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SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GG-CONVEX FUNCTIONS

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Abstract. Some related integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for exponential integral mean are also provided.

1. Introduction

We recall first some facts on GG-convex functions and Hermite-Hadamard type inequalities.

The function $f : I \subset (0,\infty) \to (0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [4]

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1.1}$$

for any $x, y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (1.1) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [50], however, the roots of the research in this area can be traced long before him [51].

It is easy to see that [51], the function $f : I \subset (0,\infty) \to (0,\infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that [51] every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients c_n is a GG-convex function on $(0,r)$, where r is the radius of convergence for f. Therefore functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x} - \cot x$ are GG-convex on $\left(0,\frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln\frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are GG-convex on $(0,1)$. Also, Γ function is a strictly GG-convex function on $[1,\infty)$.

It is also known that [51], if a function f is GG-convex, then so is $x^{\alpha}f^{\beta}(x)$ for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is GG-convex, then so is the other.

As pointed out in [51], the *Lobacevski's function*

$$L(x) := -\int_0^x \ln(\cos t)\,dt$$

is GG-convex on $(0,\pi/2)$ and the *integral sine*

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt$$

is GG-concave on $(0,\pi/2)$.

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We recall the classical Hermite-Hadamard inequality that states that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2} \tag{1.2}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1]-[20], [23]-[28], [29]-[38] and [39]-[54].

We define the *logarithmic mean* $L(a,b)$ of two positive numbers a,b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a\\ \\ b \text{ if } b=a.\end{cases}$$

In 2010, Zhang and Zheng [63] proved the following inequality for a GG-convex function f on $[a,b]$:

$$\frac{1}{\ln b-\ln a}\int_a^b f(t)\,dt\leq L(af(a),bf(b)). \tag{1.3}$$

In 2011, Mitroi and Spiridon [49] established amongst other the following double inequality:

$$f(I(a,b))\leq\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\leq[f(b)]^{\frac{b-L(a,b)}{b-a}}[f(a)]^{\frac{L(a,b)-a}{b-a}}, \tag{1.4}$$

where $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex and $I(a,b)$ is the *identric mean* of positive numbers a,b

$$I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } b\neq a\\ \\ b \text{ if } b=a.\end{cases}$$

In 2013, Işcan [42] also proved the following result:

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt \\ &\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq L(f(a),f(b))\end{aligned} \tag{1.5}$$

provided that $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex.

In the recent paper [27], by using some results for GA-convex functions from [25], we proved amongst others the following results:

Theorem 1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then for any $\lambda\in[0,1]$ we have*

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq\left[f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\right]^{\lambda} \\ &\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right) \\ &\leq\sqrt{f\left(a^{1-\lambda}b^{\lambda}\right)[f(b)]^{1-\lambda}[f(a)]^{\lambda}}\leq\sqrt{f(a)\,f(b)}\end{aligned} \tag{1.6}$$

and

Theorem 2. *Let* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *be a GG-convex function on* $[a, b]$. *Then we have the inequalities*

$$1 \leq \frac{\sqrt{f(a) f(b)}}{\exp\left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(s)}{s} ds\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right)} \tag{1.7}$$

and

$$1 \leq \frac{\exp\left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(s)}{s} ds\right)}{f\left(\sqrt{ab}\right)} \leq \left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right)}. \tag{1.8}$$

Motivated by the above results we establish in this paper some new inequalities of Hermite-Hadamard type for GG-convex functions. Applications for exponential integral mean are also provided.

2. New Results

We start with the following inequality for powers of GG-convex functions:

Theorem 3. *Let* $f : [a, b] \subset (0, \infty) \to (0, \infty)$ *be a GG-convex function on* $[a, b]$.
(i) If $p \in \left(0, \frac{1}{2}\right]$, *then we have*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt\right] \\ &\leq \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f^p(t) f^p\left(\frac{\sqrt{ab}}{t}\right) dt\right)^{\frac{1}{2p}} \\ &\leq \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f^{2p}(t) dt\right)^{\frac{1}{2p}} \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq L(f(a), f(b)). \end{aligned} \tag{2.1}$$

(ii) For $p > 0$, *but* $p \neq \frac{1}{2}$, *we also have*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt\right] \\ &\leq \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f^p(t) f^p\left(\frac{\sqrt{ab}}{t}\right) dt\right)^{\frac{1}{2p}} \\ &\leq \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f^{2p}(t) dt\right)^{\frac{1}{2p}} \\ &\leq \left[L_{2p-1}(f(a), f(b))\right]^{1-\frac{1}{2p}} \left[L(f(a), f(b))\right]^{\frac{1}{2p}}. \end{aligned} \tag{2.2}$$

If we take $p=\frac{1}{4}$ in (2.1), then we get

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right]\tag{2.3}$$
$$\leq\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt[4]{f(t)f\left(\frac{\sqrt{ab}}{t}\right)}dt\right)^2$$
$$\leq\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt{f(t)}dt\right)^2$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq L(f(a),f(b)).$$

The case $p=\frac{1}{2}$ produces

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right]\tag{2.4}$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\sqrt{f(t)f\left(\frac{\sqrt{ab}}{t}\right)}dt\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt$$
$$\leq L(f(a),f(b)).$$

Also, if we take $p=1$ in (2.2), then we get

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right]\tag{2.5}$$
$$\leq\sqrt{\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f(t)f\left(\frac{\sqrt{ab}}{t}\right)dt}$$
$$\leq\sqrt{\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f^2(t)\,dt}\leq\sqrt{A(f(a),f(b))}\sqrt{L(f(a),f(b))}.$$

We also have:

Theorem 4. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then for every $s\in[a,b]$ we have the inequality*

$$(\ln b-\ln s)f(b)+(\ln s-\ln a)f(a)-\int_a^b\frac{f(t)}{t}dt\tag{2.6}$$
$$\geq\int_a^b\frac{f(t)\ln f(t)}{t}dt-\ln f(s)\int_a^b\frac{f(t)}{t}dt.$$

If we take in (2.6) $s=G(a,b)=\sqrt{ab}$, then we get

$$(\ln b-\ln a)\frac{f(b)+f(a)}{2}-\int_a^b\frac{f(t)}{t}dt\tag{2.7}$$
$$\geq\int_a^b\frac{f(t)\ln f(t)}{t}dt-\ln f(G(a,b))\int_a^b\frac{f(t)}{t}dt.$$

Also, if we take $s = I(a,b)$, then we get from (2.6) that

$$(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a) - \int_a^b \frac{f(t)}{t} dt \tag{2.8}$$
$$\geq \int_a^b \frac{f(t) \ln f(t)}{t} dt - \ln f(I(a,b)) \int_a^b \frac{f(t)}{t} dt.$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)}$$

and

$$\ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

then the inequality (2.8) can be written as

$$\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a) - \int_a^b \frac{f(t)}{t} dt \tag{2.9}$$
$$\geq \int_a^b \frac{f(t) \ln f(t)}{t} dy - \ln f(I(a,b)) \int_a^b \frac{f(t)}{t} dt.$$

Moreover, if we take the integral mean in (2.6), then we get

$$(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a) - \int_a^b \frac{f(t)}{t} dt \tag{2.10}$$
$$\geq \int_a^b \frac{f(t) \ln f(t)}{t} dt - \frac{1}{b-a} \int_a^b \ln f(s) ds \int_a^b \frac{f(t)}{t} dt.$$

This can be also written as

$$\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a) - \int_a^b \frac{f(t)}{t} dt \tag{2.11}$$
$$\geq \int_a^b \frac{f(t) \ln f(t)}{t} dt - \frac{1}{b-a} \int_a^b \ln f(s) ds \int_a^b \frac{f(t)}{t} dt.$$

From a different perspective we have:

Theorem 5. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$\frac{f(b) + f(a)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \tag{2.12}$$
$$\geq \int_a^b \frac{f(t)}{t} \ln f(t) dt - \int_a^b \frac{f(t)}{t} dt \frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt \geq 0.$$

Also, we can state the following result as well:

Theorem 6. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$f(b) \left(\ln b - \frac{\int_a^b \frac{f(t) \ln t}{t} dt}{\int_a^b \frac{f(t)}{t} dt} \right) + f(a) \left(\frac{\int_a^b \frac{f(t) \ln t}{t} dt}{\int_a^b \frac{f(t)}{t} dt} - \ln a \right) - \int_a^b \frac{f(t)}{t} dt \tag{2.13}$$
$$\geq \int_a^b \frac{f(t)}{t} \ln f(t) dt - \int_a^b \frac{f(t)}{t} \ln f \left(\exp \left(\frac{\int_a^b \frac{f(t) \ln t}{t} dt}{\int_a^b \frac{f(t)}{t} dt} \right) \right) \geq 0.$$

Finally, we have

Theorem 7. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$
\begin{aligned}
(2.14)\qquad & \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{f^{2p}(t)}{t} dt \right)^{\frac{1}{2p}} \\
& \geq \left(\frac{1}{\ln b - \ln a} \int_a^b \left[f^{2p}(t) \right]^{1-\alpha} \left[f^{2p}\left(\frac{ab}{t} \right) \right]^{\alpha} dt \right)^{\frac{1}{2p}} \\
& \geq \left(\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{f^{2p}(t)}{t} dt \right)^{\frac{1}{2p}} \\
& \geq \left(\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{f^{p}(t) f^{p}\left(\frac{ab}{t}\right)}{t} dt \right)^{\frac{1}{2p}} \\
& \geq \exp \left[\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{\ln f(t)}{t} dt \right] \geq f\left(\sqrt{ab}\right),
\end{aligned}
$$

for every $\alpha \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ and $p > 0$.

For $p = \frac{1}{4}$ in (2.14) we get

$$
\begin{aligned}
(2.15)\qquad & \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\sqrt{f(t)}}{t} dt \right)^{2} \\
& \geq \left(\frac{1}{\ln b - \ln a} \int_a^b \sqrt{f^{1-\alpha}(t)} \left[\sqrt{f^{\alpha}\left(\frac{ab}{t} \right)} \right] dt \right)^{2} \\
& \geq \left(\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{\sqrt{f(t)}}{t} dt \right)^{2} \\
& \geq \left(\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{\sqrt[4]{f(t) f\left(\frac{ab}{t}\right)}}{t} dt \right)^{2} \\
& \geq \exp \left[\frac{1}{(1-2\alpha)(\ln b - \ln a)} \int_{a^{1-\alpha} b^{\alpha}}^{a^{\alpha} b^{1-\alpha}} \frac{\ln f(t)}{t} dt \right] \geq f\left(\sqrt{ab}\right).
\end{aligned}
$$

If we take $p=\frac{1}{2}$ in (2.14), then we get

$$
\begin{aligned}
&\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}dt \\
&\geq \frac{1}{\ln b-\ln a}\int_a^b [f(t)]^{1-\alpha}\left[f\left(\frac{ab}{t}\right)\right]^{\alpha}dt \\
&\geq \frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f(t)}{t}dt \\
&\geq \frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\sqrt{f(t)f\left(\frac{ab}{t}\right)}}{t}dt \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}dt\right] \geq f\left(\sqrt{ab}\right).
\end{aligned}
\tag{2.16}
$$

Finally, by taking $p=1$ in (2.14) we get

$$
\begin{aligned}
&\sqrt{\frac{1}{\ln b-\ln a}\int_a^b \frac{f^2(t)}{t}dt} \\
&\geq \sqrt{\frac{1}{\ln b-\ln a}\int_a^b [f^2(t)]^{1-\alpha}\left[f^2\left(\frac{ab}{t}\right)\right]^{\alpha}dt} \\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f^2(t)}{t}dt} \\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f(t)f\left(\frac{ab}{t}\right)}{t}dt} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}dt\right] \geq f\left(\sqrt{ab}\right).
\end{aligned}
\tag{2.17}
$$

3. Proofs

In the recent paper [26] we established the following inequalities for a log-convex function $g:[c,d]\to(0,\infty)$:

If $p \in \left(0, \frac{1}{2}\right]$, then we have

$$g\left(\frac{c+d}{2}\right) \leq \exp\left[\frac{1}{d-c}\int_c^d \ln g(x)\,dx\right] \tag{3.1}$$
$$\leq \left(\frac{1}{d-c}\int_c^d g^p(x)\,g^p(c+d-x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \frac{1}{d-c}\int_c^d g(x)\,dx$$
$$\leq L(g(c), g(d)).$$

For $p > 0$, but $p \neq \frac{1}{2}$, we also have

$$g\left(\frac{c+d}{2}\right) \leq \exp\left[\frac{1}{d-c}\int_c^d \ln g(x)\,dx\right] \tag{3.2}$$
$$\leq \left(\frac{1}{d-c}\int_c^d g^p(x)\,g^p(c+d-x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \left[L_{2p-1}(g(c), g(d))\right]^{1-\frac{1}{2p}}\left[L(g(c), g(d))\right]^{\frac{1}{2p}}.$$

If $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is a GG-convex function on $[a,b]$, then by taking $g := f \circ \exp$; $c = \ln a$ and $d = \ln b$ we have that g is log-convex on $[c,d]$ and by (3.1) and (3.2) we get

$$f \circ \exp\left(\frac{\ln a + \ln b}{2}\right) \tag{3.3}$$
$$\leq \exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right]dx\right]$$
$$\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f \circ \exp(\ln a + \ln b - x)\right]^p dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}}$$
$$\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f \circ \exp(x)\,dx \leq L(f \circ \exp(\ln a), f \circ \exp(d))$$

for $p \in \left(0, \frac{1}{2}\right]$ and

$$
\begin{aligned}
(3.4) \qquad & f \circ \exp\left(\frac{\ln a + \ln b}{2}\right) \\
& \leq \exp\left[\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\
& \leq \left(\frac{1}{\ln b - \ln a} \left[f \circ \exp(x)\right]^p \left[f \circ \exp(\ln a + \ln b - x)\right]^p dx\right)^{\frac{1}{2p}} \\
& \leq \left(\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\
& \leq \left[L_{2p-1}\left(f \circ \exp(\ln a), f \circ \exp(\ln b)\right)\right]^{1-\frac{1}{2p}} \\
& \times \left[L\left(f \circ \exp(\ln a), f \circ \exp(\ln b)\right)\right]^{\frac{1}{2p}}
\end{aligned}
$$

for $p > 0$, but $p \neq \frac{1}{2}$.

The inequalities (3.3) and (3.4) can be equivalently written as

$$
\begin{aligned}
(3.5) \qquad f\left(\sqrt{ab}\right) & \leq \exp\left[\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\
& \leq \left(\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f\left(\frac{\sqrt{ab}}{\exp(x)}\right)\right]^p dx\right)^{\frac{1}{2p}} \\
& \leq \left(\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\
& \leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} f \circ \exp(x)\, dx \leq L\left(f(a), f(b)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6) \qquad f\left(\sqrt{ab}\right) & \leq \exp\left[\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\
& \leq \left(\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f\left(\frac{\sqrt{ab}}{\exp(x)}\right)\right]^p dx\right)^{\frac{1}{2p}} \\
& \leq \left(\frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\
& \leq \left[L_{2p-1}\left(f(a), f(b)\right)\right]^{1-\frac{1}{2p}} \left[L\left(f(a), f(b)\right)\right]^{\frac{1}{2p}}.
\end{aligned}
$$

Now, by making the change of variable $\exp(x) = t$ in the integrals from (3.5) and (3.6) we obtain the desired results (2.1) and (2.2).

In [26] we also obtained the following result:

Let $g:[c,d]\to(0,\infty)$ be a log-convex function on $[c,d]$. Then for any $x\in[c,d]$ we have

$$g(d)(d-x)+g(c)(x-c)-\int_c^d g(y)\,dy \geq \int_c^d g(y)\ln g(y)\,dy-\ln g(x)\int_c^d g(y)\,dy. \tag{3.7}$$

A simple proof of this fact is as follows.

Since the function $\ln g$ is convex on $[c,d]$, then by the gradient inequality we have

$$\ln g(x)-\ln g(y)\geq \frac{g'_+(y)}{g(y)}(x-y) \tag{3.8}$$

for any $x\in[c,d]$ and $y\in(c,d)$.

If we multiply (3.8) by $g(y)>0$ and integrate on $[c,d]$ over y we get

$$\begin{aligned}
&\ln g(x)\int_c^d g(y)\,dy-\int_c^d g(y)\ln g(y)\,dy\\
&\geq \int_c^d g'_+(y)(x-y)\,dy = g(y)(x-y)\big|_c^d+\int_c^d g(y)\,dy\\
&= g(d)(x-d)+g(c)(c-x)+\int_c^d g(y)\,dy,
\end{aligned}$$

which is equivalent to (3.7).

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, $d=\ln b$ and $x=\ln s$, $s\in[a,b]$, we have that g is log-convex on $[c,d]$ and by (3.7) we get

$$\begin{aligned}
&(\ln b-\ln s)\,f\circ\exp(\ln b)+(\ln s-\ln a)\,f\circ\exp(\ln a)-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\\
&\geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy-\ln(f\circ\exp(\ln s))\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy,
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&(\ln b-\ln s)\,f(b)+(\ln s-\ln a)\,f(a)-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\\
&\geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy-\ln f(s)\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy,
\end{aligned}$$

that holds for each $s\in[a,b]$.

Now, if in this last inequality we make the change of variable $\exp(y)=t$, then we obtain the desired result (2.6).

We have the following result for log-convex functions that improves the trapezoid inequality for the convex function $h:[c,d]\to\mathbb{R}$

$$\frac{h(d)+h(c)}{2}-\frac{1}{d-c}\int_c^d h(y)\,dy\geq 0.$$

We have [26] :

Lemma 1. *Let $g:[c,d]\to(0,\infty)$ be a log-convex function on $[c,d]$. Then*

$$\begin{aligned}\tag{3.9} &\frac{g(d)+g(c)}{2}-\frac{1}{d-c}\int_c^d g(y)\,dy \\ &\geq \int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\frac{1}{d-c}\int_c^d \ln g(y)\,dy\geq 0.\end{aligned}$$

Proof. If we take the integral mean over x in (3.7), then we get

$$\begin{aligned}&\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,dx-\int_c^d g(y)\,dy \\ &\geq \int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\frac{1}{d-c}\int_c^d \ln g(x)\,dx\end{aligned}$$

and since

$$\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,dx=\frac{g(d)+g(c)}{2}-\frac{1}{d-c}\int_c^d g(y)\,dy$$

then the first inequality in (3.9) is proved.

Since ln is an increasing function on $(0,\infty)$, then we have

$$(g(x)-g(y))(\ln g(x)-\ln g(y))\geq 0$$

for any $x,y\in[c,d]$, showing that the functions g and $\ln g$ are synchronous on $[c,d]$.

By making use of the Čebyšev integral inequality for synchronous functions $g,h:[c,d]\to\mathbb{R}$, namely

$$\frac{1}{d-c}\int_c^d g(x)h(x)\,dx\geq\frac{1}{d-c}\int_c^d g(x)\,dx\frac{1}{d-c}\int_c^d h(x)\,dx,$$

then we have

$$\frac{1}{d-c}\int_c^d g(x)\ln g(x)\,dx\geq\frac{1}{d-c}\int_c^d g(x)\,dx\frac{1}{d-c}\int_c^d \ln g(x)\,dx,$$

which proves the last part of (3.9). □

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (3.9) we get

$$\begin{aligned}\tag{3.10} &\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy \\ &\geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy \\ &-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln(f\circ\exp(y))\,dy \\ &\geq 0.\end{aligned}$$

By changing the variable $\exp(y)=t$ in (3.10) we deduce the desired inequality (2.12).

We have [26]:

Lemma 2. *Let $g:[c,d]\to(0,\infty)$ be a log-convex function on $[c,d]$. Then*

$$g(d)\left(d-\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\right)+g(c)\left(\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}-c\right)-\int_c^d g(y)\,dy \tag{3.11}$$
$$\geq\int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\ln g\left(\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\right)\geq 0.$$

Proof. The first inequality follows by (3.7) on taking

$$x=\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\in[c,d]$$

since $g(y)>0$ for any $y\in[c,d]$.

By Jensen's inequality for the convex function $\ln g$ and the positive weight g we have

$$\frac{\int_c^d g(y)\ln g(y)\,dy}{\int_c^d g(y)\,dy}\geq g\left(\frac{\int_c^d g(y)\,ydy}{\int_c^d g(y)\,dy}\right),$$

which proves the second inequality in (3.11). □

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (3.11) we have

$$f(b)\left(\ln b-\frac{\int_{\ln a}^{\ln b} yf\circ\exp(y)\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}\right)+f(c)\left(\frac{\int_{\ln a}^{\ln b} yf\circ\exp(y)\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}-\ln a\right) \tag{3.12}$$
$$-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy$$
$$\geq\int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy$$
$$-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\times\ln(f\circ\exp)\left(\frac{\int_{\ln a}^{\ln b} y(f\circ\exp(y))\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}\right)$$
$$\geq 0.$$

By changing the variable $\exp(y)=t$ in (3.12) we get (2.13).

In [26] we also proved the following result:

Let $g:[c,d]\to(0,\infty)$ be a log-convex function. Then for every $\alpha\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p>0$ that

$$\left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}} \tag{3.13}$$
$$\geq\left(\frac{1}{d-c}\int_c^d\left[g^{2p}(x)\right]^{1-\alpha}\left[g^{2p}(c+d-x)\right]^{\alpha}dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d} g^{2p}(u)\,du\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d} g^{p}(u)\,g^{p}(c+d-u)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\exp\left[\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d}\ln g(u)\,du\right]\geq g\left(\frac{c+d}{2}\right).$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (3.13) we have

$$\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f^{2p}\circ\exp(x)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f^{2p}\circ\exp(x)\right]^{1-\alpha}\left[f^{2p}\left(\frac{ab}{\exp(x)}\right)\right]^{\alpha}dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)} f^{2p}\circ\exp(x)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)} f^{p}\circ\exp(x)\,f^{p}\left(\frac{ab}{\exp(x)}\right)dx\right)^{\frac{1}{2p}}$$
$$\geq\exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)}\ln f\circ\exp(x)\,dx\right]\geq f\left(\sqrt{ab}\right),$$

which, by changing the variable $t=\exp x$, is equivalent to (2.14).

4. Applications for Exponential Integral Mean

First, we consider the *exponential integral* $\operatorname{Ei}:\mathbb{R}\to\mathbb{R}$ defined by

$$\operatorname{Ei}(x)=\int_{-\infty}^{x}\frac{e^{t}}{t}dt,\ \ x\in\mathbb{R}$$

and for $b>a>0$ define the *exponential integral mean* by

$$\operatorname{Ei}_{\mu}(a,b):=\frac{\operatorname{Ei}(b)-\operatorname{Ei}(a)}{\ln b-\ln a}=\frac{\int_{a}^{b}\frac{e^{t}}{t}dt}{\ln b-\ln a}.$$

If we use the inequality (2.3) in the form

$$\exp\left[\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{\ln f(t)}{t}dt\right]\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f(t)}{t}dt\leq L(f(a),f(b)) \tag{4.1}$$

for the GG-convex function $f(t)=\exp t$, then we get the basic inequality

$$\exp(L(a,b))\leq\operatorname{Ei}_{\mu}(a,b)\leq L(\exp(a),\exp(b))=E(a,b), \tag{4.2}$$

where

$$E(a,b):=\frac{\exp b-\exp a}{b-a},\ \ b>a>0.$$

From (2.12) for the GG-convex function $f(t)=\exp t$, we have

$$A(\exp(a),\exp(b))-\operatorname{Ei}_{\mu}(a,b)\geq(b-a)\left[E(a,b)-\operatorname{Ei}_{\mu}(a,b)\right]\geq 0. \tag{4.3}$$

If we use the inequality (2.16) in the form

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f(t)}{t}dt\\ &\geq\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f(t)}{t}dt\\ &\geq\exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{\ln f(t)}{t}dt\right]\geq f\left(\sqrt{ab}\right)\end{aligned}$$

for the GG-convex function $f(t)=\exp t$, then we also have

$$\begin{aligned}\operatorname{Ei}_{\mu}(a,b)&\geq\operatorname{Ei}_{\mu}\left(a^{1-\alpha}b^{\alpha},a^{\alpha}b^{1-\alpha}\right)\\ &\geq\exp\left(L\left(a^{1-\alpha}b^{\alpha},a^{\alpha}b^{1-\alpha}\right)\right)\geq\exp(G(a,b))\end{aligned} \tag{4.4}$$

for every $\alpha\in[0,1]\setminus\left\{\frac{1}{2}\right\}$.

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