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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GG-CONVEX FUNCTIONS

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Abstract. Some new inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. Introduction

We recall the classical *Hermite-Hadamard inequality* that states that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1]-[21], [24]-[30], [31]-[40] and [41]-[56].

The function $f:I\subset(0,\infty)\to(0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [4]

$$f\left(x^{1-\lambda}y^{\lambda}\right)\leq[f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1.2}$$

for any $x,y\in I$ and $\lambda\in[0,1]$. If the inequality is reversed in (1.2) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [52], however, the roots of the research in this area can be traced long before him [53].

It is easy to see that [53], the function $f:I\subset(0,\infty)\to(0,\infty)$ is *GG-convex* if and only if the the function $g:\ln I\to\mathbb{R}$, $g=\ln\circ f\circ\exp$ is convex on $\ln I$.

It is known that [53] every real analytic function $f(x)=\sum_{n=0}^{\infty}c_nx^n$ with nonnegative coefficients c_n is a GG-convex function on $(0,r)$, where r is the radius of convergence for f. Therefore functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x}-\cot x$ are GG-convex on $\left(0,\frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln\frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are GG-convex on $(0,1)$. Also, Γ function is a strictly GG-convex function on $[1,\infty)$.

It is also known that [53], if a function f is GG-convex, then so is $x^{\alpha}f^{\beta}(x)$ for all $\alpha\in\mathbb{R}$ and all $\beta>0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is GG-convex, then so is the other.

We define the *logarithmic mean* $L(a,b)$ of two positive numbers a,b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a\\ \\ b \text{ if } b=a.\end{cases}$$

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In 2010, Zhang and Zheng [65] proved the following inequality for a GG-convex function f on $[a,b]$:

$$\frac{1}{\ln b-\ln a}\int_a^b f(t)\,dt\leq L(af(a),bf(b)). \tag{1.3}$$

In 2011, Mitroi and Spiridon [51] established amongst other the following double inequality:

$$f(I(a,b))\leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\leq [f(b)]^{\frac{b-L(a,b)}{b-a}}[f(a)]^{\frac{L(a,b)-a}{b-a}}, \tag{1.4}$$

where $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex and $I(a,b)$ is the *identric mean* of positive numbers a, b

$$I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b\neq a\\ \\ b \text{ if } b=a.\end{cases}$$

In 2013, Işcan [44] also proved the following result:

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt \\ &\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq L(f(a),f(b))\end{aligned} \tag{1.5}$$

provided that $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex.

The function $g:I\subset(0,\infty)\to\mathbb{R}$ is called GA-convex if [4]

$$g\left(x^{1-\lambda}y^{\lambda}\right)\leq(1-\lambda)g(x)+\lambda g(y) \tag{1.6}$$

for any $x,y\in I$ and $\lambda\in[0,1]$.

One can observe that the function $f:I\subset(0,\infty)\to(0,\infty)$ is GG-convex if and only if $g:=\ln f$ is GA-convex on I.

In the recent paper [29], by using some results for GA-convex functions from [26], we proved amongst other the following results:

Theorem 1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then for any $\lambda\in[0,1]$ we have*

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq\left[f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\right]^{\lambda} \\ &\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right) \\ &\leq\sqrt{f\left(a^{1-\lambda}b^{\lambda}\right)[f(b)]^{1-\lambda}[f(a)]^{\lambda}}\leq\sqrt{f(a)f(b)}\end{aligned} \tag{1.7}$$

and

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequalities*

$$1\leq\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)}\leq\left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right)} \tag{1.8}$$

and

$$1 \leq \frac{\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right)}{f\left(\sqrt{ab}\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right)}. \tag{1.9}$$

Motivated by the above results we establish in this paper some new inequalities of Hermite-Hadamard type for GG-convex functions. Applications for special means are also provided.

2. Main Results

We have the following generalization of (1.7) for an arbitrary division:

Theorem 3. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a GG-convex function on $[a, b]$. For a division $0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0, 1]$, we have*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \prod_{i=0}^{n-1}\left[f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right)\right]^{\lambda_{i+1}-\lambda_i} \\ &\leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right) \\ &\leq \prod_{i=0}^{n-1}\left[f\left(a^{1-\lambda_i} b^{\lambda_i}\right) f\left(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}\right)\right]^{\frac{\lambda_{i+1}-\lambda_i}{2}} \leq \sqrt{f(a) f(b)}. \end{aligned} \tag{2.1}$$

We observe that by choosing $0 = \lambda_0 < \lambda = \lambda_1 < \lambda_2 = 1$ in (2.1) we deduce the inequality (1.7) from Introduction.

For a division of the interval $[a, b]$ we also have:

Theorem 4. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a GG-convex function on $[a, b]$. Then for any division $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ with $n \geq 1$ we have the inequalities*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \left(\prod_{i=0}^{n-1}\left[f\left(\sqrt{t_i t_{i+1}}\right)\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}} \\ &\leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right) \\ &\leq \prod_{i=0}^{n-1}\left(\left[\sqrt{f(t_i) f(t_{i+1})}\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}} \leq \sqrt{f(a) f(b)}. \end{aligned} \tag{2.2}$$

If we write the inequality (2.2) for $a = t_0 < t_1 = t < t_2 = b$, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \left[f\left(\sqrt{at}\right)\right]^{\frac{\ln t - \ln a}{\ln b - \ln a}}\left[f\left(\sqrt{tb}\right)\right]^{\frac{\ln b - \ln t}{\ln b - \ln a}} \\ &\leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right) \\ &\leq \sqrt{f(t)}\sqrt{[f(a)]^{\frac{\ln t - \ln a}{\ln b - \ln a}}[f(b)]^{\frac{\ln b - \ln t}{\ln b - \ln a}}} \leq \sqrt{f(a) f(b)} \end{aligned} \tag{2.3}$$

for any $t \in [a, b]$.

From a different perspective we have:

Theorem 5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GG-convex function on $[a,b]$. Then for any division $a=t_0<t_1<...<t_{n-1}<t_n=b$ with $n\geq 1$ we have the inequalities*

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right]\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{1}{s}\sqrt{f(s)f\left(\frac{t_it_{i+1}}{s}\right)}ds\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds. \tag{2.4}$$

If we take $n=1$ in (2.4), then we have

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right]\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{s}\sqrt{f(s)f\left(\frac{ab}{s}\right)}ds\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds. \tag{2.5}$$

The first inequality in (2.5) provides a refinement of (1.5).
If $a\leq t\leq b$, then by (2.5) written for $n=2$ we get

$$\begin{aligned}&f\left(\sqrt{ab}\right)\\&\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right]\\&\leq\frac{1}{\ln b-\ln a}\left[\int_a^t\frac{1}{s}\sqrt{f(s)f\left(\frac{at}{s}\right)}ds+\int_t^b\frac{1}{s}\sqrt{f(s)f\left(\frac{tb}{s}\right)}ds\right]\\&\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds.\end{aligned} \tag{2.6}$$

We have the following results for differentiable functions:

Theorem 6. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable GG-convex function on $[a,b]$. Then we have the inequalities*

$$\frac{\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt}{f\left(\sqrt{ab}\right)}\geq L\left(\left(\frac{b}{a}\right)^{\frac{f'(\sqrt{ab})\sqrt{ab}}{2f(\sqrt{ab})}},\left(\frac{b}{a}\right)^{-\frac{f'(\sqrt{ab})\sqrt{ab}}{2f(\sqrt{ab})}}\right)\geq 1 \tag{2.7}$$

and

$$\frac{\sqrt{f(a)f(b)}}{\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}dt} \geq 1+\log\left[\frac{\int_a^b \frac{f(t)}{t}dt}{\int_a^b \frac{f(t)}{t}\left(\frac{\sqrt{ab}}{t}\right)^{\frac{f'(t)t}{f(t)}}dt}\right] \geq 1+\log\left[\frac{\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}dt}{f\left(\sqrt{ab}\right)}\right] \geq 1 \tag{2.8}$$

where $L(\cdot,\cdot)$ *is the logarithmic mean.*

Now we can state the following result for weighted integrals:

Theorem 7. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *an integrable function on* $[a,b]$, *then*

$$\begin{aligned} & f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right) \\ & \leq \exp\left(\frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \\ & \leq [f(a)]^{\frac{\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b-\ln a}}[f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a}{\ln b-\ln a}}. \end{aligned} \tag{2.9}$$

We define the *p-logarithmic mean* of two positive numbers a,b by

$$L_p(a,b):=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} \text{ if } b\neq a,\\ \\ b \text{ if } b=a.\end{cases}$$

In particular we have for any GG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$ that

$$\begin{aligned} & \left[f\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\right]^{L_p^p(a,b)} \\ & \leq \exp\left(\frac{1}{b-a}\int_a^b t^p\ln f(t)\,dt\right) \\ & \leq [f(a)]^{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{(p+1)(b-a)}}[f(b)]^{\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{(p+1)(b-a)}}, \end{aligned} \tag{2.10}$$

for any $p\in\mathbb{R}$ with $p\neq 0,-1$.

If $p=0$, namely we take $w(t)=1$ in (2.9), then we get (1.4).

If $p=-1$, namely we take $w(t)=\frac{1}{t}$ in (2.9), then we get

$$f\left(\sqrt{ab}\right)\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right)\leq\sqrt{f(a)f(b)}. \tag{2.11}$$

This also proves the first inequality in (2.4).

If $p=1$ in (2.10) then we also have

$$f\left(\sqrt{I\left(a^{2},b^{2}\right)}\right)\leq\exp\left(\frac{1}{b-a}\int_{a}^{b}t\ln f\left(t\right)dt\right)\leq\left[f\left(a\right)\right]^{\frac{A(a,b)L(a,b)-a^{2}}{2(b-a)}}\left[f\left(b\right)\right]^{\frac{b^{2}-A(a,b)L(a,b)}{2(b-a)}}. \tag{2.12}$$

3. The Proofs of Main Results

In the paper [27] we obtained the following result for a division of the interval $[0,1]$:

Lemma 1. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. For any division $0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1$ with $n\geq 1$ of the interval $[0,1]$, we have*

$$\begin{aligned}g\left(\sqrt{ab}\right)&\leq\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_{i}\right)g\left(a^{1-\frac{\lambda_{i}+\lambda_{i+1}}{2}}b^{\frac{\lambda_{i}+\lambda_{i+1}}{2}}\right)\\&\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{g\left(s\right)}{s}ds\\&\leq\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_{i}\right)\frac{g\left(a^{1-\lambda_{i}}b^{\lambda_{i}}\right)+g\left(a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}\right)}{2}\\&\leq\frac{1}{2}\left[g\left(a\right)+g\left(b\right)\right].\end{aligned}\tag{3.1}$$

If we write the inequality (3.1) for the GA-convex function $g=\ln f$, where f is GG-convex, then we have

$$\begin{aligned}\ln f\left(\sqrt{ab}\right)&\leq\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_{i}\right)\ln f\left(a^{1-\frac{\lambda_{i}+\lambda_{i+1}}{2}}b^{\frac{\lambda_{i}+\lambda_{i+1}}{2}}\right)\\&\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{\ln f\left(s\right)}{s}ds\\&\leq\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_{i}\right)\frac{\ln f\left(a^{1-\lambda_{i}}b^{\lambda_{i}}\right)+\ln f\left(a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}\right)}{2}\\&\leq\frac{1}{2}\left[\ln f\left(a\right)+\ln f\left(b\right)\right]\end{aligned}$$

that is equivalent to

$$\begin{aligned}\ln f\left(\sqrt{ab}\right)&\leq\ln\left[\prod_{i=0}^{n-1}f\left(a^{1-\frac{\lambda_{i}+\lambda_{i+1}}{2}}b^{\frac{\lambda_{i}+\lambda_{i+1}}{2}}\right)\right]^{\left(\lambda_{i+1}-\lambda_{i}\right)}\\&\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{\ln f\left(s\right)}{s}ds\\&\leq\ln\prod_{i=0}^{n-1}\left[f\left(a^{1-\lambda_{i}}b^{\lambda_{i}}\right)f\left(a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}\right)\right]^{\frac{\lambda_{i+1}-\lambda_{i}}{2}}\\&\leq\ln\sqrt{f\left(a\right)f\left(b\right)}.\end{aligned}\tag{3.2}$$

By taking the exponential in (3.2) we get the desired result (3.1).

In [27] we also proved the following result for a division of the interval $[a,b]$:

Lemma 2. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then for any division $a=t_0<t_1<...<t_{n-1}<t_n=b$ with $n\geq 1$ we have the inequalities*

$$g\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right)g\left(\sqrt{t_it_{i+1}}\right)\tag{3.3}$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{g(s)}{s}ds$$
$$\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right)\frac{g(t_i)+g(t_{i+1})}{2}$$
$$\leq\frac{1}{2}\left[f(a)+f(b)\right].$$

If we write the inequality for the GA-convex function $g=\ln f$, where f is GG-convex, then we have

$$\ln f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right)\ln\left[f\left(\sqrt{t_it_{i+1}}\right)\right]$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds$$
$$\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right)\frac{\ln f(t_i)+\ln f(t_{i+1})}{2}$$
$$\leq\frac{1}{2}\left[\ln f(a)+\ln f(b)\right].$$

This is equivalent to

$$\ln f\left(\sqrt{ab}\right)\leq\ln\left(\prod_{i=0}^{n-1}\left[f\left(\sqrt{t_it_{i+1}}\right)\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}}\tag{3.4}$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds$$
$$\leq\ln\prod_{i=0}^{n-1}\left(\left[f(t_i)f(t_{i+1})\right]^{\frac{1}{2}\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}}\leq\ln\sqrt{f(a)f(b)}.$$

By taking the exponential in (3.4) we get the desired result (2.2).

In paper [28] we have established the following result for log-convex functions:

Lemma 3. *Let $g:[c,d]\to(0,\infty)$ be a log-convex function on $[c,d]$ and $c=y_0<y_1<...<y_{n-1}<y_n=d$ an arbitrary division of $[c,d]$ with $n\geq 1$. Then*

$$\exp\left[\frac{1}{d-c}\int_c^d\ln g(y)\,dy\right]\leq\frac{1}{d-c}\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}}\sqrt{g(y)\,g(y_i+y_{i+1}-y)}dy\tag{3.5}$$
$$\leq\frac{1}{d-c}\int_c^d g(y)\,dy.$$

If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a GG-convex function on $[a,b]$ and $a=t_0<t_1<...<t_{n-1}<t_n=b$ with $n\geq 1$ is a division of $[a,b]$ then by taking $g:=f\circ\exp$,

$y_i := \ln t_i$, $i \in \{0, ..., n-1\}$, $c = \ln a$ and $d = \ln b$ we have that g is log-convex on $[c, d]$ and $y_i := \ln t_i$, $i \in \{0, ..., n-1\}$ is a division of $[c, d]$.

By writing the inequality (3.5) for these choices, we have

$$\exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^y)\, dy\right] \tag{3.6}$$
$$\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\int_{\ln t_i}^{\ln t_{i+1}} \sqrt{f(e^y) f\left(e^{\ln t_i + \ln t_{i+1} - y}\right)} dy$$
$$\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^y)\, dydy,$$

that is equivalent to

$$\exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^y)\, dy\right] \tag{3.7}$$
$$\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\int_{\ln t_i}^{\ln t_{i+1}} \sqrt{f(e^y) f\left(\frac{t_i t_{i+1}}{e^y}\right)} dy$$
$$\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^y)\, dy.$$

If we make the change of variable $e^y = t$ in (3.7), then we deduce the desired result (2.4).

In [19] we proved the following result for log-convex functions:

Lemma 4. *Let $g : I \to (0, \infty)$ be a differentiable log-convex function on the interval of real numbers $\mathring{I}$ (the interior of I) and $c, d \in \mathring{I}$ with $c < d$. Then the following inequalities hold:*

$$\frac{\frac{1}{d-c}\int_c^d g(x)\, dx}{g\left(\frac{c+d}{2}\right)} \tag{3.8}$$
$$\geq L\left(\exp\left[\frac{g'\left(\frac{c+d}{2}\right)}{g\left(\frac{c+d}{2}\right)}\left(\frac{d-c}{2}\right)\right], \exp\left[-\frac{g'\left(\frac{c+d}{2}\right)}{g\left(\frac{c+d}{2}\right)}\left(\frac{d-c}{2}\right)\right]\right)$$
$$\geq 1.$$

If $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is a GG-convex function on $[a, b]$, then by taking $g := f \circ \exp$, $c = \ln a$ and $d = \ln b$ we have that g is log-convex on $[c, d]$ we have

$$\frac{\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^x)\, dx}{f \circ \exp\left(\frac{\ln a + \ln b}{2}\right)} \tag{3.9}$$
$$\geq L\left(\exp\left[\frac{f' \circ \exp\left(\frac{\ln a + \ln b}{2}\right)\exp\left(\frac{\ln a + \ln b}{2}\right)}{f \circ \exp\left(\frac{\ln a + \ln b}{2}\right)}\left(\frac{\ln b - \ln a}{2}\right)\right],\right.$$
$$\left.\exp\left[-\frac{f' \circ \exp\left(\frac{\ln a + \ln b}{2}\right)\exp\left(\frac{\ln a + \ln b}{2}\right)}{f \circ \exp\left(\frac{\ln a + \ln b}{2}\right)}\left(\frac{\ln b - \ln a}{2}\right)\right]\right)$$
$$\geq 1,$$

that is equivalent to

$$
\begin{aligned}
(3.10)\qquad & \frac{\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\left(e^{x}\right) dx}{f\left(\sqrt{ab}\right)} \\
& \geq L\left(\exp\left[\frac{f'\left(\sqrt{ab}\right)\sqrt{ab}}{f\left(\sqrt{ab}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right],\right. \\
& \left.\exp\left[-\frac{f'\left(\sqrt{ab}\right)\sqrt{ab}}{f\left(\sqrt{ab}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right]\right) \\
& \geq 1,
\end{aligned}
$$

and the inequality (2.7) is proved.

The inequality (2.8) follows by the following result [19] that holds for the differentiable log-convex function $g:[c,d]\to\mathbb{R}$

$$
\begin{aligned}
(3.11)\qquad \frac{\frac{g(c)+g(d)}{2}}{\frac{1}{d-c}\int_{c}^{d} g(x)\,dx} & \geq 1+\log\left[\frac{\int_{c}^{d} g(x)\,dx}{\int_{c}^{d} g(x)\exp\left[\frac{g'(x)}{g(x)}\left(\frac{c+d}{2}-x\right)\right]dx}\right] \\
& \geq 1+\log\left[\frac{\frac{1}{d-c}\int_{c}^{d} g(x)\,dx}{g\left(\frac{c+d}{2}\right)}\right]\geq 1.
\end{aligned}
$$

We omit the details.

We have the following result for GA-convex functions [27]:

Lemma 5. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, then*

$$
\begin{aligned}
(3.12)\qquad & g\left(\exp\left(\frac{\int_{a}^{b} w(t)\ln t dt}{\int_{a}^{b} w(t)\,dt}\right)\right) \\
& \leq \frac{\int_{a}^{b} w(t)\,g(t)\,dt}{\int_{a}^{b} w(t)\,dt} \\
& \leq \frac{\left(\ln b-\frac{\int_{a}^{b} w(t)\ln t dt}{\int_{a}^{b} w(t)dt}\right)g(a)+\left(\frac{\int_{a}^{b} w(t)\ln t dt}{\int_{a}^{b} w(t)dt}-\ln a\right)g(b)}{\ln b-\ln a}.
\end{aligned}
$$

If we take $g=\ln f$, then we get from (3.12) that

$$\begin{aligned}
&\ln f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right) \\
&\leq \frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt} \\
&\leq \frac{\left(\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}\right)\ln f(a)+\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a\right)\ln f(b)}{\ln b-\ln a} \\
&=\ln\left([f(a)]^{\frac{\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b-\ln a}}[f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a}{\ln b-\ln a}}\right)
\end{aligned} \tag{3.13}$$

and by taking the exponential in (3.13) we get the desired result (2.9).

In [27] we obtained the following particular inequalities for $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, a GA-convex function

$$\begin{aligned}
&L_p^p(a,b)\,g\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) \\
&\leq \frac{1}{b-a}\int_a^b t^p g(t)\,dt \\
&\leq \frac{\left(L\left(a^{p+1},b^{p+1}\right)-a^{p+1}\right)g(a)+\left(b^{p+1}-L\left(a^{p+1},b^{p+1}\right)\right)g(b)}{(p+1)(b-a)},
\end{aligned} \tag{3.14}$$

for any $p\in\mathbb{R}$ with $p\neq 0,-1$.

If $p=0$, namely we take $w(t)=1$ in (3.12), then we have

$$g(I(a,b))\leq\frac{1}{b-a}\int_a^b g(t)\,dt\leq\frac{(L(a,b)-a)\,g(a)+(b-L(a,b))\,f(b)}{b-a}. \tag{3.15}$$

If $p=-1$, namely we take $w(t)=\frac{1}{t}$ in (3.12), then we have

$$f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq\frac{f(a)+f(b)}{2}. \tag{3.16}$$

If $p=1$ (3.14), then we have

$$\begin{aligned}
&f\left(\sqrt{I\left(a^2,b^2\right)}\right) \\
&\leq\frac{1}{b-a}\int_a^b tf(t)\,dt \\
&\leq\frac{\left(A(a,b)\,L(a,b)-a^2\right)f(a)+\left(b^2-A(a,b)\,L(a,b)\right)f(b)}{2(b-a)}.
\end{aligned} \tag{3.17}$$

If we take in these inequalities $g=\ln f$, then we get the desired results (1.4), (2.11) and (2.12).

4. Applications

If we take the simple example of GG-convex functions $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp t$ and use the inequality (2.1) for a division $0=\lambda_0<\lambda_1<...<$

$\lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0,1]$, then we have

$$\begin{aligned}\exp\left(\sqrt{ab}\right) &\leq \exp\left(\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_i\right)a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}}b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right)\\ &\leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln\exp(s)}{s}ds\right)\\ &\leq \exp\sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_i\right)\left(\frac{a^{1-\lambda_i}b^{\lambda_i}+a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}}{2}\right)\\ &\leq \exp\left(\frac{a+b}{2}\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}G(a,b) &\leq \sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_i\right)a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}}b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\\ &\leq L(a,b) \leq \sum_{i=0}^{n-1}\left(\lambda_{i+1}-\lambda_i\right)\left(\frac{a^{1-\lambda_i}b^{\lambda_i}+a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}}{2}\right)\\ &\leq A(a,b),\end{aligned}\tag{4.1}$$

where $G(a,b)=\sqrt{ab}$ is the *geometric mean.*

For any division $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ with $n \geq 1$, by employing Theorem 4 we have the inequalities

$$\begin{aligned}\exp\left(\sqrt{ab}\right) &\leq \exp\left(\frac{1}{\ln b-\ln a}\ln\left(\prod_{i=0}^{n-1}\left(\frac{t_{i+1}}{t_i}\right)^{\sqrt{t_it_{i+1}}}\right)\right)\\ &\leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln\exp(s)}{s}ds\right)\\ &\leq \exp\left(\frac{1}{\ln b-\ln a}\ln\left(\prod_{i=0}^{n-1}\left(\frac{t_{i+1}}{t_i}\right)^{\frac{t_i+t_{i+1}}{2}}\right)\right)\\ &\leq \exp\left(\frac{a+b}{2}\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}G(a,b) &\leq \frac{1}{\ln b-\ln a}\ln\left(\prod_{i=0}^{n-1}\left(\frac{t_{i+1}}{t_i}\right)^{\sqrt{t_it_{i+1}}}\right)\\ &\leq L(a,b)\\ &\leq \frac{1}{\ln b-\ln a}\ln\left(\prod_{i=0}^{n-1}\left(\frac{t_{i+1}}{t_i}\right)^{\frac{t_i+t_{i+1}}{2}}\right) \leq A(a,b).\end{aligned}\tag{4.2}$$

Finally, if we use (2.9) for the GG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp t$, then for any integrable nonnegative weight w we have

$$\begin{aligned}
&\exp\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right)\\
&\le \exp\left(\frac{\int_a^b w(t)\,t dt}{\int_a^b w(t)\,dt}\right)\\
&\le \exp\left(a\frac{\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b-\ln a}+b\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a}{\ln b-\ln a}\right)
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right) &\le \frac{\int_a^b w(t)\,t dt}{\int_a^b w(t)\,dt}\\
&\le a\frac{\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b-\ln a}+b\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a}{\ln b-\ln a}.
\end{aligned}\tag{4.3}$$

From (2.10) we have for $w(t)=t^p$ that

$$\begin{aligned}
&L_p^p(a,b)\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\\
&\le L_{p+1}^{p+1}(a,b)\\
&\le a\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{(p+1)(b-a)}+b\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{(p+1)(b-a)},
\end{aligned}\tag{4.4}$$

for any $p\in\mathbb{R}$ with $p\neq 0,-1,-2$.

The interested reader may obtain other particular inequalities from (4.3) by choosing different examples of w.

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