Received 03/06/15

WEIGHTED INTEGRAL INEQUALITIES FOR GG-CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. Some weighted integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. Introduction

The function $f : I \subset (0,\infty) \to (0,\infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [4]

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1.1}$$

for any $x, y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (1.1) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [53], however, the roots of the research in this area can be traced long before him [54].

It is easy to see that [54], the function $f : I \subset (0,\infty) \to (0,\infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that [54] every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients c_n is a GG-convex function on $(0,r)$, where r is the radius of convergence for f. Therefore functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x} - \cot x$ are GG-convex on $\left(0, \frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln \frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are GG-convex on $(0,1)$. Also, the Γ function is a strictly GG-convex function on $[1,\infty)$.

It is also known that [54], if a function f is GG-convex, then so is $x^{\alpha} f^{\beta}(x)$ for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is GG-convex, then so is the other.

Recently [30], we obtained the following weighted inequality:

Theorem 1. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$, then*

$$\begin{aligned} & f\left(\exp\left(\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)\,dt}\right)\right) \\ & \leq \exp\left(\frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \\ & \leq [f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)dt}}{\ln b - \ln a}} [f(b)]^{\frac{\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)dt} - \ln a}{\ln b - \ln a}}. \end{aligned} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Integral inequalities, GG-Convex functions.

One can observe that, by the weighted geometric mean - arithmetic mean inequality

$$\alpha^{1-\lambda}\beta^{\lambda} \leq (1-\lambda)\alpha + \lambda\beta, \ \ \alpha, \beta > 0 \text{ and } \lambda \in [0,1],$$

we have the further upper bound

$$[f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b - \ln a}} [f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt} - \ln a}{\ln b - \ln a}} \leq \left(\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b - \ln a} \right) f(a) + \left(\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt} - \ln a}{\ln b - \ln a} \right) f(b).$$

We define the *p-logarithmic mean* of two positive numbers a, b by

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}} \text{ if } b \neq a, \\ \\ b \text{ if } b = a. \end{cases}$$

In particular, by taking $w(t) = t^p$ in (1.2), we have for any GG-convex function $f : [a,b] \subset (0,\infty) \to (0,\infty)$ that

$$\left[f\left(\left[I\left(a^{p+1}, b^{p+1}\right) \right]^{\frac{1}{p+1}} \right) \right]^{L_p^p(a,b)} \leq \exp\left(\frac{1}{b-a} \int_a^b t^p \ln f(t)\, dt \right) \leq [f(a)]^{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{(p+1)(b-a)}} [f(b)]^{\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{(p+1)(b-a)}}, \tag{1.3}$$

for any $p \in \mathbb{R}$ with $p \neq 0, -1$.

We recall that the *logarithmic mean* is defined by $L(p,q) := \frac{p-q}{\ln p - \ln q}$ if $p \neq q$ and $L(p,p) := p$.

If $p = 0$, namely we take $w(t) = 1$ in (1.2), then we get

$$f(I(a,b)) \leq \exp\left(\frac{1}{b-a} \int_a^b \ln f(t)\, dt \right) \leq [f(b)]^{\frac{b-L(a,b)}{b-a}} [f(a)]^{\frac{L(a,b)-a}{b-a}} \tag{1.4}$$

that has been obtained by Mitroi and Spiridon in [52].

If $p = -1$, namely we take $w(t) = \frac{1}{t}$ in (1.2), then we get

$$f\left(\sqrt{ab}\right) \leq \exp\left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt \right) \leq \sqrt{f(a) f(b)}. \tag{1.5}$$

If $p = 1$ in (2.11) then we also have

$$f\left(\sqrt{I(a^2,b^2)}\right) \leq \exp\left(\frac{1}{b-a} \int_a^b t \ln f(t)\, dt \right) \leq [f(a)]^{\frac{A(a,b)L(a,b)-a^2}{2(b-a)}} [f(b)]^{\frac{b^2-A(a,b)L(a,b)}{2(b-a)}}. \tag{1.6}$$

We recall that the *identric mean* is defined by $I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ for $b \neq a$ and $I(a,a) := a$.

We also recall the classical *Hermite-Hadamard inequality* that states that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{f(a)+f(b)}{2} \tag{1.7}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1]-[21], [24]-[31], [32]-[41] and [42]-[57].

Motivated by the above results we establish here some new weighted integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals. Applications for special means are also provided.

2. New Results for General Weights

We have:

Theorem 2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, then*

$$f\left(\sqrt{ab}\right)\leq\left(\frac{\int_{a}^{b}w(t)\,f^{p}(t)\,f^{p}\left(\frac{ab}{t}\right)dt}{\int_{a}^{b}w(t)\,dt}\right)^{\frac{1}{2p}}\leq\sqrt{f(a)\,f(b)} \tag{2.1}$$

for any $p>0$.

Proof. From the definition of GG-convex functions we have

$$f\left(x^{1-\lambda}y^{\lambda}\right)\leq[f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{2.2}$$

and

$$f\left(x^{\lambda}y^{1-\lambda}\right)\leq[f(x)]^{\lambda}[f(y)]^{1-\lambda} \tag{2.3}$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$.

By multiplication of (2.2) with (2.3) we get

$$f\left(x^{1-\lambda}y^{\lambda}\right)f\left(x^{\lambda}y^{1-\lambda}\right)\leq f(x)\,f(y)$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$.

Therefore

$$f\left(a^{1-\lambda}b^{\lambda}\right)f\left(a^{\lambda}b^{1-\lambda}\right)\leq f(a)\,f(b) \tag{2.4}$$

for any $\lambda\in[0,1]$.

From (2.2) we also have

$$f(\sqrt{xy})\leq\sqrt{f(x)\,f(y)} \tag{2.5}$$

for any $x,y\in[a,b]$.

By taking $x=a^{1-\lambda}b^{\lambda}$, $y=a^{\lambda}b^{1-\lambda}$ in (2.5) and then squaring we get

$$f^{2}\left(\sqrt{ab}\right)\leq f\left(a^{1-\lambda}b^{\lambda}\right)f\left(a^{\lambda}b^{1-\lambda}\right). \tag{2.6}$$

Since for any $t\in[a,b]$ there is a unique $\lambda\in[0,1]$ such that $t=a^{1-\lambda}b^{\lambda}$, we obtain from (2.4) and (2.6) that

$$f^{2}\left(\sqrt{ab}\right)\leq f(t)\,f\left(\frac{ab}{t}\right)\leq f(a)\,f(b) \tag{2.7}$$

for any $t\in[a,b]$.

If we take the power $p>0$ in (2.7), multiply by $w(t)\geq 0$ for $t\in[a,b]$ and integrate, we get

$$f^{2p}\left(\sqrt{ab}\right)\int_a^b w(t)\,dt \leq \int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt \leq f^p(a)\,f^p(b)\int_a^b w(t)\,dt \tag{2.8}$$

that is equivalent to

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt} \leq f^p(a)\,f^p(b) \tag{2.9}$$

and by taking the power $\frac{1}{2p}$ we get the desired result (2.1). □

We observe that for $p=1$ we get the inequality

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}} \leq \sqrt{f(a)\,f(b)}, \tag{2.10}$$

while from $p=\frac{1}{2}$ we get

$$f\left(\sqrt{ab}\right) \leq \frac{\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt} \leq \sqrt{f(a)\,f(b)}. \tag{2.11}$$

If we take $p=\frac{1}{4}$ in (2.1), then we get

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t)\sqrt[4]{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\right)^2 \leq \sqrt{f(a)\,f(b)}. \tag{2.12}$$

Using *Jensen's inequality* for the power $p\geq 1$ ($p\in(0,1)$), namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^p \leq (\geq) \frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can state the following more precise result:

Corollary 1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$.*

(i) If $p\geq 1$, then

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}} \leq \left(\frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)\,f(b)}. \tag{2.13}$$

(ii) If $p \in (0,1)$, *then*

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt}\right)^{\frac{1}{2p}} \leq \left(\frac{\int_a^b w(t) f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}. \tag{2.14}$$

If we take in Corollary 1 $w(t) = 1$, $t \in [a,b]$, then for any $f : [a,b] \subset (0,\infty) \to (0,\infty)$ a GG-convex function, we have for $p \geq 1$ that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{b-a}\int_a^b f(t) f\left(\frac{ab}{t}\right) dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{b-a}\int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) dt\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)} \tag{2.15}$$

and for $p \in (0,1)$, that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{b-a}\int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) dt\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f(t) f\left(\frac{ab}{t}\right) dt\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}. \tag{2.16}$$

If we take in Corollary 1 $w(t) = \frac{1}{t}$, $t \in [a,b]$, then for any $f : [a,b] \subset (0,\infty) \to (0,\infty)$ a GG-convex function, we have for $p \geq 1$ that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)} \tag{2.17}$$

and for $p \in (0,1)$, that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2p}} \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}. \tag{2.18}$$

If we take $p = \frac{1}{2}$ in the first inequality in (2.18), then we get

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t) f\left(\frac{ab}{t}\right)} dt \tag{2.19}$$

that has been obtained by Işcan in [45].

Theorem 3. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, then*

$$f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right) \tag{2.20}$$
$$\leq \exp\left(\frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right)$$
$$\leq \frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt} \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}} \frac{\int_a^b w(t)\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}} dt}{\int_a^b w(t)\,dt}$$
$$\leq \left(\frac{\ln b-\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b-\ln a}\right) f(a) + \left(\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}-\ln a}{\ln b-\ln a}\right) f(b).$$

Proof. If we use Jensen's inequality for the exponential function and nonnegative weight w, we have

$$\exp\left(\frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \leq \frac{\int_a^b w(t)\exp(\ln f(t))\,dt}{\int_a^b w(t)\,dt} = \frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt},$$

and the second inequality in (2.20) is proved.

Let $t=a^{1-\lambda}b^{\lambda}\in[a,b]$ with $\lambda\in[0,1]$, then $\lambda=\frac{\ln t-\ln a}{\ln b-\ln a}$. By the GG-convexity of f we have

$$f(t)=f\left(a^{1-\lambda}b^{\lambda}\right)\leq[f(a)]^{1-\lambda}[f(b)]^{\lambda} \tag{2.21}$$
$$=[f(a)]^{\frac{\ln b-\ln t}{\ln b-\ln a}}[f(b)]^{\frac{\ln t-\ln a}{\ln b-\ln a}}$$
$$=\left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}}\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}}$$
$$\leq\frac{\ln b-\ln t}{\ln b-\ln a}f(a)+\frac{\ln t-\ln a}{\ln b-\ln a}f(b)$$

for any $t\in[a,b]$.

Now, if we take the weighted integral mean in (2.21), then we get the last part of (2.20). □

By choosing $w(t)=1$, $t\in[a,b]$ in (2.20), we deduce

$$\begin{aligned}
&f\left(\exp\left(\frac{1}{b-a}\int_a^b \ln t dt\right)\right)\\
&\le \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\\
&\le \frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}} dt}{b-a}\\
&\le \left(\frac{\ln b-\frac{1}{b-a}\int_a^b \ln t dt}{\ln b-\ln a}\right) f(a) + \left(\frac{\frac{1}{b-a}\int_a^b \ln t dt-\ln a}{\ln b-\ln a}\right) f(b),
\end{aligned}$$

and since $\frac{1}{b-a}\int_a^b \ln t dt=\ln I(a,b)$, hence

$$\begin{aligned}
(2.22)\quad f(I(a,b)) &\le \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\\
&\le \frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}} dt}{b-a}\\
&\le \frac{\ln b-\ln I(a,b)}{\ln b-\ln a} f(a) + \frac{\ln I(a,b)-\ln a}{\ln b-\ln a} f(b).
\end{aligned}$$

If we take $w(t)=\frac{1}{t}$, $t\in[a,b]$ in (2.20), then we get

$$\begin{aligned}
(2.23)\quad &f\left(\exp\left(\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}\right)\right)\\
&\le \exp\left(\frac{\int_a^b \frac{1}{t}\ln f(t)\,dt}{\int_a^b \frac{1}{t}dt}\right)\\
&\le \frac{\int_a^b \frac{1}{t} f(t)\,dt}{\int_a^b \frac{1}{t}dt} \le \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}} \frac{\int_a^b \frac{1}{t}\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}} dt}{\int_a^b w(t)\,dt}\\
&\le \left(\frac{\ln b-\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}}{\ln b-\ln a}\right) f(a) + \left(\frac{\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}-\ln a}{\ln b-\ln a}\right) f(b).
\end{aligned}$$

This is equivalent, after suitable calculations, to

$$\begin{aligned}
(2.24)\quad f\left(\sqrt{ab}\right) &\le \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}\ln f(t)\,dt\right)\\
&\le \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t} f(t)\,dt \le L(f(a),f(b)) \left(\le \frac{f(a)+f(b)}{2}\right).
\end{aligned}$$

The third inequality in (2.24) has been obtained in a different way by Işcan in [45].

3. Other Weighted Inequalities

We have:

Theorem 4. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{2} \cdot \frac{\int_a^b \left(1+\frac{ab}{t^2}\right) w(t) \ln f(t)\, dt}{\int_a^b w(t)\, dt}\right) \le \sqrt{f(a) f(b)}. \tag{3.1}$$

Proof. By taking the log in (2.7) we get

$$2 \ln f\left(\sqrt{ab}\right) \le \ln f(t) + \ln f\left(\frac{ab}{t}\right) \le \ln f(a) + \ln f(b) \tag{3.2}$$

for any $t \in [a,b]$.

If we multiply (3.2) by $w(t) \ge 0$ with $t \in [a,b]$ and use the fact that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$, then we get

$$\begin{aligned} 2w(t) \ln f\left(\sqrt{ab}\right) &\le w(t) \ln f(t) + w\left(\frac{ab}{t}\right) \ln f\left(\frac{ab}{t}\right) \\ &\le w(t) \left[\ln f(a) + \ln f(b)\right] \end{aligned} \tag{3.3}$$

for any $t \in [a,b]$.

If we integrate the inequality (3.3) on $[a,b]$ we get

$$\begin{aligned} &2 \ln f\left(\sqrt{ab}\right) \int_a^b w(t)\, dt \\ &\le \int_a^b w(t) \ln f(t)\, dt + \int_a^b w\left(\frac{ab}{t}\right) \ln f\left(\frac{ab}{t}\right) dt \\ &\le \left[\ln f(a) + \ln f(b)\right] \int_a^b w(t)\, dt. \end{aligned} \tag{3.4}$$

By changing the variable $u = \frac{ab}{t}$, we have $dt = -\frac{ab}{u^2} du$ and

$$\begin{aligned} \int_a^b w\left(\frac{ab}{t}\right) \ln f\left(\frac{ab}{t}\right) dt &= -\int_b^a w(u) \ln f(u) \frac{ab}{u^2} du \\ &= \int_a^b w(t) \ln f(t) \frac{ab}{t^2} dt \end{aligned}$$

and by (3.4) we get

$$\begin{aligned} 2 \ln f\left(\sqrt{ab}\right) \int_a^b w(t)\, dt &\le \int_a^b w(t) \ln f(t)\, dt + \int_a^b w(t) \ln f(t) \frac{ab}{t^2} dt \\ &\le \left[\ln f(a) + \ln f(b)\right] \int_a^b w(t)\, dt, \end{aligned}$$

which is equivalent to the desired result (3.1). □

If we take in (3.1) $w(t) = 1$, $t \in [a,b]$, then we get

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{2} \cdot \frac{\int_a^b \left(1+\frac{ab}{t^2}\right) \ln f(t)\, dt}{b-a}\right) \le \sqrt{f(a) f(b)}. \tag{3.5}$$

Another example of weight w that satisfies the condition $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$ is $w(t)=\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|$, with $t\in[a,b]\subset(0,\infty)$.

The following result also holds:

Theorem 5. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right)\le\exp\left(\frac{\int_a^b\frac{1}{t}w(t)\ln f(t)\,dt}{\int_a^b\frac{1}{t}w(t)\,dt}\right)\le\sqrt{f(a)f(b)}. \tag{3.6}$$

Proof. From (3.2) for $t=a^{1-\lambda}b^{\lambda}$ with $\lambda\in[0,1]$, we have

$$2\ln f\left(\sqrt{ab}\right)\le\ln f\left(a^{1-\lambda}b^{\lambda}\right)+\ln f\left(a^{\lambda}b^{1-\lambda}\right)\le\ln f(a)+\ln f(b) \tag{3.7}$$

for any $\lambda\in[0,1]$.

Since $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$, then $w\left(a^{1-\lambda}b^{\lambda}\right)=w\left(a^{\lambda}b^{1-\lambda}\right)$ for any $\lambda\in[0,1]$ and by (3.7) we have

$$\begin{aligned}&2\ln f\left(\sqrt{ab}\right)w\left(a^{1-\lambda}b^{\lambda}\right)\\&\le w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)+w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right)\\&\le w\left(a^{1-\lambda}b^{\lambda}\right)\left[\ln f(a)+\ln f(b)\right]\end{aligned} \tag{3.8}$$

for any $\lambda\in[0,1]$.

Integrating the inequality over $\lambda\in[0,1]$ we have

$$\begin{aligned}&2\ln f\left(\sqrt{ab}\right)\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda\\&\le\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda+\int_0^1 w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda\\&\le\left[\ln f(a)+\ln f(b)\right]\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda\end{aligned} \tag{3.9}$$

and since

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\int_0^1 w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda,$$

hence by (3.9) we get

$$\ln f\left(\sqrt{ab}\right)\le\frac{\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda}{\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda}\le\ln\left(\sqrt{f(a)f(b)}\right). \tag{3.10}$$

By changing the variable $a^{1-\lambda}b^{\lambda}=t$, then $(1-\lambda)\ln a+\lambda\ln b=\ln t$ which gives that

$$\lambda=\frac{\ln t-\ln a}{\ln b-\ln a}.$$

Therefore $d\lambda=\frac{1}{t}dt$,

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}w(t)\ln f(t)\,dt$$

and

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}w(t)\,dt$$

and by (3.10) we get the desired result (3.6). □

If we take in (3.6) $w(t) = 1$, $t \in [a,b]$, then we recapture (1.5).
If we take in (3.6)

$$w(t) = \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| = \left|\ln t - \frac{\ln a + \ln b}{2}\right|$$

and since

$$\begin{aligned}\int_a^b \frac{1}{t}w(t)\,dt &= \int_a^b \frac{1}{t}\left|\ln t - \frac{\ln a + \ln b}{2}\right|dt \\ &= \int_a^b \left|\ln t - \frac{\ln a + \ln b}{2}\right|d\ln t \\ &= \int_{\ln a}^{\ln b}\left|x - \frac{\ln a + \ln b}{2}\right|dx = \frac{1}{4}(\ln b - \ln a)^2\end{aligned}$$

then we get

$$\begin{aligned}\frac{1}{4}(\ln b - \ln a)^2 \ln f\left(\sqrt{ab}\right) &\le \int_a^b \frac{1}{t}\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|\ln f(t)\,dt \\ &\le \frac{1}{4}(\ln b - \ln a)^2 \ln\left(\sqrt{f(a)f(b)}\right).\end{aligned} \tag{3.11}$$

We also have:

Theorem 6. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$. Then we have the inequalities*

$$\begin{aligned}f\left(\sqrt{ab}\right) &\le \exp\left(\frac{1}{2}\cdot\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \\ &\le \frac{\int_a^b w(t)\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt} \\ &\le \frac{\sqrt{ab\int_a^b \frac{w(t)f(t)}{t^2}dt\int_a^b w(t)f(t)\,dt}}{\int_a^b w(t)\,dt} \\ &\le \frac{1}{2}\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)f(t)\,dt}{\int_a^b w(t)\,dt}.\end{aligned} \tag{3.12}$$

Proof. As in the proof of Theorem 4 we have

$$\frac{1}{2}\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt = \int_a^b w(t)\ln\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt.$$

Then by Jensen's inequality for the exponential and the weight w we have

$$\begin{aligned}
&\exp\left(\frac{1}{2}\cdot\frac{\int_a^b \left(1+\frac{ab}{t^2}\right) w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \\
&= \exp\left(\frac{\int_a^b w(t)\ln\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\right) \\
&\le \frac{\int_a^b w(t)\exp\left(\ln\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}\right)dt}{\int_a^b w(t)\,dt} = \frac{\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}
\end{aligned}$$

that proves the second part of (3.12).

By Cauchy-Bunyakowsky-Schwarz inequality and the property of w we have

$$\begin{aligned}
\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt &\le \sqrt{\int_a^b w(t)\,f(t)\,dt\int_a^b w(t)\,f\left(\frac{ab}{t}\right)dt} \\
&= \sqrt{\int_a^b w(t)\,f(t)\,dt\int_a^b w\left(\frac{ab}{t}\right)f\left(\frac{ab}{t}\right)dt} \\
&= \sqrt{ab\int_a^b \frac{w(t)\,f(t)}{t^2}dt\int_a^b w(t)\,f(t)\,dt},
\end{aligned}$$

which proves the third inequality in (3.12).

By the geometric mean - arithmetic mean inequality we also have

$$\begin{aligned}
&\sqrt{ab\int_a^b \frac{w(t)\,f(t)}{t^2}dt\int_a^b w(t)\,f(t)\,dt} \\
&\le \frac{1}{2}\left(ab\int_a^b \frac{w(t)\,f(t)}{t^2} + \int_a^b w(t)\,f(t)\,dt\right) \\
&= \frac{1}{2}\int_a^b \left(1+\frac{ab}{t^2}\right) w(t)\,f(t)\,dt
\end{aligned}$$

that proves the last part of (3.12). □

If we take $w(t) = 1$, $t \in [a,b]$ in (3.12), then we get

$$\begin{aligned}
f\left(\sqrt{ab}\right) &\le \exp\left(\frac{1}{2(b-a)}\int_a^b \left(1+\frac{ab}{t^2}\right)\ln f(t)\,dt\right) \\
&\le \frac{1}{b-a}\int_a^b \sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt \\
&\le \frac{1}{b-a}\sqrt{ab\int_a^b \frac{f(t)}{t^2}dt\int_a^b f(t)\,dt} \\
&\le \frac{1}{2(b-a)}\int_a^b \left(1+\frac{ab}{t^2}\right) f(t)\,dt.
\end{aligned} \tag{3.13}$$

We observe that, if in the first inequality in (2.16) we take $p=\frac{1}{2}$, then we have

$$f\left(\sqrt{ab}\right)\leq\frac{1}{b-a}\int_{a}^{b}\sqrt{f\left(t\right)f\left(\frac{ab}{t}\right)}dt. \tag{3.14}$$

Therefore the first part of (3.13) is a refinement of (3.14).

4. Some Particular Cases

We consider a simple example of GG-convex functions, namely $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\exp x$. By Corollary 1, we have for any $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, that

$$\begin{aligned}\exp\left(G\left(a,b\right)\right)&\leq\left(\frac{\int_{a}^{b}w\left(t\right)\exp\left(t+\frac{ab}{t}\right)dt}{\int_{a}^{b}w\left(t\right)dt}\right)^{\frac{1}{2}}\\&\leq\left(\frac{\int_{a}^{b}w\left(t\right)\exp\left[p\left(t+\frac{ab}{t}\right)\right]dt}{\int_{a}^{b}w\left(t\right)dt}\right)^{\frac{1}{2p}}\leq\exp\left(A\left(a,b\right)\right),\end{aligned} \tag{4.1}$$

for $p\geq 1$.

(ii) If $p\in(0,1)$, then

$$\begin{aligned}\exp\left(G\left(a,b\right)\right)&\leq\left(\frac{\int_{a}^{b}w\left(t\right)\exp\left[p\left(t+\frac{ab}{t}\right)\right]dt}{\int_{a}^{b}w\left(t\right)dt}\right)^{\frac{1}{2p}}\\&\leq\left(\frac{\int_{a}^{b}w\left(t\right)\exp\left(t+\frac{ab}{t}\right)dt}{\int_{a}^{b}w\left(t\right)dt}\right)^{\frac{1}{2}}\leq\exp\left(A\left(a,b\right)\right).\end{aligned} \tag{4.2}$$

From the inequality (2.20) applied for the GG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\exp x$ we have

$$\begin{aligned}&\exp\left(\exp\left(\frac{\int_{a}^{b}w\left(t\right)\ln tdt}{\int_{a}^{b}w\left(t\right)dt}\right)\right)\\&\leq\exp\left(\frac{\int_{a}^{b}w\left(t\right)tdt}{\int_{a}^{b}w\left(t\right)dt}\right)\leq\frac{\int_{a}^{b}w\left(t\right)\exp\left(t\right)dt}{\int_{a}^{b}w\left(t\right)dt}\\&\leq\frac{\int_{a}^{b}w\left(t\right)t^{L(a,b)}dt}{\int_{a}^{b}w\left(t\right)dt}\exp\left(\frac{a\ln b-b\ln a}{\ln b-\ln a}\right),\end{aligned} \tag{4.3}$$

for any $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$.

If $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$, then from (3.1) and (3.6) we get

$$G\left(a,b\right)\leq\frac{1}{2}\cdot\frac{\int_{a}^{b}\left(1+\frac{ab}{t^{2}}\right)tw\left(t\right)dt}{\int_{a}^{b}w\left(t\right)dt}\leq A\left(a,b\right) \tag{4.4}$$

and

$$G\left(a,b\right)\leq\frac{\int_{a}^{b}w\left(t\right)dt}{\int_{a}^{b}\frac{1}{t}w\left(t\right)dt}\leq A\left(a,b\right). \tag{4.5}$$

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[5] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 `[Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf]`.

[6] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[7] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[8] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[9] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[10] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[11] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[12] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[13] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[14] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[15] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[16] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[17] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[18] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[19] S. S. Dragomir, Refinements of the Hermite-Hadamard integral inequality for log-convex functions, *Austral. Math. Soc. Gaz.* **28** (2001), no. 3, 129–134.

[20] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[21] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[22] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[23] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No.3, Article 35.
[24] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.
[25] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1
[26] S. S. Dragomir, Inequalities of Hermite-Hadamard type for *GA*-convex functions, Preprint *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 30. [http://rgmia.org/papers/v18/v18a30.pdf].
[27] S. S. Dragomir, Some new inequalities of Hermite-Hadamard type for *GA*-convex functions, Preprint *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 33. [http://rgmia.org/papers/v18/v18a33.pdf].
[28] S. S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions, Preprint *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 42. [http://rgmia.org/papers/v18/v18a42.pdf].
[29] S. S. Dragomir, Inequalities of Hermite-Hadamard type for *GG*-convex functions, Preprint *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 71. [http://rgmia.org/papers/v18/v18a71.pdf].
[30] S.S. Dragomir, New inequalities of Hermite-Hadamard type for *GG*-convex functions, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 72. [Online http://rgmia.org/papers/v18/v18a72.pdf].
[31] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.
[32] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.
[33] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.
[34] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.
[35] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.
[36] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.
[37] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.
[38] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.
[39] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.
[40] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.
[41] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.
[42] A. El Farissi, Simple proof and refeinment of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.
[43] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985
[44] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.
[45] I. Işcan, Some new Hermite-Hadamard type inequalities for geometrically convex functions, *Mathematics and Statistics* **1** (2) (2013) 86-91.
[46] I. Işcan, On some new Hermite-Hadamard type inequalities for s-geometrically convex functions, *International Journal of Mathematics and Mathematical Sciences,* Volume **2014**, Article ID 163901, 8 pages.

[47] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* Volume **13** (2010), Number 1, 1–32
[48] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.
[49] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.
[50] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
[51] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.
[52] F.-C. Mitroi and C. I. Spiridon, A Hermite-Hadamard type inequality for multiplicatively convex functions, *Annals of the University of Craiova, Mathematics and Computer Science Series,* Volume **38**(1), 2011, Pages 96-99.
[53] P. Montel, Sur les functions convexes et les fonctions sousharmoniques, *Journal de Math.*, **9** (1928), 7, 29–60.
[54] C. P. Niculescu, Convexity according to the geometric mean, *Math. Inequal. Appl.,* **3**, (2000), 2, 155–167.
[55] M. A. Noor, K. I. Noor and M. U. Awan, Some inequalities for geometrically-arithmetically h-convex functions, *Creat. Math. Inform.* **23** (2014), No. 1, 91 - 98.
[56] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
[57] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.
[58] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.
[59] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.
[60] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
[61] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
[62] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.
[63] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.
[64] M. Tunç, On Hadamard type inequalities for s-geometrically convex functions, Preprint, *RGMIA Research Report Collection*, vol. **15**, article 70, 6 pages, 2012.
[65] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.
[66] X.-M. Zhang and N.-G. Zheng, Geometrically convex functions and estimation of remainder terms for Taylor expansion of some functions, *J. Math. Inequal.* **4** (2010), 1, 15-25.
[67] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa