int_0^1 (1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt. \qquad (2.16)\end{aligned}$$

By applying Lemma 3 in inequalities (2.15) and (2.16) and then using the resulting inequalities in (2.14), we get the required inequality. □

Corollary 4. *If the assumptions of Theorem 6 are satisfied and if $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^{2}\left(\frac{1}{2}\right)^{1/q} \\ &\quad\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\zeta_2(a,b;q)\left|f^{'}(a)\right|^{q}+\zeta_1(a,b;q)\left|f^{'}(b)\right|^{q}\right]^{1/q}\right. \\ &\quad\left.+\left[\zeta_1(b,a;q)\left|f^{'}(a)\right|^{q}+\zeta_2(b,a;q)\left|f^{'}(b)\right|^{q}\right]^{1/q}\right\}, \qquad (2.17)\end{aligned}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ and $\zeta_2(\cdot,\cdot;\cdot)$ are defined in Lemma 3.

Theorem 7. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and a, $b\in I^{\circ}$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f^{'}\in L([a,b])$. If $\left|f^{'}\right|^{q}$ is harmonically-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right| \\ &\le\left(\frac{b-a}{b+a}\right)^{2}\left(\frac{1}{2}\right)^{2/q-1}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\|g\|_{\infty}\left\{[\zeta_2(a,b;q)+\zeta_1(b,a;q)]\left|f^{'}(a)\right|^{q}\right. \\ &\quad\left.+[\zeta_1(a,b;q)+\zeta_2(b,a;q)]\left|f^{'}(b)\right|^{q}\right\}^{1/q}, \qquad (2.18)\end{aligned}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ and $\zeta_2(\cdot,\cdot;\cdot)$ are defined in Lemma 3.

Proof. From Lemma 1 and Hölder's inequality, we have

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_\infty \left(\int_0^1 t^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}. \tag{2.19}$$

By the power-mean inequality ($a^r+b^r\leq 2^{1-r}(a+b)^r$ for $a>0,b>0$ and $r<1$), we have

$$\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{2q} dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{2q} dt\right)^{1/q}$$
$$\leq 2^{1-1/q}\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt+\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt\right)^{1/q}. \tag{2.20}$$

Since $\left|f'\right|^q$ is harmonically-convex on $[a,b]$ for $q>1$, we obtain

$$\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt+\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt$$
$$\leq \frac{1}{2}\left|f'(a)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt$$
$$+\frac{1}{2}\left|f'(b)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt$$
$$+\frac{1}{2}\left|f'(a)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt$$
$$+\frac{1}{2}\left|f'(b)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt$$
$$=\frac{1}{2}\left(\frac{2ab}{b+a}\right)^{2q}\Big\{\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(a)\right|^q$$
$$+\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(b)\right|^q\Big\}. \tag{2.21}$$

Using (2.20) in (2.21), we get

$$\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^q dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^q dt\right)^{1/q}$$
$$\leq 2^{1-2/q}\left(\frac{2ab}{b+a}\right)^2\Big\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(a)\right|^q$$
$$+\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(b)\right|^q\Big\}^{1/q}. \tag{2.22}$$

Applying (2.22) in (2.19), we obtain the required inequality (2.18). □

Corollary 5. *If the assumptions of Theorem 7 are satisfied and if* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$*, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1-2/q}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(a)\right|^q\right.$$
$$\left.+\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(b)\right|^q\right\}^{1/q},\quad(2.23)$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 3.*

Theorem 8. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *and* $a,b\in I^\circ$ *with* $a<b$ *and let* $g:[a,b]\to[0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f'\in L([a,b])$*. If* $\left|f'\right|$ *is harmonically-convex on* $[a,b]$*, then the following inequality holds for* $q>1$

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\le\frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\|g\|_\infty$$
$$\times\left[\left[\varsigma(a,b;q)\right]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(a)\right|+\left[\omega(q)\right]^{1/q}\left|f'(a)\right|\right\}\right.$$
$$\left.+\left[\varsigma(b,a;q)\right]^{1-1/q}\left\{\left[\omega(q)\right]^{1/q}\left|f'(a)\right|+\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(b)\right|\right\}\right],\quad(2.24)$$

where $\varsigma(\cdot,\cdot;\cdot)$ *is defined in Lemma 3 and*

$$\omega(q)=\frac{\sqrt{\pi}\Gamma(q+1)}{2\Gamma\left(q+\frac{3}{2}\right)}.$$

Proof. From Lemma 1 and by using the harmonic-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|$$
$$\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\int_0^1\left[(1-t)(U(t))^2\left|f'(U(t))\right|+(1-t)(L(t))^2\left|f'(L(t))\right|\right]dt$$
$$\le\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\int_0^1(U(t))^2\left[\frac{(1-t)^2}{2}\left|f'(a)\right|+\left(\frac{1-t^2}{2}\right)\left|f'(b)\right|\right]dt\right.$$
$$\left.+\int_0^1(L(t))^2\left[\left(\frac{1-t^2}{2}\right)\left|f'(a)\right|+\frac{(1-t)^2}{2}\left|f'(b)\right|\right]dt\right\}.\quad(2.25)$$

Since

$$\omega(q)=\int_0^1\left(1-t^2\right)^q dt=\int_0^{\frac{\pi}{2}}\cos^{2q+1}(t)\,dt=\frac{\sqrt{\pi}\Gamma(q+1)}{2\Gamma\left(q+\frac{3}{2}\right)}\quad(2.26)$$

and

$$\int_0^1(1-t)^{2q}dt=\frac{1}{2q+1}.\quad(2.27)$$

Now by using Hölder integral inequality, (2.26), (2.27) and Lemma 3, we have

$$\int_0^1 \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 \left[\frac{(1-t)^2}{2}\left|f'(a)\right| + \left(\frac{1-t^2}{2}\right)\left|f'(b)\right|\right] dt$$
$$\le \left(\int_0^1 \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q/(q-1)} dt\right)^{1-1/q}$$
$$\times \left\{\left[\int_0^1 \frac{(1-t)^{2q}}{2^q} dt\right]^{1/q}\left|f'(a)\right| + \left[\int_0^1 \frac{(1-t^2)^q}{2^q} dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$= \frac{1}{2}\left(\frac{2ab}{a+b}\right)^2 \left[\varsigma(a,b;q)\right]^{1-\frac{1}{q}} \left\{\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left|f'(a)\right| + \left[\omega(q)\right]^{\frac{1}{q}}\left|f'(b)\right|\right\}. \quad (2.28)$$

Similarly, one has

$$\int_0^1 \left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 \left[\left(\frac{1-t^2}{2}\right)\left|f'(a)\right| + \frac{(1-t)^2}{2}\left|f'(b)\right|\right] dt$$
$$\le \left(\int_0^1 \left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q/(q-1)} dt\right)^{1-1/q}$$
$$\times \left\{\left[\int_0^1 \frac{(1-t^2)^q}{2^q} dt\right]^{1/q}\left|f'(a)\right| + \left[\int_0^1 \frac{(1-t)^{2q}}{2^q} dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$= \frac{1}{2}\left(\frac{2ab}{a+b}\right)^2 \left[\varsigma(b,a;q)\right]^{1-\frac{1}{q}} \left\{\left[\omega(q)\right]^{\frac{1}{q}}\left|f'(a)\right| + \left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left|f'(b)\right|\right\}. \quad (2.29)$$

Using (2.28) and (2.29) in (2.25), we obtain the required result (2.24). □

Corollary 6. *Under the assumptions of Theorem 8, if $g(x) = \frac{ab}{b-a}$ for all $x \in [a,b]$, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{1}{2}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{a+b}\right)^2$$
$$\times \left[\left[\varsigma(a,b;q)\right]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(a)\right| + \left[\omega(q)\right]^{1/q}\left|f'(a)\right|\right\}\right.$$
$$\left. + \left[\varsigma(b,a;q)\right]^{1-1/q}\left\{\left[\omega(q)\right]^{1/q}\left|f'(a)\right| + \left(\frac{1}{2q+1}\right)^{1/q}\left|f'(b)\right|\right\}\right], \quad (2.30)$$

where $\varsigma(\cdot,\cdot;\cdot)$ and $\omega(\cdot)$ are defined in Theorem 8.

Theorem 9. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and* *harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f^{'} \in L([a,b])$. *If* $\left|f^{'}\right|$ *is harmonically-convex on* $[a,b]$, *then the following inequality holds for* $q>1$

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right| \leq \frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\|g\|_\infty$$
$$\times\left\{[\nu(a,b;q)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|\right]\right.$$
$$\left.+[\nu(b,a;q)]^{1-1/q}\left[\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|+\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|\right]\right\}, \quad (2.31)$$

where

$$\nu(a,b;q)=\frac{\Gamma\left(\frac{2q-1}{q-1}\right)}{\Gamma\left(\frac{3q-2}{q-1}\right)}\left[a^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{a(b-a)}{a+b}\right)\right.$$
$$\left.-b^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{b(b-a)}{a+b}\right)\right],$$

$\Gamma(\cdot)$ *is the Gamma function and* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

Proof. From Lemma 1 and by using the harmonic-convexity of $\left|f^{'}\right|$ on $[a,b]$, we have

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\int_0^1\left[(1-t)(U(t))^2\left|f^{'}(U(t))\right|+(1-t)(L(t))^2\left|f^{'}(L(t))\right|\right]dt$$
$$\leq\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\int_0^1(1-t)(U(t))^2\left[\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|\right]dt\right.$$
$$\left.+\int_0^1(1-t)(L(t))^2\left[\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|\right]dt\right\}. \quad (2.32)$$

Application of Hölder integral inequality yields

$$\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\left[\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|\right]dt$$
$$\leq\left(\int_0^1(1-t)^{q/(q-1)}\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(a)\right|+\left[\int_0^1\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(b)\right|\right\}$$
$$=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\nu(a,b;q)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|\right]. \quad (2.33)$$

Similarly, one has

$$\int_0^1 (1-t)\left[\frac{2ab}{(1-t)\,a+(1+t)\,b}\right]^2 \left[\left(\frac{1+t}{2}\right)\left|f'(a)\right| + t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right] dt$$
$$\leq \left(\int_0^1 t^{q/(q-1)}\left[\frac{2ab}{(1-t)\,a+(1+t)\,b}\right]^{2q/(q-1)} dt\right)^{1-1/q}$$
$$\times \left\{\left[\int_0^1 \left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right| + \left[\int_0^1 \left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$= \frac{1}{2}\left(\frac{2ab}{a+b}\right)^2 \left[\nu(b,a;q)\right]^{1-\frac{1}{q}} \left[\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}}\left|f'(a)\right| + \left|f'(b)\right|\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right]. \tag{2.34}$$

Using (2.33) and (2.34) in (2.32), we obtain the required inequality (2.31). □

Corollary 7. *Suppose the assumptions of Theorem 8 are satisfied and if $g(x) = \frac{ab}{b-a}$ for all $x \in [a,b]$, then the following inequality holds*

$$\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{1}{2}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{a+b}\right)^2$$
$$\times \left\{\left[\nu(a,b;q)\right]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(a)\right| + \left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(b)\right|\right]\right.$$
$$\left. + \left[\nu(b,a;q)\right]^{1-\frac{1}{q}}\left[\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}}\left|f'(a)\right| + \left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left|f'(b)\right|\right]\right\}, \tag{2.35}$$

where $\nu(\cdot,\cdot;\cdot)$ is defined in Theorem 9.

Remark 1. *Some further results can be obtained from (2.25) but we omit the details for the interested readers.*

3. Applications to Special Means

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a harmonically convex function and a harmonically symmetric function to construct inequalities for special means.

For positive numbers $a > 0$ and $b > 0$ with $a \neq b$

$$A(a,b) = \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2ab}{a+b}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ L(a,b), & p = -1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0 \end{cases}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively.

For further information on means, we refer the readers to [1] and the references therein.

Let $g : [a, b] \to \mathbb{R}_0$ be defined as

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a, b].$$

It is obvious that

$$g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}}\right) = g(x)$$

for all $x \in [a, b]$. Hence $g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2$, $x \in [a, b]$ is harmonically symmetric with respect to $x = \frac{2ab}{a+b}$.

Now applications of our results are given in the following theorems to come.

Theorem 10. *Let $0 < a < b$. Then the following inequality holds*

$$\left|\frac{A^2(a,b) - G^2(a,b)}{3A^2(a,b)} - \frac{A^2(a,b) + G^2(a,b)}{G^2(a,b)} + \frac{2A(a,b)}{L(a,b)}\right| \\ \leq \frac{a}{b}\ln\left(\frac{a}{A(a,b)}\right) + \frac{b}{a}\ln\left(\frac{A(a,b)}{b}\right) + \frac{a\ln\left(\frac{b}{A(a,b)}\right) + b\ln\left(\frac{A(a,b)}{a}\right)}{H(a,b)}. \quad (3.1)$$

Proof. Applying Corollary 1 to the functions

$$f(x) = x^2 \text{ for } x > 0$$

and

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a, b]$$

we get the desired result. □

Theorem 11. *Let $0 < a < b$ and $r \in (-1, \infty) \setminus \{0\}$. Then the following inequality holds*

$$\left|H^{r+2}(a,b) - G^2(a,b)\, L_r^r(a,b)\right| \\ \leq (r+2)\left(\frac{1}{2}\right)\left(\frac{2ab}{b-a}\right)^2 \left\{\left(\frac{a^{r+1}}{H(a,b)} - \frac{b^{r+1}}{a}\right)\ln\left(\frac{b}{A(a,b)}\right) \right. \\ \left. + \left(\frac{a^{r+1}}{b} - \frac{b^{r+1}}{H(a,b)}\right)\ln\left(\frac{a}{A(a,b)}\right)\right\}. \quad (3.2)$$

Proof. The assertion follows from the inequality proved in Corollary 3 for $f(x) = x^{r+2}$, $x > 0$, $r \in (-1, \infty) \setminus \{0\}$. □

Theorem 12. *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$\left|H^2(a,b) - G^2(a,b)\right| \leq \left(\frac{1}{2}\right)^{1/q-1}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^2\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left(\left[\frac{2ab^{2q}\left((1-2q)(a-b)-b\right)H^{-2q}(a,b) - \left((1-2q)(a-b)-2b\right)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]a^q\right.\right.$$
$$\left.+\left[\frac{2a^2b^{2q}H^{-2q}(a,b) + \left((2q-1)(a-b)-2a\right)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]b^q\right)^{1/q}$$
$$+\left(\left[\frac{2a^{2q}b\left((1-2q)(b-a)-a\right)H^{-2q}(a,b) - \left((1-2q)(b-a)-2a\right)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]b^q\right.$$
$$\left.\left.+\left[\frac{2a^{2q}b^2H^{-2q}(a,b) + \left((2q-1)(b-a)-2b\right)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]a^q\right)^{1/q}\right\}. \tag{3.3}$$

Proof. Applying Corollary 4 to the function

$$f(x) = x^2 \text{ for } x > 0$$

we get the desired result. □

Theorem 13. *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$\left|\frac{A^2(a,b) - G^2(a,b)}{3A^2(a,b)} - \frac{A^2(a,b) + G^2(a,b)}{G^2(a,b)} + \frac{2A(a,b)}{L(a,b)}\right|$$
$$\leq \left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1/q-2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{b-a}{2ab}\right)^2$$
$$\times\left\{\left[\frac{\left(a^2b^{2q} + a^{2q}b\left((2q-1)(a-b)-a\right)\right)H^{-2q}(a,b)}{(a-b)^2(q-1)(2q-1)}\right]a^q\right.$$
$$\left.+\left[\frac{\left(a^{2q}b^2 + ab^{2q}\left((2q-1)(b-a)-b\right)\right)H^{-2q}(a,b)}{(a-b)^2(q-1)(2q-1)}\right]b^q\right\}^{1/q}. \tag{3.4}$$

Proof. Applying Theorem 7 to the functions

$$f(x) = x^2 \text{ for } x > 0$$

and

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a,b]$$

we get the desired result. □

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