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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GG-CONVEX FUNCTIONS

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ABSTRACT. Some inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. INTRODUCTION

The function $f : I \subset (0,\infty) \to (0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [4]

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1.1}$$

for any $x, y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (1.1) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [48], however, the roots of the research in this area can be traced long before him [49].

It is easy to see that [49], the function $f : I \subset (0,\infty) \to (0,\infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that [49] every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients c_n is a GG-convex function on $(0,r)$, where r is the radius of convergence for f. Therefore functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x} - \cot x$ are GG-convex on $\left(0, \frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln \frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are GG-convex on $(0,1)$. Also, Γ function is a strictly GG-convex function on $[1,\infty)$.

It is also known that [49], if a function f is GG-convex, then so is $x^{\alpha} f^{\beta}(x)$ for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/logx})$ is GG-convex, then so is the other.

The following result is due to P. Montel [48]:

Proposition 1. *Let $f : [0,a] \to [0,\infty)$ be a continuous GG-convex on $(0,a)$, then $F(x) := \int_0^x f(t)\,dt$ is continuous GG-convex on $(0,a)$.*

Therefore, as pointed out in [49], the *Lobacevski's function*

$$L(x) := -\int_0^x \ln(\cos t)\,dt$$

is GG-convex on $(0,\pi/2)$ and the integral sine

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt$$

is GG-concave on $(0,\pi/2)$.

The following characterizations hold [49].

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Proposition 2. *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a differentiable function on the interior $\mathring{I}$ of I. The following statements are equivalent:*

(i) The function f is GG-convex on I;

(ii) The function $\frac{xf'(x)}{f(x)}$ is nondecreasing on $\mathring{I}$;

(iii) We have the inequality

$$\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}} \tag{1.2}$$

for any $x \in I$ and $y \in \mathring{I}$;

and

Proposition 3. *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable function on the interior $\mathring{I}$ of I. The following statements are equivalent:*

(i) The function f is GG-convex on I;

(ii) We have the inequality

$$x\left[f(x) f''(x) - (f'(x))^2\right] + f(x) f'(x) \geq 0 \tag{1.3}$$

for any $x \in \mathring{I}$.

We recall the classical Hermite-Hadamard inequality that states that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

For related results, see [1]-[20], [23]-[26], [27]-[36] and [37]-[52].

We define the *logarithmic mean* $L(a,b)$ of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a \\ \\ b \text{ if } b = a. \end{cases}$$

In 2010, Zhang and Zheng [61] proved the following inequality for a GG-convex function f on $[a,b]$:

$$\frac{1}{\ln b - \ln a}\int_a^b f(t)\,dt \leq L(af(a), bf(b)). \tag{1.5}$$

In 2011, Mitroi and Spiridon [47] established amongst other the following double inequality:

$$f(I(a,b)) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \leq [f(b)]^{\frac{b-L(a,b)}{b-a}}[f(a)]^{\frac{L(a,b)-a}{b-a}}, \tag{1.6}$$

where $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is GG-convex and $I(a,b)$ is the *identric mean* of positive numbers a, b

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } b \neq a \\ \\ b \text{ if } b = a. \end{cases}$$

In 2013, Işcan [40] also proved the following result:

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)\, f\left(\frac{ab}{t}\right)}dt \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \leq L(f(a), f(b)) \tag{1.7}$$

provided that $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is GG-convex.

The function $g : I \subset (0, \infty) \to \mathbb{R}$ is called GA-convex if [4]

$$g\left(x^{1-\lambda}y^{\lambda}\right) \leq (1-\lambda)\, g(x) + \lambda g(y) \tag{1.8}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.

One can observe that the function $f : I \subset (0, \infty) \to (0, \infty)$ is GG-convex if and only if $g := \ln f$ is GA-convex on I.

In order to prove in a different way (1.6) by using GA-convexity we use the following result:

Lemma 1. *If $b > a > 0$ and $g : [a, b] \to \mathbb{R}$ is a GA-convex function on $[a, b]$, then*

$$g(I(a, b)) \leq \frac{1}{b-a}\int_a^b g(t)\, dt \leq \frac{b - L(a, b)}{b-a}g(b) + \frac{L(a, b) - a}{b-a}g(a). \tag{1.9}$$

This result was proved in the case of differentiable functions in [62] and in general case in [25].

Since f is GG-convex, then $g := \ln f$ is GA-convex and by (1.9) we have

$$\ln f(I(a, b)) \leq \frac{1}{b-a}\int_a^b \ln f(t)\, dt \leq \frac{b - L(a, b)}{b-a}\ln f(b) + \frac{L(a, b) - a}{b-a}\ln f(a)$$

that is equivalent to

$$\ln f(I(a, b)) \leq \frac{1}{b-a}\int_a^b \ln f(t)\, dt \leq \ln\left([f(b)]^{\frac{b-L(a,b)}{b-a}}[f(a)]^{\frac{L(a,b)-a}{b-a}}\right)$$

and by taking the exponential, with the desired inequality (1.6).

Motivated by the above results we establish in this paper some new inequalities of Hermite-Hadamard type for GG-convex functions. Applications for special means are also provided.

2. Main Results

We have the following generalization of Zhang and Zheng result (1.5):

Theorem 1. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be a GG-convex function on $[a, b]$. Then for every $p > 0$ and $q \in \mathbb{R}$ we have the inequality:*

$$\frac{1}{\ln b - \ln a}\int_a^b t^{q-1}f^p(t)\, dt \leq L\left(a^q f^p(a), b^q f^p(b)\right). \tag{2.1}$$

For $p = 1$ we then have (see (1.7))

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \leq L(f(a), f(b)) \tag{2.2}$$

for $q=0$, and (see (1.5))

$$\frac{1}{\ln b-\ln a}\int_a^b f(t)\,dt \le L(af(a),bf(b)) \tag{2.3}$$

for $q=1$.

From a different perspective we have:

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. If $p>0$ and*

(i) $q\in\mathbb{R}\setminus\{0,1\}$, then we have the inequalities

$$\begin{aligned} f\left(\sqrt{ab}\right)\left(L_{q-1}^{q-1}(a,b)\right)^{\frac{1}{2p}} &\le \left(\frac{1}{b-a}\int_a^b f^p(t)\,f^p\left(\frac{ab}{t}\right)t^{q-1}dt\right)^{\frac{1}{2p}} \\ &\le \sqrt{f(a)f(b)}\left(L_{q-1}^{q-1}(a,b)\right)^{\frac{1}{2p}}. \end{aligned} \tag{2.4}$$

(ii) We have the inequalities

$$f\left(\sqrt{ab}\right)\le\left(\frac{1}{b-a}\int_a^b f^p(t)\,f^p\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}}\le\sqrt{f(a)f(b)}. \tag{2.5}$$

(iii) We have the inequalities

$$f\left(\sqrt{ab}\right)\le\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{f^p(t)\,f^p\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2p}}\le\sqrt{f(a)f(b)}. \tag{2.6}$$

If we take $p=\frac{1}{2}$ above, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right)L_{q-1}^{q-1}(a,b) &\le \frac{1}{b-a}\int_a^b t^{q-1}\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt \\ &\le L_{q-1}^{q-1}(a,b)\sqrt{f(a)f(b)} \end{aligned} \tag{2.7}$$

for $q\in\mathbb{R}\setminus\{0,1\}$,

$$f\left(\sqrt{ab}\right)\le\frac{1}{b-a}\int_a^b\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt\le\sqrt{f(a)f(b)} \tag{2.8}$$

and

$$f\left(\sqrt{ab}\right)\le\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt\le\sqrt{f(a)f(b)}. \tag{2.9}$$

The inequality (2.9) was stated for GG-convex and monotonically decreasing functions in [41].

If we take $p=1$ above, then we also have

$$\begin{aligned} f\left(\sqrt{ab}\right)\sqrt{L_{q-1}^{q-1}(a,b)} &\le \sqrt{\frac{1}{b-a}\int_a^b f(t)\,f\left(\frac{ab}{t}\right)t^{q-1}dt} \\ &\le \sqrt{f(a)f(b)}\sqrt{L_{q-1}^{q-1}(a,b)}, \end{aligned} \tag{2.10}$$

$$f\left(\sqrt{ab}\right) \leq \sqrt{\frac{1}{b-a}\int_a^b f(t) f\left(\frac{ab}{t}\right) dt} \leq \sqrt{f(a) f(b)} \tag{2.11}$$

for $q \in \mathbb{R}\setminus\{0,1\}$, and

$$f\left(\sqrt{ab}\right) \leq \sqrt{\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t} dt} \leq \sqrt{f(a) f(b)}. \tag{2.12}$$

The inequality (2.11) was stated for GG-convex and monotonically decreasing functions in [59].

The following result holds:

Theorem 3. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then for any $\lambda \in [0,1]$ we have*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \left[f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda} \left[f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right)\right]^{\lambda} \\ &\leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t} dt\right) \\ &\leq \sqrt{f\left(a^{1-\lambda} b^{\lambda}\right) [f(b)]^{1-\lambda} [f(a)]^{\lambda}} \leq \sqrt{f(a) f(b)}. \end{aligned} \tag{2.13}$$

If we take in (2.13) $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \sqrt{f\left(\sqrt[4]{ab^3}\right) f\left(\sqrt[4]{a^3 b}\right)} \\ &\leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t} dt\right) \\ &\leq \sqrt{f\left(\sqrt{ab}\right)} \sqrt[4]{f(b) f(a)} \leq \sqrt{f(a) f(b)}. \end{aligned} \tag{2.14}$$

We also have:

Theorem 4. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequalities*

$$1 \leq \frac{\sqrt{f(a) f(b)}}{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b) b}{f(b)} - \frac{f'_+(a) a}{f(a)}\right]} \tag{2.15}$$

and

$$1 \leq \frac{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right)}{f\left(\sqrt{ab}\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b) b}{f(b)} - \frac{f'_+(a) a}{f(a)}\right]}. \tag{2.16}$$

We also have:

Theorem 5. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GG-convex function on $[a,b]$. Then for any $t \in [a,b]$ we have*

$$\begin{aligned} &\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right) \\ &\leq \sqrt{f(t)} \sqrt{f(b)^{\frac{\ln b - \ln t}{\ln b - \ln a}} f(a)^{\frac{\ln t - \ln a}{\ln b - \ln a}}} \leq \sqrt{f(a) f(b)}. \end{aligned} \tag{2.17}$$

If we take in (2.17) $t = G(a,b) = \sqrt{ab}$, then we get the second part of (2.14), namely

$$\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right) \le \sqrt{f\left(\sqrt{ab}\right)}\sqrt[4]{f(b)f(a)} \le \sqrt{f(a)f(b)}.$$

If we take $t = I(a,b)$ in (2.17) we also get

$$\begin{aligned}(2.18)\qquad &\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\\ &\le \sqrt{f(I(a,b))}\sqrt{f(b)^{\frac{\ln b - \ln I(a,b)}{\ln b - \ln a}} f(a)^{\frac{\ln I(a,b) - \ln a}{\ln b - \ln a}}} \le \sqrt{f(a)f(b)}.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\ln b - \ln I(a,b)}{\ln b - \ln a} &= \frac{\ln b - \frac{b\ln b - a\ln a}{b-a} + 1}{\ln b - \ln a}\\ &= \frac{b-a}{(b-a)(\ln b - \ln a)} - \frac{a}{b-a} = \frac{L(a,b) - a}{b-a}\end{aligned}$$

and, similarly,

$$\frac{\ln I(a,b) - \ln a}{\ln b - \ln a} = \frac{b - L(a,b)}{b-a}.$$

Therefore the inequality (2.18) is equivalent to:

$$\begin{aligned}(2.19)\qquad &\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\\ &\le \sqrt{f(I(a,b))}\sqrt{f(b)^{\frac{L(a,b)-a}{b-a}} f(a)^{\frac{b-L(a,b)}{b-a}}} \le \sqrt{f(a)f(b)}.\end{aligned}$$

We also have
(2.20)

$$\begin{aligned}&\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\\ &\le \exp\left(\frac{1}{2}\cdot\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\sqrt{f(b)^{\frac{L(a,b)-a}{b-a}} f(a)^{\frac{b-L(a,b)}{b-a}}} \le \sqrt{f(a)f(b)}.\end{aligned}$$

A proof of this inequality is done at the end of next section.

3. Proofs

Since : $[a,b] \subset (0,\infty) \to (0,\infty)$ is a GG-convex function on $[a,b]$, then for any $p > 0$ we have

$$(3.1)\qquad f^p\left(a^{1-\lambda}b^{\lambda}\right) \le \left[f^p(a)\right]^{1-\lambda}\left[f^p(b)\right]^{\lambda}$$

for all $\lambda \in [0,1]$.

Now if we multiply (3.1) by $\left(a^{1-\lambda}b^{\lambda}\right)^q > 0$ for $q \in \mathbb{R}$ we get

$$\begin{aligned}(3.2)\qquad f^p\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q &\le \left[f^p(a)\right]^{1-\lambda}\left[f^p(b)\right]^{\lambda}\left(a^{1-\lambda}b^{\lambda}\right)^q\\ &= \left[a^q f^p(a)\right]^{1-\lambda}\left[b^q f^p(b)\right]^{\lambda}\end{aligned}$$

for all $\lambda \in [0,1]$.

If we integrate the inequality (3.2) over $\lambda \in [0,1]$, then we get

$$\int_0^1 f^p\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda \le \int_0^1 \left[a^q f^p(a)\right]^{1-\lambda}\left[b^q f^p(b)\right]^{\lambda} d\lambda. \tag{3.3}$$

Observe that

$$\int_0^1 \left[a^q f^p(a)\right]^{1-\lambda}\left[b^q f^p(b)\right]^{\lambda} d\lambda = L\left(a^q f^p(a), b^q f^p(b)\right)$$

and by the change of variable $a^{1-\lambda}b^{\lambda} = t$ we have

$$\int_0^1 f^p\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda = \frac{1}{\ln b - \ln a}\int_a^b t^{q-1} f^p(t)\,dt,$$

which, together with (3.3) produces the desired result (2.1)

This proves Theorem 1.

To prove Theorem 2, we observe that if f is GG-convex, then we have, as in [40]:

$$f\left(\sqrt{ab}\right) \le \sqrt{f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right)} \le \sqrt{f(a)\,f(b)} \tag{3.4}$$

for all $\lambda \in [0,1]$.

If we take the power $2p > 0$ in (3.4) we get

$$f^{2p}\left(\sqrt{ab}\right) \le f^p\left(a^{1-\lambda}b^{\lambda}\right) f^p\left(a^{\lambda}b^{1-\lambda}\right) \le f^p(a)\,f^p(b) \tag{3.5}$$

for all $\lambda \in [0,1]$.

By multiplying with $\left(a^{1-\lambda}b^{\lambda}\right)^q > 0$ for $q \in \mathbb{R}\setminus\{0\}$ and integrating over λ on $[0,1]$ we get

$$\begin{aligned} f^{2p}\left(\sqrt{ab}\right)\int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda &\le \int_0^1 f^p\left(a^{1-\lambda}b^{\lambda}\right) f^p\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda \\ &\le f^p(a)\,f^p(b)\int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} \int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda &= \int_0^1 \left(a^q\right)^{1-\lambda}\left(b^q\right)^{\lambda} d\lambda = L\left(a^q, b^q\right) \\ &= \frac{a^q - b^q}{\ln a^q - \ln b^q} = \frac{a^q - b^q}{q(a-b)}\frac{a-b}{\ln a - \ln b} \\ &= L_{q-1}^{q-1}(a,b)\,L(a,b) \end{aligned}$$

for $q \neq 1$ and for $q = 1$, $\int_0^1 a^{1-\lambda}b^{\lambda} d\lambda = L(a,b)$, while

$$\begin{aligned} &\int_0^1 f^p\left(a^{1-\lambda}b^{\lambda}\right) f^p\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda \\ &= \int_0^1 f^p\left(a^{1-\lambda}b^{\lambda}\right) f^p\left(\frac{ab}{a^{1-\lambda}b^{\lambda}}\right)\left(a^{1-\lambda}b^{\lambda}\right)^q d\lambda \\ &= \frac{1}{\ln b - \ln a}\int_a^b f^p(t)\, f^p\left(\frac{ab}{t}\right) t^{q-1} dt, \end{aligned}$$

then from (3.6) we have for $q \neq 0,1$ that

$$f^{2p}\left(\sqrt{ab}\right) L_{q-1}^{q-1}(a,b) L(a,b) \leq \frac{1}{\ln b - \ln a} \int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) t^{q-1} dt \leq f^p(a) f^p(b) L_{q-1}^{q-1}(a,b) L(a,b). \tag{3.7}$$

For $q = 1$ we also have

$$f^{2p}\left(\sqrt{ab}\right) L(a,b) \leq \frac{1}{\ln b - \ln a} \int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) dt \leq f^p(a) f^p(b) L(a,b),$$

which is equivalent to

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{1}{b-a} \int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) dt \leq f^p(a) f^p(b). \tag{3.8}$$

If $q = 0$, then by (3.5) we have

$$f^{2p}\left(\sqrt{ab}\right) \leq \int_0^1 f^p\left(a^{1-\lambda} b^\lambda\right) f^p\left(a^\lambda b^{1-\lambda}\right) d\lambda \leq f^p(a) f^p(b),$$

which is equivalent to

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t} dt \leq f^p(a) f^p(b) \tag{3.9}$$

and Theorem 2 is completely proved.

In order to prove Theorem 3 we use the following result for GA-convex functions [25].

Lemma 2. *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$. Then for any $\lambda \in [0,1]$ we have*

$$\begin{aligned} g\left(\sqrt{ab}\right) &\leq (1-\lambda) g\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda g\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2}\left[g\left(a^{1-\lambda} b^\lambda\right) + (1-\lambda) g(b) + \lambda g(a)\right] \leq \frac{g(a) + g(b)}{2}. \end{aligned} \tag{3.10}$$

If f is a GG-convex function, then by writing the inequality (3.10) for $g = \ln f$ we have

$$\begin{aligned} \ln f\left(\sqrt{ab}\right) &\leq (1-\lambda) \ln f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda \ln f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt \\ &\leq \frac{1}{2}\left[\ln f\left(a^{1-\lambda} b^\lambda\right) + (1-\lambda) \ln f(b) + \lambda \ln f(a)\right] \\ &\leq \frac{\ln f(a) + \ln f(b)}{2} \end{aligned}$$

that is equivalent to

$$\ln f\left(\sqrt{ab}\right) \leq \ln\left(\left[f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\right]^{\lambda}\right) \tag{3.11}$$
$$\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}dt$$
$$\leq \ln\sqrt{f\left(a^{1-\lambda}b^{\lambda}\right)[f(b)]^{1-\lambda}[f(a)]^{\lambda}} \leq \ln\sqrt{f(a)f(b)}.$$

By taking the exponential in (3.11) we get the desired result (2.13).

We have the following reverse inequalities [25].

Lemma 3. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then we have*

$$0 \leq \frac{g(a)+g(b)}{2} - \frac{1}{\ln b-\ln a}\int_a^b \frac{g(s)}{s}ds \tag{3.12}$$
$$\leq \frac{1}{8}\left[g'_-(b)\,b - g'_+(a)\,a\right](\ln b-\ln a)$$

and

$$0 \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{g(s)}{s}ds - g\left(\sqrt{ab}\right) \tag{3.13}$$
$$\leq \frac{1}{8}\left[g'_-(b)\,b - g'_+(a)\,a\right](\ln b-\ln a).$$

If we take in (3.12) $g=\ln f$, then we get

$$\frac{\ln f(a)+\ln f(b)}{2} - \frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds$$
$$\leq \frac{1}{8}\left[\frac{f'_-(b)\,b}{f(b)} - \frac{f'_+(a)\,a}{f(a)}\right](\ln b-\ln a),$$

which is equivalent to

$$\frac{\ln f(a)+\ln f(b)}{2}$$
$$\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds + \frac{1}{8}\left[\frac{f'_-(b)\,b}{f(b)} - \frac{f'_+(a)\,a}{f(a)}\right]\ln\left(\frac{b}{a}\right),$$

or to

$$\ln\sqrt{f(a)f(b)} \tag{3.14}$$
$$\leq \ln\left(\exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right]}\right).$$

Taking the exponential in (3.14) we get

$$\sqrt{f(a)f(b)} \leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right]}$$

and the inequality (2.15).

The inequality (2.16) follows by (3.13) in a similar way and the details are omitted.

Further, we use the following result for GA-convex functions [25].

Lemma 4. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then for any $t\in[a,b]$ we have*

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{g(s)}{s}ds &\le(\ge)\frac{1}{2}\left[g(t)+\frac{g(b)(\ln b-\ln t)+g(a)(\ln t-\ln a)}{\ln b-\ln a}\right]\\ &\le(\ge)\frac{g(a)+g(b)}{2}.\end{aligned}\tag{3.15}$$

If f is GG-convex, then if we write (3.15) for $g=\ln f$, we have

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds &\le\frac{1}{2}\left[\ln f(t)+\frac{(\ln b-\ln t)\ln f(b)+(\ln t-\ln a)\ln f(a)}{\ln b-\ln a}\right]\\ &\le\frac{\ln f(a)+\ln f(b)}{2}\end{aligned}$$

that is equivalent to

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds &\le\ln\left(\sqrt{f(t)}\sqrt{f(b)^{\frac{\ln b-\ln t}{\ln b-\ln a}}f(a)^{\frac{\ln t-\ln a}{\ln b-\ln a}}}\right)\le\ln\sqrt{f(a)f(b)}\end{aligned}\tag{3.16}$$

and by taking the exponential in (3.16) we get the desired result (2.17).

If we take the integral mean in (3.15) we get

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{g(s)}{s}ds &\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right.\\ &\left.+\frac{g(b)\left(\ln b-\frac{1}{b-a}\int_a^b\ln t dt\right)+g(a)\left(\frac{1}{b-a}\int_a^b\ln t dt-\ln a\right)}{\ln b-\ln a}\right]\\ &\le\frac{g(a)+g(b)}{2},\end{aligned}\tag{3.17}$$

namely

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{g(s)}{s}ds &\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{g(b)(\ln b-\ln I(a,b))+g(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right]\\ &\le\frac{g(a)+g(b)}{2}.\end{aligned}\tag{3.18}$$

Since

$$\begin{aligned}&\frac{g(b)(\ln b-\ln I(a,b))+g(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\\&=\frac{L(a,b)-a}{b-a}g(b)+\frac{b-L(a,b)}{b-a}g(a),\end{aligned}$$

then by (3.18) we have

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_a^b\frac{g(s)}{s}ds\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{L(a,b)-a}{b-a}g(b)+\frac{b-L(a,b)}{b-a}g(a)\right]\\&\leq\frac{g(a)+g(b)}{2}.\end{aligned}$$

Writing this inequality for $\ln f$ we have

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\ln f(t)\,dt+\frac{L(a,b)-a}{b-a}\ln f(b)+\frac{b-L(a,b)}{b-a}\ln f(a)\right]\\&\leq\frac{\ln f(a)+\ln f(b)}{2}\end{aligned}$$

and the inequality (2.20) is thus proved.

4. Applications

If we take the simple example of GG-convex functions $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp t$ and use the inequality (2.13), then we have

$$\begin{aligned}\exp\left(\sqrt{ab}\right)&\leq\left[\exp\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[\exp\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\right]^{\lambda}\\&\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln\exp(t)}{t}dt\right)\\&\leq\sqrt{\exp\left(a^{1-\lambda}b^{\lambda}\right)[\exp(b)]^{1-\lambda}[\exp(a)]^{\lambda}}\leq\sqrt{\exp(a)\exp(b)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}G(a,b)&\leq(1-\lambda)a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}+\lambda a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\leq L(a,b)\\&\leq\frac{1}{2}\left(a^{1-\lambda}b^{\lambda}+(1-\lambda)b+\lambda a\right)\leq A(a,b)\end{aligned}\tag{4.1}$$

for any $\lambda\in[0,1]$.

If we use the inequalities (2.15) and (2.16) for $f(t)=\exp t$, then we get

$$1\leq\frac{\sqrt{\exp(a)\exp f(b)}}{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln\exp(s)}{s}ds\right)}\leq\left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}$$

and

$$1 \le \frac{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right)}{\exp\left(\sqrt{ab}\right)} \le \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}.$$

By taking the ln in these inequalities, we obtain

$$0 \le A(a,b) - L(a,b) \le \frac{1}{8}(b-a)^2 \frac{1}{L(a,b)} \tag{4.2}$$

and

$$0 \le L(a,b) - G(a,b) \le \frac{1}{8}(b-a)^2 \frac{1}{L(a,b)}. \tag{4.3}$$

By making use of (2.17) for $f(t) = \exp t$, we get

$$\begin{aligned} &\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln \exp(s)}{s} ds\right) \\ &\le \sqrt{\exp(t)}\sqrt{\exp(b)^{\frac{\ln b - \ln t}{\ln b - \ln a}} \exp(a)^{\frac{\ln t - \ln a}{\ln b - \ln a}}} \le \sqrt{\exp(a)\exp(b)}, \end{aligned}$$

that is equivalent to

$$L(a,b) \le \frac{1}{2}\left[t + \frac{b(\ln b - \ln t) + a(\ln t - \ln a)}{\ln b - \ln a}\right] \le A(a,b) \tag{4.4}$$

for any $t \in [a,b]$.

If we use the inequality (2.8) for the function for $f(t) = \exp(t)$ then we get

$$\exp\left(\sqrt{ab}\right) \le \frac{1}{b-a}\int_a^b \exp\left[\frac{1}{2}\left(t + \frac{ab}{t}\right)\right] dt \le \exp\left(\frac{a+b}{2}\right). \tag{4.5}$$

It is an open problem for the author to calculate the integral from the middle term in the inequality (4.5).

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