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ON INEQUALITIES OF JENSEN-OSTROWSKI TYPE

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ABSTRACT. We provide inequalities of Jensen-Ostrowski type, by considering bounds for the magnitude of

$$\int_{\Omega} f\circ g\,d\mu - f(\zeta) - \left(\int_{\Omega} g d\mu - \zeta\right) f'(\zeta) - \frac{1}{2}\lambda \int_{\Omega} (g-\zeta)^2\,d\mu, \quad \zeta \in [a.b],$$

with various assumptions on the absolutely continuous function $f:[a,b]\to\mathbb{C}$ and a μ-measurable function g, and $\lambda \in \mathbb{C}$. Inequalities of Ostrowski and Jensen type are obtained as special cases, by setting $\lambda = 0$ and $\zeta = \int_{\Omega} g\,d\mu$, respectively. In particular, we obtain some bounds for the discrepancy in Jensen's integral inequality. Applications of these inequalities for f-divergence measures are also given.

1. Introduction

In 1905 (1906) Jensen defined convex functions as follows [18] : f is convex if

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a,b \in D(f)$ (here $D(f)$ is the domain of f). Inequality (1.1) is the simplest form of Jensen's inequality. Jensen's inequality has been widely applied in many areas of research, e.g. probability theory, statistical physics, and information theory.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space such that $\int_{\Omega} d\mu = 1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers. Jensen's inequality now takes the following form: for a μ-integrable function $g:\Omega\to[m,M]\subset\mathbb{R}$, and a convex function $f:[m,M]\to\mathbb{R}$, we have

$$f\left(\int_{\Omega} g\,d\mu\right) \le \int_{\Omega} f\circ g\,d\mu. \tag{1.2}$$

Costarelli and Spigler [4] considered the sharpness of Jensen's integral inequality (for real-valued convex function f and non-negative function g) by studying bounds for the discrepancy in Jensen's inequality. Further inequalities involving the discrepancy in Jensen's inequality for general integrals are given in [7] and [8]. We summarise these results in Section 2.

In 1938, Ostrowski [17], proved an inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$ ($x\in[a,b]$):

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Proposition 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.3}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Ostrowski's inequality has been extended to approximate the integral mean of n-time differentiable mappings and other classes of functions. We refer the readers to the book by Mitrinović, Pečarić, and Fink [16] and the book by Dragomir and Rassias [11] for these generalisations. In what follows, we recall a generalisation of Ostrowski's inequality for twice differentiable mappings.

Proposition 2 (Cerone, Dragomir, and Roumeliotis [2])**.** *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f' : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then, we have the inequality:*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)f(x) + (b-a)\left(x - \frac{a+b}{2}\right) f'(x) \right| \\ &\le \left[\frac{1}{24} + \frac{1}{2}\frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)^3 \|f''\|_\infty, \end{aligned} \tag{1.4}$$

for all $x \in [a,b]$.

Dragomir [9] introduced some inequalities which combine the two aforementioned inequalities, referred to as the Jensen-Ostrowski type inequalities. These inequalities are established by obtaining bounds for the magnitude of

$$\int_\Omega f \circ g \, d\mu - f(\zeta) - \lambda \left(\int_\Omega g d\mu - \zeta \right), \quad \zeta \in [a,b],$$

for various assumptions on the absolutely continuous function $f : [a,b] \to \mathbb{C}$ and a μ-measurable function g, and $\lambda \in \mathbb{C}$. Inequalities of Jensen and Ostrowski type are obtained by setting $\zeta = \int_\Omega g\,d\mu$ and $\lambda = 0$, respectively. Further Jensen-Ostrowski type inequalities are given in [3], by investigating the magnitude of:

$$\int_\Omega (f \circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta) f' \circ g \, d\mu + \frac{1}{2}\lambda \int_\Omega (g-\zeta)^2 \, d\mu, \quad \zeta \in [a,b].$$

In this paper, we study the magnitude of:

$$\int_\Omega f \circ g \, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta \right) f'(\zeta) - \frac{1}{2}\lambda \int_\Omega (g-\zeta)^2 \, d\mu, \quad \zeta \in [a.b],$$

to provide new inequalities of Jensen-Ostrowski type. Our results stem on the use of the Taylor's approximation with integral remainders (cf. Lemma 1 of Section 3). We obtain inequalities with bounds involving the p-norms ($1 \le p \le \infty$) (Section 4), as well as inequalities for functions with bounded second derivatives and convex second derivatives (Section 5). An application for f-divergence measure in information theory are provided in Section 6.

Similarly to the results in [3] and [9], we obtain inequalities of Ostrowski type by setting $\lambda = 0$. We provide a generalised version of the Ostrowski inequality

(1.4) (cf. Proposition 2 above) in the measure-theoretic (and probabilistic) form in Remark 4.

We obtain inequalities of Jensen type by setting $\zeta = \int_\Omega g\,d\mu$. In particular, we obtain in Corollary 2, a result on the discrepancy in Jensen's inequality (cf. inequality (4.4)), without the assumption of convexity. We connect this result with the results of Costarelli and Spigler [4] (cf. Proposition 3 of Section 2) in Remark 3. Costarelli and Sprigler noted that the bound in (2.4) is better than (2.3) due to a stronger assumption of C^2 smoothness. Under the assumptions of Proposition 3, our result gives a better upper bound than (2.3), although (2.4) still gives the better upper bound. However, our result holds in a more general setting, that is, for differentiable functions with absolutely continuous derivatives, in a measure-theoretic (and probabilistic) form.

2. The sharpness of Jensen type inequalities

In this section, we recall some results concerning the discrepancy in Jensen's inequality.

Proposition 3 (Costarelli and Spigler [4])**.** *Let $\varphi : I \to \mathbb{R}$ be real-valued convex function, where I is a connected bounded set in $\mathbb{R}$, and $f : [0,1] \to I$ a real-valued nonnegative function where $f \in L^1(0,1)$. If φ is a C^2 function, then*

$$\varphi(f(x)) = \varphi(c) + \varphi'(c)[f(x)-c] + \frac{1}{2}\varphi''\left(c^*(x)\right)[f(x)-c]^2, \quad x \in [0,1], \tag{2.1}$$

where $c = f(x_0)$ which can be chosen arbitrarily in the domain of φ such that $f(x_0) \in \mathring{I}$, and $c^*(x)$ is a suitable value between $f(x)$ and $f(x_0)$. Furthermore,

$$\int_0^1 \varphi(f(x))\,dx = \varphi(c)+\varphi'(c)\int_0^1 [f(x)-c]\,dx+\frac{1}{2}\int_0^1 \varphi''\left(c^*(x)\right)[f(x)-c]^2\,dx. \tag{2.2}$$

The discrepancy in the Jensen inequality is given by the following estimates:

$$\begin{aligned} 0 &\le \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right) \\ &\le \frac{1}{2}\|\varphi''\|_{L^\infty(I_2)}\left[\|f-c\|_{L^2}^2 + \|f-c\|_{L^1}^2\right], \end{aligned} \tag{2.3}$$

where I_2 denotes the domain of φ''; and

$$\begin{aligned} 0 &\le \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right) \\ &\le \frac{1}{2}\|\varphi''\|_{L^\infty(I^2)}\|f-c\|_{L^2}^2 - \frac{1}{2}\inf_{I_2}\varphi''\left[\int_0^1 ((f(x)-c)\,dx\right]^2, \end{aligned} \tag{2.4}$$

where φ is a C^2-smooth function.

Consider the Lebesgue space

$$L(\Omega,\mu) := \left\{ f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\,d\mu(t) < \infty \right\}.$$

For simplicity of notation, we write in the text $\int_\Omega w\,d\mu$ instead of $\int_\Omega w(t)d\mu(t)$. Dragomir [7] obtained in 2002, the following result on the discrepancy in the Jensen inequality for general integrals.

Theorem 1. *Let $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $g : \Omega \to [m, M]$ so that $f \circ g,\ g,\ f' \circ g,\ (f' \circ g) \cdot g \in L(\Omega, \mu)$. Then,*

$$0 \leq \int_\Omega f \circ g \, d\mu - f\left(\int_\Omega g \, d\mu\right) \tag{2.5}$$
$$\leq \int_\Omega g \cdot (f' \circ g) \, d\mu - \int_\Omega f' \circ g d\mu \int_\Omega g \, d\mu$$
$$\leq \frac{1}{2} [f'(M) - f'(m)] \int_\Omega \left| g - \int_\Omega g \, d\mu \right| d\mu.$$

Further result on the discrepancy in the Jensen inequality for general integrals is given in the following result.

Theorem 2 (Dragomir [8])**.** *Let $f : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq g(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $g,\ f \circ g \in L(\Omega, \mu)$, then

$$0 \leq \int_\Omega f \circ g \, d\mu - f\left(\int_\Omega g \, d\mu\right) \tag{2.6}$$
$$\leq \left(M - \int_\Omega g \, d\mu\right)\left(\int_\Omega g \, d\mu - m\right) \frac{f'_-(M) - f'_+(m)}{M - m}$$
$$\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right],$$

where f'_- is the left and f'_+ is the right derivative of the convex function f.

We refer the readers to [8] for further results on the discrepancy in the Jensen inequality and their applications to divergence measures.

3. Some estimates

We start with the following lemma to assist us in our calculations throughout the paper.

Lemma 1. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a, b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a, b]$ and $\zeta \in [a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω such that $f \circ g,\ g, (g - \zeta)^2 \in L(\Omega, \mu)$, with $\int_\Omega d\mu = 1$, then*

$$\int_\Omega f \circ g \, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta) - \frac{1}{2}\lambda \int_\Omega (g - \zeta)^2 \, d\mu \tag{3.1}$$
$$= \int_\Omega (g - \zeta)^2 \left(\int_0^1 (1 - s)\left[f''((1 - s)\zeta + sg) - \lambda\right] ds\right) d\mu$$
$$= \int_0^1 (1 - s)\left(\int_\Omega (g - \zeta)^2 \left[f''((1 - s)\zeta + sg) - \lambda\right] d\mu\right) ds,$$

for any $\lambda \in \mathbb{C}$.

Proof. Making use of the Taylor's expansion with integral remainder we have

$$f(x) = f(\zeta) + (x - \zeta) f'(\zeta) + (x - \zeta)^2 \int_0^1 (1 - s) f''((1 - s)\zeta + sx) \, ds \tag{3.2}$$

for any $\zeta, x \in [a,b]$. We observe that for $\lambda \in \mathbb{C}$ we have

$$
\begin{aligned}
(3.3)\qquad & (x-\zeta)^2 \int_0^1 (1-s)\left[f''((1-s)\zeta + sx) - \lambda\right] ds \\
&= (x-\zeta)^2 \int_0^1 (1-s) f''((1-s)\zeta + sx)\, ds - (x-\zeta)^2 \lambda \int_0^1 (1-s)\, ds \\
&= (x-\zeta)^2 \int_0^1 (1-s) f''((1-s)\zeta + sx)\, ds - \frac{1}{2}(x-\zeta)^2 \lambda
\end{aligned}
$$

and by (3.2) we get

$$
\begin{aligned}
f(x) &= f(\zeta) + (x-\zeta) f'(\zeta) + \frac{1}{2}\lambda (x-\zeta)^2 \\
&\quad + (x-\zeta)^2 \int_0^1 (1-s)\left[f''((1-s)\zeta + sx) - \lambda\right] ds
\end{aligned}
$$

for any $\zeta, x \in [a,b]$ and $\lambda \in \mathbb{C}$. Now, if we replace x with $g(t) \in [a,b]$ we get

$$
\begin{aligned}
(3.4)\qquad f(g(t)) &= f(\zeta) + (g(t)-\zeta) f'(\zeta) + \frac{1}{2}\lambda (g(t)-\zeta)^2 \\
&\quad + (g(t)-\zeta)^2 \int_0^1 (1-s)\left[f''((1-s)\zeta + sg(t)) - \lambda\right] ds
\end{aligned}
$$

for any $\zeta \in [a,b], t \in \Omega$ and $\lambda \in \mathbb{C}$. If we integrate (3.4) on Ω and use the fact that $\int_\Omega d\mu = 1$ we obtain the first result in (3.1) by rearranging the terms. The second part follows by Fubini's theorem. □

We denote by $\sigma^2(g)$, the dispersion of g on Ω, that is,

$$
\sigma^2(g) := \int_\Omega g^2\, d\mu - \left(\int_\Omega g\, d\mu\right)^2 = \int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 d\mu.
$$

Corollary 1. *Under the assumptions of Lemma 1, we have the following identities by when $\zeta = \int_\Omega g\, d\mu$:*

$$
\begin{aligned}
(3.5)\qquad & \int_\Omega f \circ g\, d\mu - f\left(\int_\Omega g\, d\mu\right) - \frac{1}{2}\lambda\sigma^2(g) \\
&= \int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 \left(\int_0^1 (1-s)\left[f''\left((1-s)\int_\Omega g\, d\mu + sg\right) - \lambda\right] ds\right) d\mu \\
&= \int_0^1 (1-s)\left(\int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 \left[f''\left((1-s)\int_\Omega g\, d\mu + sg\right) - \lambda\right] d\mu\right) ds,
\end{aligned}
$$

for any $\lambda \in \mathbb{C}$.

Remark 1. Following the main idea for some estimates obtained by Costarelli and Spigler [4], another estimate one may obtain is to consider the mean value form of the remainder in (3.2)

$$
(3.6)\qquad f(x) = f(\zeta) + (x-\zeta) f'(\zeta) + \frac{1}{2} f''(\xi)(x-\zeta)^2,
$$

where ξ is between x and ζ. By setting $x = g(t)$, and integrate (3.6) on Ω, we obtain

$$
(3.7)\qquad \int_\Omega f \circ g\, d\mu = f(\zeta) + f'(\zeta)\left(\int_\Omega g\, d\mu - \zeta\right) + \frac{1}{2}\int_\Omega f''(\xi)(g-\zeta)^2\, d\mu,
$$

where $\xi = \xi(t)$ is between $g(t)$ and ζ.

Let $\varphi : I \to \mathbb{R}$ be a real-valued convex function, where I is a connected bounded set in $\mathbb{R}$, and $f : [0,1] \to I$ a real-valued nonnegative function where $f \in L^1(0,1)$. Suppose that φ is a C^2 function. Set $f \equiv \varphi$, $g \equiv f$, and $\zeta = c = f(x_0)$ (x_0 can be chosen arbitrarily such that $f(x_0) \in \mathring{I}$) in (3.7), we have

$$\begin{aligned} &\int_0^1 \varphi\left(f(x)\right)\,dx \\ &= \varphi\left(c\right) + \varphi'\left(c\right)\int_0^1 \left(f(x) - c\right)\,dx + \frac{1}{2}\int_0^1 \varphi''(c^*(x))(g(x)-c)^2\,dx, \end{aligned}$$

where $c^*(x)$ is between $f(x)$ and $\zeta = f(x_0)$. This estimate is given in the paper by Costarelli and Spigler [4, p. 2] to investigate the sharpness of the Jensen inequality (cf. Proposition 3).

4. Bounds in terms of p-norms

We use the notation

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p\,d\mu(t)\right)^{1/p}, & p \geq 1,\ k \in L_p(\Omega,\mu); \\ \operatorname*{ess\,sup}_{t\in\Omega} |k(t)|, & p = \infty,\ k \in L_\infty(\Omega,\mu); \end{cases}$$

and

$$\|f\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |f(s)|^p\,ds\right)^{1/p}, & p \geq 1,\ f \in L_p([0,1]); \\ \operatorname*{ess\,sup}_{s\in[0,1]} |f(s)|, & p = \infty,\ f \in L_\infty([0,1]). \end{cases}$$

We denote by ℓ, the identity function on $[0,1]$, namely, $\ell(t) = t$ ($t \in [0,1]$); and for $t \in \Omega$, $\zeta \in [a,b]$, and $\lambda \in \mathbb{C}$, we have

$$\operatorname*{ess\,sup}_{s\in[0,1]} |f''\big((1-s)\zeta + sg(t)\big) - \lambda| = \|f''\big((1-\ell)\zeta + \ell g\big) - \lambda\|_{[0,1],\infty}.$$

Theorem 3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a,b]$ and $\zeta \in [a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g,\ g, (g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$, then for any $\lambda \in \mathbb{C}$,*

$$\begin{aligned} (4.1)\ &\left|\int_\Omega f\circ g d\mu - f\left(\zeta\right) - \left(\int_\Omega g d\mu - \zeta\right) f'\left(\zeta\right) - \frac{1}{2}\lambda \int_\Omega \left(g-\zeta\right)^2 d\mu\right| \\ &\leq \frac{1}{2}\int_\Omega \left(g-\zeta\right)^2 \|f''\left(\left(1-\ell\right)\zeta + \ell g\right) - \lambda\|_{[0,1],\infty}\,d\mu \\ &\leq \begin{cases} \frac{1}{2}\|g-\zeta\|^2_{\Omega,\infty}\big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\big\|_{\Omega,1}; \\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,p}\big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\big\|_{\Omega,q},\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,1}\big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\big\|_{\Omega,\infty}. \end{cases} \end{aligned}$$

Proof. Taking the modulus in (3.1), we have

$$
\begin{aligned}
(4.2)\qquad & \left|\int_\Omega f\circ g d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta) - \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2\, d\mu\right| \\
\le\ & \int_0^1 (1-s)\left(\int_\Omega (g-\zeta)^2 \left|f''((1-s)\zeta + sg) - \lambda\right| d\mu\right) ds \\
\le\ & \int_0^1 (1-s)\left(\int_\Omega (g-\zeta)^2 \left\|f''((1-\ell)\zeta + \ell g) - \lambda\right\|_{[0,1],\infty} d\mu\right) ds \\
=\ & \int_0^1 (1-s)\, ds\left(\int_\Omega (g-\zeta)^2 \left\|f''((1-\ell)\zeta + \ell g) - \lambda\right\|_{[0,1],\infty} d\mu\right) \\
=\ & \frac{1}{2}\int_\Omega (g-\zeta)^2 \left\|f''((1-\ell)\zeta + \ell g) - \lambda\right\|_{[0,1],\infty} d\mu,
\end{aligned}
$$

for any $\lambda \in \mathbb{C}$. Utilising Hölder's inequality for the μ-measurable functions $F, G : \Omega \to \mathbb{C}$,

$$
\left|\int_\Omega FG\, d\mu\right| \le \left(\int_\Omega |F|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |G|^q\, d\mu\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1,
$$

and

$$
\left|\int_\Omega FG\, d\mu\right| \le \operatorname*{ess\,sup}_{t\in\Omega} |F(t)| \int_\Omega |G|\, d\mu,
$$

we get (4.1) from (4.2). □

Remark 2. One obtains Ostrowski type and Jensen type inequalities from Theorem 3, by letting $\lambda = 0$ and $\zeta = \int_\Omega g\, d\mu$ in (4.1), respectively.

Corollary 2. *Under the assumptions of Theorem 3, we have the following Ostrowski type inequality:*

$$
\begin{aligned}
(4.3)\qquad & \left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\
& \le \frac{1}{2}\|f''\|_{[a,b],\infty}\left[\sigma^2(g) + \left(\int_\Omega g\, d\mu - \zeta\right)^2\right].
\end{aligned}
$$

We also have the following Jensen type inequality:

$$
(4.4)\qquad \left|\int_\Omega (f\circ g)\, d\mu - f\left(\int_\Omega g\, d\mu\right)\right| \le \frac{1}{2}\|f''\|_{[a,b],\infty}\ \sigma^2(g)
$$

which is the best inequality one can get from (4.3).

Proof. We have from (4.1) with $\lambda = 0$

$$
\begin{aligned}
& \left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\
& \le \frac{1}{2}\int_\Omega (g-\zeta)^2 \|f''\big((1-\ell)\zeta + \ell g\big)\|_{[0,1],\infty}\, d\mu.
\end{aligned}
$$

For any $t \in \Omega$ and almost every $s \in [0,1]$, we have

$$
\left|f''\big((1-s)\zeta + sg(t)\big)\right| \le \operatorname*{ess\ sup}_{u\in[a,b]} |f''(u)| = \|f''\|_{[a,b],\infty},
$$

which implies that

$$\left\|f''\big((1-\ell)\zeta+\ell g\big)\right\|_{[0,1],\infty}=\operatorname{ess}\sup_{s\in[0,1]}\left|f''\big((1-s)\zeta+sg(t)\big)\right|\leq\|f''\|_{[a,b],\infty}.$$

Therefore, we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta)-\left(\int_\Omega g d\mu-\zeta\right)f'(\zeta)\right|\leq\frac{1}{2}\|f''\|_{[a,b],\infty}\int_\Omega (g-\zeta)^2\,d\mu.$$

We also note that

$$\begin{aligned}\int_\Omega (g-\zeta)^2\,d\mu &= \int_\Omega\left(g-\int_\Omega g\,d\mu+\int_\Omega g\,d\mu-\zeta\right)^2 d\mu \\ &= \int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu+\left(\int_\Omega g\,d\mu-\zeta\right)^2 \\ &= \sigma^2(g)+\left(\int_\Omega g\,d\mu-\zeta\right)^2.\end{aligned}\tag{4.5}$$

and this proves (4.3). By choosing $\zeta=\int_\Omega g\,d\mu$ in (4.3), we obtain (4.4). □

Remark 3. We may also obtained the results in Corollary 2 from (3.7), which uses the mean value form of the remainder, so that,

$$\begin{aligned}&\left|\int_\Omega f\circ g\,d\mu-f(\zeta)-f'(\zeta)\left(\int_\Omega g\,d\mu-\zeta\right)\right| \\ &\leq\frac{1}{2}\int_\Omega |f''(\xi)|(g-\zeta)^2\,d\mu \\ &\leq\frac{1}{2}\|f''\|_{[a,b],\infty}\int_\Omega (g-\zeta)^2\,d\mu \\ &=\frac{1}{2}\|f''\|_{[a,b],\infty}\left(\sigma^2(g)+\left(\int_\Omega g\,d\mu-\zeta\right)^2\right).\end{aligned}\tag{4.6}$$

Let $\varphi:I\to\mathbb{R}$ be a real-valued convex function, where I is a connected bounded set in $\mathbb{R}$, and $f:[0,1]\to I$ a real-valued nonnegative function where $f\in L^1(0,1)$. Suppose that φ is a C^2 function. Set $f\equiv\varphi$, $g\equiv f$, and $\zeta=\int_0^1 g(t)\,dt$ in (4.6), we have

$$\begin{aligned}&\left|\int_0^1\varphi(f(x))\,dx-\varphi\left(\int_0^1 f(x)\,dx\right)\right| \\ &\leq\frac{1}{2}\|\varphi''\|_{I_2,\infty}\int_0^1\left(f(x)-\int_0^1 f(x)\,dx\right)^2 dt,\end{aligned}$$

where I_2 is the domain of φ''. Furthermore, if φ is convex and f is continuous, then the mean value theorem for integration asserts that there exists $x_0\in[0,1]$ such that $\int_0^1 f(t)\,dt = f(x_0) =: c$, and thus

$$\begin{aligned} 0 &\leq \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right) \\ &\leq \frac{1}{2}\|\varphi''\|_{I_2,\infty} \int_0^1 (f(x)-c)^2\, dx \qquad (4.7) \\ &= \frac{1}{2}\|\varphi''\|_{I_2,\infty}\|f-c\|^2_{[0,1],2} \\ &\leq \frac{1}{2}\|\varphi''\|_{I_2,\infty}\left(\|f-c\|^2_{[0,1],2} + \|f-c\|^2_{[0,1],1}\right) \end{aligned}$$

where the last estimate is given by Costarelli and Spigler in (2.3). Here, our result is shown to be sharper than the result by Costarelli and Spigler. When, φ is assumed to be C^2-smooth, the result (2.4) by Costarelli and Spigler is sharper than our estimate:

$$\begin{aligned} 0 &\leq \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right) \\ &\leq \frac{1}{2}\|\varphi''\|_{I_2,\infty}\|f-c\|^2_{[0,1],2} - \frac{1}{2}\inf_{I_2}\varphi''\left[\int_0^1 ((f(x)-c)\,dx\right]^2 \\ &\leq \frac{1}{2}\|\varphi''\|_{I_2,\infty}\|f-c\|^2_{[0,1],2}. \end{aligned}$$

Costarelli and Spigler provide an example to compare the two bounds given in (2.3) and (2.4) [4, Example 3.1, p. 5]. In what follows, we recall the example and provide a comparison to the bound obtained in (4.7). Let $\varphi(y) = -\sin \pi y$ and $f(x) = x^2$. The true discrepancy E between the two sides of the Jensen inequality is $E \approx 0.3612$. Using (2.3), the estimate for E is: $E \leq 1.3627....$ Noting that $\inf_{I_2}\varphi''\left[\int_0^1((f(x)-c)\,dx\right]^2 = 0$, the estimate for E by using (2.4) is the same to that of (4.7), which is closer to the true discrepancy, that is,

$$E \leq \frac{1}{2}\|\varphi''\|_{I_2,\infty}\|f-c\|^2_{[0,1],2} = \frac{\pi^2}{2}\left[\frac{1}{9} - \frac{2}{9} + \frac{1}{5}\right] \approx 0.4386.$$

Remark 4. We recall the quantity:

$$\int_\Omega (g-\zeta)^2\,d\mu = \int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu - \zeta\right)^2.$$

In the case that $\Omega = [a,b]$, $g : [a,b] \to [a,b]$ defined by $g(t) = t$, and $\mu(t) = \frac{t}{b-a}$, we have

$$\int_\Omega g\,d\mu = \frac{1}{b-a}\int_a^b t\,dt = \frac{a+b}{2},$$

and

$$\begin{aligned} &\int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu - \zeta\right)^2 \\ &= \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)^2 dt + \left(\zeta - \frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12} + \left(\zeta - \frac{a+b}{2}\right)^2. \end{aligned}$$

Under this assumption, the left-hand side of (4.3) becomes

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(\zeta) - \left(\frac{a+b}{2}-\zeta\right)f'(\zeta)\right|$$

and the right-hand side of (4.3) becomes

$$\begin{aligned}&\frac{1}{2}\|f''\|_{[a,b],\infty}\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu-\zeta\right)^2\right]\\&=\frac{1}{2}\|f''\|_{[a,b],\infty}\left[\frac{1}{12}(b-a)^2+\left(\zeta-\frac{a+b}{2}\right)^2\right].\end{aligned}$$

Thus, (4.3) becomes

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(\zeta) + \left(\zeta-\frac{a+b}{2}\right)f'(\zeta)\right|\\&\le\left[\frac{1}{24}+\frac{1}{2}\,\frac{(\zeta-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)^2\|f''\|_{[a,b],\infty}\end{aligned}$$

for $\zeta\in[a,b]$, which recovers the result by Cerone, Dragomir, and Roumeliotis [2] (cf. Dragomir and Rassias [11, p. 87]), by multiplying the above inequality by $(b-a)$ and setting $\zeta = x\in[a,b]$.

5. Inequalities for functions with bounded and convex second derivatives

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [9]

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{h:[a,b]\to\mathbb{C}\middle|\mathrm{Re}\left[(\Gamma-h(t))(\overline{h(t)}-\overline{\gamma})\right]\ge 0 \text{ for a.e. } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{h:[a,b]\to\mathbb{C}\middle|\,\left|h(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}\left|\Gamma-\gamma\right| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation results may be stated [9].

Proposition 4. *For any $\gamma,\Gamma\in\mathbb{C}$ and $\gamma\ne\Gamma$, we have*

(i) *$\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets;*
(ii) *$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$; and*
(iii) $\bar{U}_{[a,b]}(\gamma,\Gamma)=\{h:[a,b]\to\mathbb{C}\mid(\mathrm{Re}(\Gamma)-\mathrm{Re}(h(t)))(\mathrm{Re}(h(t))-\mathrm{Re}(\gamma)) + (\mathrm{Im}(\Gamma)-\mathrm{Im}(h(t)))(\mathrm{Im}(h(t))-\mathrm{Im}(\gamma))\ge 0$ *for a.e.* $t\in[a,b]\}$.

We have the following Jensen-Ostrowski inequality:

Theorem 4. *Let $f: I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, $f':[a,b]\subset\mathring{I}\to\mathbb{C}$ is absolutely continuous on $[a,b]$ and $\zeta\in[a,b]$. For some $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$, assume that $f''\in\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on*

Ω such that $f\circ g,\ g,(g-\zeta)^2\in L(\Omega,\mu)$, with $\int_\Omega d\mu=1$, then

$$\left|\int_\Omega (f\circ g)\,d\mu-f(\zeta)-\left(\int_\Omega gd\mu-\zeta\right)f'(\zeta)-\frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu\right| \le \frac{1}{4}|\Gamma-\gamma|\left[\sigma^2(g)+\left(\int_\Omega g\,d\mu-\zeta\right)^2\right]. \tag{5.1}$$

In particular, we have the following Ostrowski type inequality:

$$\left|\int_\Omega (f\circ g)\,d\mu-f\left(\frac{a+b}{2}\right) -\left(\int_\Omega gd\mu-\frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right)-\frac{\gamma+\Gamma}{4}\int_\Omega\left(g-\frac{a+b}{2}\right)^2 d\mu\right| \le \frac{1}{4}|\Gamma-\gamma|\left[\sigma^2(g)+\left(\int_\Omega g\,d\mu-\frac{a+b}{2}\right)^2\right], \tag{5.2}$$

and we have the following Jensen type inequality:

$$\left|\int_\Omega (f\circ g)\,d\mu-f\left(\int_\Omega g\,d\mu\right)-\frac{\gamma+\Gamma}{4}\sigma^2(g)\right| \le \frac{1}{4}|\Gamma-\gamma|\sigma^2(g). \tag{5.3}$$

Proof. By equality (3.1), for $\lambda=\frac{\gamma+\Gamma}{2}$ we have

$$\int_\Omega (f\circ g)\,d\mu-f(\zeta)-\left(\int_\Omega gd\mu-\zeta\right)f'(\zeta)-\frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu = \int_\Omega\left[(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg\big)-\frac{\gamma+\Gamma}{2}\right]ds\right]d\mu. \tag{5.4}$$

Since $f''\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

$$\left|f''\big((1-s)\zeta+sg\big)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma|, \tag{5.5}$$

for almost every $s\in[0,1]$ and any $t\in\Omega$. Multiply (5.5) with $s>0$ and integrate over $[0,1]$, we obtain

$$\int_0^1 s\left|f''\big((1-s)\zeta+sg\big)-\frac{\gamma+\Gamma}{2}\right|ds\le\frac{1}{2}|\Gamma-\gamma|\int_0^1 s\,ds=\frac{1}{4}|\Gamma-\gamma|, \tag{5.6}$$

for any $t\in\Omega$.

Taking the modulus of (5.4), we get the following, by (5.6)

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu-f(\zeta)-\left(\int_\Omega gd\mu-\zeta\right)f'(\zeta)-\frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu\right|\\ &\le\int_\Omega\left[(g-\zeta)^2\int_0^1 s\left|f''\big((1-s)\zeta+sg\big)-\frac{\gamma+\Gamma}{2}\right|ds\right]d\mu\\ &\le\frac{1}{4}|\Gamma-\gamma|\int_\Omega (g-\zeta)^2\,d\mu,\end{aligned}$$

and the proof is completed (note the use of (4.5)). We obtain (5.2) and (5.2), by setting $\zeta=(a+b)/2$ and $\zeta=\int_\Omega g\,d\mu$, respectively. □

Remark 5. If f' is convex in Theorem 4, then $\gamma = f''_+(a)$ and $\Gamma = f''_-(b)$.

Recall the following definitions of convexity:

Definition 1. *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then,*

(1) h is convex, if for any $x, y \in I$ and $s \in [0,1]$, we have

$$h\big((1-s)x + sy\big) \le (1-s)h(x) + sh(y).$$

(2) h is quasi-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have

$$h\big((1-s)x + sy\big) \le \max\{h(x), h(y)\}.$$

(3) h is log-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have

$$h\big((1-s)x + sy\big) \le h(x)^{1-s} h(y)^s.$$

(4) for a fixed $q \in (0,1]$, h is q-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have

$$h\big((1-s)x + sy\big) \le (1-s)^q h(x) + s^q h(y).$$

For further background on these notions of convexity and some integral inequalities for functions with some convexity properties, we refer the reader to the paper by Dragomir [10].

Theorem 5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a,b]$ and $\zeta \in [a,b]$. Suppose that $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $g, (g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$.*

(i) If $|f''|$ is convex, then we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\ &\le \frac{1}{3}\left[|f''(\zeta)| \int_\Omega (g-\zeta)^2\, d\mu + \frac{1}{2}\int_\Omega (g-\zeta)^2 |f''\circ g|\, d\mu\right].\end{aligned} \tag{5.7}$$

(ii) If $|f''|$ is quasi-convex, then we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\ &\le \frac{1}{2}\|f''\|_{[a,b],\infty} \int_\Omega (g-\zeta)^2\, d\mu.\end{aligned} \tag{5.8}$$

(iii) If $|f''|$ is log-convex, then we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\ &\le \int_\Omega (g-\zeta)^2 \left|\frac{-|f''(\zeta)| + |f''\circ g| + |f''(\zeta)|\left[\log(|f''(\zeta)|) - \log(|f''\circ g|)\right]}{\left[\log(|f''(\zeta)|) - \log(|f''\circ g|)\right]^2}\right| d\mu.\end{aligned} \tag{5.9}$$

(iv) If $|f''|$ is q-convex (for a fixed $q \in (0,1]$), then we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\, d\mu - f(\zeta) - \left(\int_\Omega g d\mu - \zeta\right) f'(\zeta)\right| \\ &\le \frac{1}{(q+2)}\left[|f''(\zeta)| \int_\Omega (g-\zeta)^2\, d\mu + \frac{1}{q+1}\int_\Omega (g-\zeta)^2 |f''\circ g|\, d\mu\right].\end{aligned} \tag{5.10}$$

Proof. (i) If $|f''|$ is convex, then

$$\left|f''\big((1-s)\zeta + sg(t)\big)\right| \le (1-s)|f''(\zeta)| + s|f''(g(t))|, \quad \text{for all } t \in \Omega,$$

which implies that

$$\begin{aligned}
&\int_0^1 (1-s)\left|f''\big((1-s)\zeta + sg(t)\big)\right| ds \\
&\le \left[\int_0^1 (1-s)^2\, ds\right] |f''(\zeta)| + \left[\int_0^1 s(1-s)\, ds\right] |f''(g(t))| \\
&= \frac{1}{3}|f''(\zeta)| + \frac{1}{6}|f''(g(t))|, \quad \text{for all } t \in \Omega.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega \left|(g-\zeta)^2 \int_0^1 (1-s)\left[f''\big((1-s)\zeta + sg(t)\big)\right] ds\right| d\mu \\
&\le \frac{1}{3}|f''(\zeta)| \int_\Omega (g-\zeta)^2\, d\mu + \frac{1}{6}\int_\Omega (g-\zeta)^2 |f''\circ g|\, d\mu.
\end{aligned}$$

The proof is completed by (3.1) with $\lambda = 0$.

(ii) If $|f''|$ is quasi-convex, then

$$\left|f''\big((1-s)\zeta + sg(t)\big)\right| \le \max\{|f''(\zeta)|, |f''(g(t))|\}, \quad \text{for all } t \in \Omega$$

which implies that

$$\begin{aligned}
&\int_0^1 (1-s)\left|f''\big((1-s)\zeta + sg(t)\big)\right| ds \\
&\le \left[\int_0^1 (1-s)\, ds\right] \max\{|f''(\zeta)|, |f''(g(t))|\} \\
&= \frac{1}{2}\max\{|f''(\zeta)|, |f''(g(t))|\}, \quad \text{for all } t \in \Omega.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega \left|(g-\zeta)^2 \int_0^1 (1-s)\left[f''\big((1-s)\zeta + sg(t)\big)\right] ds\right| d\mu \\
&\le \frac{1}{2}\int_\Omega (g-\zeta)^2 \max\{|f''(\zeta)|, |f''\circ g|\}\, d\mu = \frac{1}{2}\|f''\|_{[a,b],\infty}\int_\Omega (g-\zeta)^2\, d\mu
\end{aligned}$$

The proof is completed by (3.1) with $\lambda = 0$.

(iii) If $|f''|$ is log-convex, then

$$\left|f''\big((1-s)\zeta + sg(t)\big)\right| \le |f''(\zeta)|^{1-s}|f''(g(t))|^s, \quad \text{for all } t \in \Omega$$

which implies that

$$\begin{aligned}
&\int_0^1 (1-s)\left|f''\big((1-s)\zeta + sg(t)\big)\right| ds \\
&\le \left[\int_0^1 (1-s)|f''(\zeta)|^{1-s}|f''(g(t))|^s ds\right] \\
&= \frac{-|f''(\zeta)| + |f''(g(t))| + |f''(\zeta)|\left[\log(|f''(\zeta)|) - \log(|f''(g(t))|)\right]}{\left[\log(|f''(\zeta)|) - \log(|f''(g(t))|)\right]^2},
\end{aligned}$$

for all $t \in \Omega$. Thus,

$$\int_\Omega \left| (g-\zeta)^2 \int_0^1 (1-s) \left[f''\big((1-s)\zeta + sg(t)\big)\right] ds \right| d\mu$$
$$\leq \int_\Omega (g-\zeta)^2 \left| \frac{-|f''(\zeta)| + |f''\circ g| + |f''(\zeta)| \left[\log(|f''(\zeta)|) - \log(|f''\circ g|)\right]}{\left[\log(|f''(\zeta)|) - \log(|f''\circ g|)\right]^2} \right| d\mu.$$

The proof is completed by (3.1) with $\lambda = 0$.

(iv) If $|f''|$ is q-convex (for a fixed $q \in (0,1]$), then

$$\left| f''\big((1-s)\zeta + sg\big)\right| \leq (1-s)^q |f''(\zeta)| + s^q |f''(g(t))|, \quad \text{for all } t \in \Omega$$

which implies that

$$\int_0^1 (1-s) \left| f''\big((1-s)\zeta + sg(t)\big)\right| ds$$
$$\leq \left[\int_0^1 (1-s)^{q+1}\, ds\right] |f''(\zeta)| + \left[\int_0^1 (1-s)s^q\, ds\right] |f''(g(t))|$$
$$= \frac{1}{q+2}|f''(\zeta)| + \frac{1}{(q+1)(q+2)}|f''(g(t))|, \quad \text{for all } t \in \Omega.$$

Thus,

$$\int_\Omega \left| (g-\zeta)^2 \int_0^1 (1-s) \left[f''\big((1-s)\zeta + sg(t)\big)\right] ds \right| d\mu$$
$$\leq \frac{1}{q+2}|f''(\zeta)| \int_\Omega (g-\zeta)^2\, d\mu + \frac{1}{(q+1)(q+2)} \int_\Omega (g-\zeta)^2 |f''\circ g|\, d\mu.$$

The proof is completed by (3.1) with $\lambda = 0$. □

Remark 6 (Jensen type inequalities)**.** Furthermore, we obtain Jensen type inequalities by letting $\zeta = \int_\Omega g\, d\mu$ in Theorem 5. The assumption of convexity on $|f''|$ provides refinements for (4.4) (cf. Corollary 2), as shown in the following: If $|f''|$ is convex, then

$$\left| \int_\Omega (f\circ g)\, d\mu - f\left(\int_\Omega g\, d\mu\right)\right|$$
$$\leq \frac{1}{3}\left[\left| f''\left(\int_\Omega g\, d\mu\right)\right| \sigma^2(g) + \frac{1}{2}\int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 |f''\circ g|\, d\mu \right]$$
$$\leq \frac{1}{3}\left[\|f''\|_{[a,b],\infty}\sigma^2(g) + \frac{1}{2}\|f''\|_{[a,b],\infty} \int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 d\mu \right]$$
$$= \frac{1}{2}\|f''\|_{[a,b],\infty}\sigma^2(g).$$

We give an example to the above comparison. Let $f(t) = e^{-t}$ and $g(t) = t$ for $t \in [1,2]$. The true discrepancy in the Jensen inequality is:

$$E = \left| \int_1^2 e^{-t}\, dt - f\left(\int_1^2 t\, dt\right)\right| = \frac{e-1}{e^2} - e^{-3/2} \approx 0.009414.$$

The estimate for the discrepancy given by Corollary 2 is:

$$E \leq \frac{1}{2}\max\{e^{-t}, t \in [1,2]\} \int_1^2 \left(t - \frac{3}{2}\right)^2 dt = \frac{1}{24}e^{-1} \approx 0.015328.$$

The estimate for the discrepancy given by Theorem 5 is closer to the true discrepancy, that is,

$$\begin{aligned} E &\leq \frac{1}{3}\left[\left|f''\left(\frac{3}{2}\right)\right|\int_1^2\left(t-\frac{3}{2}\right)^2 dt+\frac{1}{2}\int_1^2\left(t-\frac{3}{2}\right)^2 e^{-t}\,dt\right] \\ &= \frac{1}{3}\left[\frac{1}{12}e^{-3/2}+\frac{1}{2}\left(\frac{5e-13}{4e^2}\right)\right]\approx 0.009533. \end{aligned}$$

6. Applications for f-Divergence

In the same spirit to that of [3], we apply our result to obtain inequalities for f-divergence measures. One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures are specific cases of the Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p|p:\Omega\to\mathbb{R},\ p(t)\geq 0,\ \int_\Omega p(t)\,d\mu(t)=1\right\}$. The Kullback-Leibler divergence [15] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t),\quad p,q\in\mathcal{P}. \tag{6.1}$$

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These include the variation distance, Hellinger distance [12], χ^2-divergence, α-divergence, Bhattacharyya distance [1], Harmonic distance, Jeffrey's distance [13], triangular discrimination [20]. We recall the definition of χ^2-divergence, due to its usage in this text:

$$D_{\chi^2}(p,q) := \int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right]d\mu(t),\quad p,q\in\mathcal{P}. \tag{6.2}$$

For other divergence measures, see the paper [14] by Kapur or the book on line [19] by Taneja.

Csiszár f-divergence is defined as follows [5]

$$I_f(p,q) := \int_\Omega p(t)f\left[\frac{q(t)}{p(t)}\right]d\mu(t),\quad p,q\in\mathcal{P}, \tag{6.3}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. The χ^2-divergence and the above mention distances are particular instances of Csiszár f-divergence. There are also many others which are not in this class [19]. For the basic properties of Csiszár f-divergence, we refer the readers to [5], [6] and [21].

Proposition 5. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,q\in\mathcal{P}$ and there exists constants $0<r<1<R<\infty$ such that*

$$r\leq\frac{q(t)}{p(t)}\leq R,\quad \textit{for } \mu\textit{-a.e. } t\in\Omega. \tag{6.4}$$

If $\zeta \in [r, R]$, then we have the inequalities

$$|I_f(p,q) - f(\zeta) - (1-\zeta) f'(\zeta)| \le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]. \tag{6.5}$$

In particular, by choosing $\zeta = (r+R)/2$, we have

$$\begin{aligned}&\left|I_f(p,q) - f\left(\frac{r+R}{2}\right) - \left(1 - \frac{r+R}{2}\right) f'\left(\frac{r+R}{2}\right)\right| \\ &\le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1\right)^2\right],\end{aligned} \tag{6.6}$$

and when $\zeta = 1$,

$$|I_f(p,q)| \le \frac{1}{2}\|f''\|_{[r,R],\infty} D_{\chi^2}(p,q). \tag{6.7}$$

Proof. We choose $g(t) = q(t)/p(t)$ and noting that $\int_\Omega p(t)\, d\mu = 1$, in inequality (4.3), we have

$$\begin{aligned}&\left|\int_\Omega f\left(\frac{q(t)}{p(t)}\right) p(t)\, d\mu - f(\zeta) - \left(\int_\Omega q(t)\, d\mu - \zeta\right) f'(\zeta)\right| \\ &= |I_f(p,q) - f(\zeta) - (1-\zeta) f'(\zeta)| \\ &\le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q(t)}{p(t)} - \int_\Omega q(t)\, d\mu\right)^2 p(t)\, d\mu + \left(\int_\Omega q(t)\, d\mu - \zeta\right)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q(t)}{p(t)} - 1\right)^2 p(t)\, d\mu + (\zeta-1)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q^2(t)}{p(t)} - 2q(t) + p(t)\right) d\mu + (\zeta-1)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \frac{q^2(t)}{p(t)}\, d\mu - 1 + (\zeta-1)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q^2(t)}{p(t)} - p(t)\right) d\mu + (\zeta-1)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega\left(\frac{q^2(t)}{p^2(t)} - 1\right) p(t)\, d\mu + (\zeta-1)^2\right] \\ &= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right];\end{aligned}$$

and this completes the proof. □

Proposition 6. *Under the assumptions of Proposition 5, if f' is convex or $f''_\pm$ exists, then we have*

$$\begin{aligned}&\left|I_f(p,q) - f(\zeta) - (1-\zeta) f'(\zeta) + \frac{f''_+(r) + f''_-(R)}{4}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]\right| \\ &\le \frac{1}{4}|f''_-(R) - f''_+(r)|\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],\end{aligned} \tag{6.8}$$

for $\zeta \in [r, R]$. Some particular cases of interest are obtained by setting $\zeta = (r+R)/2$ and $\zeta = 1$.

Proof. When f' is convex, we set $\gamma = f''_+(r)$ and $\Gamma = f''_-(R)$ (cf. Remark 5). For the case where $f''_\pm$ exists, we set γ and Γ appropriately to the values of $f''_+(r)$ and $f''_-(R)$, with $\gamma \le \Gamma$. Utilising (5.1) for $g(t) = q(t)/p(t)$ and the measure $\int_\Omega p(t)\,d\mu = 1$, we have

$$\begin{aligned}
&\left|\int_\Omega f\left(\frac{q(t)}{p(t)}\right) p(t)\,d\mu - f(\zeta) - \left(\int_\Omega q(t)d\mu - \zeta\right) f'(\zeta)\right.\\
&\left. + \frac{f''_+(r) + f''_-(R)}{4}\int_\Omega \left(\frac{q(t)}{p(t)} - \zeta\right)^2 p(t)\,d\mu\right|\\
&= \left| I_f(p,q) - f(\zeta) - (1-\zeta)f'(\zeta) + \frac{f''_+(r) + f''_-(R)}{4}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]\right|\\
&\le \frac{1}{4}\left|f''_-(R) - f''_+(r)\right| \left[\int_\Omega \left(\frac{q(t)}{p(t)} - 1\right)^2 p(t)\,d\mu + (\zeta-1)^2\right]\\
&= \frac{1}{4}\left|f''_-(R) - f''_+(r)\right| \left[D_{\chi^2}(p,q) + (\zeta-1)^2\right].
\end{aligned}$$

Note that we make use of the following:

$$\begin{aligned}
\int_\Omega \left(\frac{q(t)}{p(t)} - \zeta\right)^2 p(t)\,d\mu &= \int_\Omega \left(\frac{q(t)}{p(t)}\right)^2 p(t)\,d\mu - 1 + (\zeta-1)^2\\
&= \int_\Omega \left(\left(\frac{q(t)}{p(t)}\right)^2 - 1\right) p(t)\,d\mu + (\zeta-1)^2\\
&= D_{\chi^2}(p,q) + (\zeta-1)^2;
\end{aligned}$$

and this completes the proof. □

Example 1. If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\log(t)$, then

$$I_f(p,q) = \int_\Omega p(t)\frac{q(t)}{p(t)}\log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = \int_\Omega q(t)\log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = D_{KL}(q,p).$$

We have $f'(t) = \log(t) + 1$ and $f''(t) = 1/t$. By Proposition 5, we have the following inequalities

$$\begin{aligned}
&|D_{KL}(q,p) - \zeta\log(\zeta) - (1-\zeta)(\log(\zeta)+1)|\\
&= |D_{KL}(q,p) - 1 + \zeta - \log(\zeta)|\\
&\le \frac{1}{2}\left[\sup_{x\in[r,R]} \frac{1}{x}\right]\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]\\
&= \frac{1}{2r}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],
\end{aligned}$$

for all $\zeta \in [r,R]$; and when $\zeta = 1$,

$$0 \le D_{KL}(q,p) \le \frac{1}{2r} D_{\chi^2}(p,q). \tag{6.9}$$

Furthermore, by Proposition 6, we have the inequalities:

$$\begin{aligned}
&\left|D_{KL}(q,p) - \log(\zeta) - 1 + \zeta + \frac{r+R}{4rR}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]\right|\\
&\le \frac{R-r}{4rR}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],
\end{aligned}$$

for $\zeta \in [r, R]$; and when $\zeta = 1$,

$$\left| D_{KL}(q,p) + \frac{r+R}{4rR} D_{\chi^2}(p,q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p,q). \tag{6.10}$$

Example 2. If we consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = -\log(t)$, then

$$I_f(p,q) = -\int_\Omega p(t) \log\left(\frac{q(t)}{p(t)}\right) d\,\mu(t) = \int_\Omega p(t) \log\left(\frac{p(t)}{q(t)}\right) d\,\mu(t) = D_{KL}(p,q).$$

We have $f'(t) = -1/t$ and $f''(t) = 1/t^2$, and we note that

$$\int_\Omega \frac{p^2(t)}{q(t)}\, d\mu = D_{\chi^2}(q,p) + 1.$$

By Proposition 5, we have the following inequalities

$$\left| D_{KL}(p,q) + \log(\zeta) + \frac{1}{\zeta} - 1 \right|$$
$$\leq \frac{1}{2}\left[\sup_{x\in[r,R]} \frac{1}{t^2}\right]\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right] = \frac{1}{2r^2}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],$$

for all $\zeta \in [r, R]$; and when $\zeta = 1$,

$$-\frac{1}{2r^2} D_{\chi^2}(p,q) \leq 0 \leq D_{KL}(p,q) \leq \frac{1}{2r^2} D_{\chi^2}(p,q). \tag{6.11}$$

Recall the following inequality from [3]:

$$\left| D_{KL}(p,q) - D_{\chi^2}(q,p) \right| \leq \frac{1}{2r^2} D_{\chi^2}(p,q),$$

or equivalently,

$$D_{\chi^2}(q,p) - \frac{1}{2r^2} D_{\chi^2}(p,q) \leq D_{KL}(p,q) \leq D_{\chi^2}(q,p) + \frac{1}{2r^2} D_{\chi^2}(p,q). \tag{6.12}$$

Thus, we have the following chain of inequalities:

$$\begin{aligned} -\frac{1}{2r^2} D_{\chi^2}(p,q) &\leq D_{\chi^2}(q,p) - \frac{1}{2r^2} D_{\chi^2}(p,q) \\ &\leq D_{KL}(p,q) \\ &\leq \frac{1}{2r^2} D_{\chi^2}(p,q) \leq D_{\chi^2}(q,p) + \frac{1}{2r^2} D_{\chi^2}(p,q). \end{aligned}$$

Furthermore, by Proposition 6, we have the inequalities:

$$\left| D_{KL}(p,q) + \log(\zeta) + \frac{1}{\zeta} - 1 + \frac{r^2+R^2}{4r^2R^2}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right] \right|$$
$$\leq \frac{R^2-r^2}{4r^2R^2}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],$$

for $\zeta \in [r, R]$; and when $\zeta = 1$,

$$-\frac{R^2-r^2}{4r^2R^2} \leq D_{KL}(p,q) + \frac{r^2+R^2}{4r^2R^2}\left[D_{\chi^2}(p,q)\right] \leq \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q). \tag{6.13}$$

Recall the following inequality from [3]:

$$\left| D_{KL}(p,q) - D_{\chi^2}(q,p) + \frac{r^2+R^2}{4r^2R^2}\left[D_{\chi^2}(p,q)\right] \right| \leq \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q),$$

or equivalently,

$$\begin{aligned} D_{\chi^2}(q,p) - \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q) &\le D_{KL}(p,q) + \frac{r^2+R^2}{4r^2R^2}\left[D_{\chi^2}(p,q)\right] \\ &\le D_{\chi^2}(q,p) + \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q). \end{aligned} \tag{6.14}$$

Thus, we have the following chain of inequalities:

$$\begin{aligned} -\frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q) &\le D_{\chi^2}(q,p) - \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q) \\ &\le D_{KL}(p,q) + \frac{r^2+R^2}{4r^2R^2}\left[D_{\chi^2}(p,q)\right] \\ &\le \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q) \\ &\le D_{\chi^2}(q,p) + \frac{R^2-r^2}{4r^2R^2} D_{\chi^2}(p,q). \end{aligned}$$

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