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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *GH*-CONVEX FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for GH-convex functions defined on positive intervals are given. Applications for special means are provided as well.

1. Introduction

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [21, p. 2], [22, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{1.1}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

For related results, see [1]-[20], [23]-[25], [26]-[35] and [36]-[45].

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R} \setminus \{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f((1-\lambda)x + \lambda y) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} = \frac{f(x) f(y)}{(1-\lambda) f(y) + \lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)} \leq (\geq) \frac{1}{f((1-\lambda)x + \lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore we can state the following fact:

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Criterion 1. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.1) for the function $\frac{1}{f}$ then we state the following result:

Proposition 1. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \le (\ge) \int_0^1 \frac{d\lambda}{f((1-\lambda)x+\lambda y)} \le (\ge) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.2}$$

for any $x, y \in C$.

Following [4], we can introduce the concept of *GH-convex (concave)* function $f : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}} = \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}. \tag{1.3}$$

Since

$$f\left(x^{1-\lambda}y^{\lambda}\right) = f\circ\exp\left[(1-\lambda)\ln x+\lambda\ln y\right]$$

and

$$\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} = \frac{f\circ\exp(\ln x)\, f\circ\exp(\ln y)}{(1-\lambda)f\circ\exp(y)+\lambda f\circ\exp(x)}$$

then $f : I \subset (0,\infty) \to \mathbb{R}$ is *GH*-convex (concave) on I if and only if $f\circ\exp$ is *AH-convex (concave)* on $\ln I := \{x|\ x = \ln t,\ t \in I\}$.

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for *GH*-convex (concave) functions. Some examples for special means are provided as well.

2. Results

As a direct consequence of Hermite-Hadamard inequality we have:

Theorem 1. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2f(a)f(b)} \le (\ge) \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{tf(t)}dt \le (\ge) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{2.1}$$

From a different perspective we have:

Theorem 2. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}dt \le (\ge) \frac{G^2(f(a),f(b))}{L(f(a),f(b))}, \tag{2.2}$$

where, for $p, q > 0$, $G(p,q) := \sqrt{pq}$ is the geometric-mean while

$$L(p,q) := \begin{cases} \frac{p-q}{\ln p-\ln q} \text{ if } p \ne q \\ q \text{ if } p = q \end{cases}$$

is the logarithmic-mean.

Using the following well known inequality $G(a,b) \leq L(a,b)$ we have a simpler upper bound

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \leq \frac{G^2(f(a),f(b))}{L(f(a),f(b))} \leq G(f(a),f(b)) \tag{2.3}$$

provided that $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GH-convex on $[a,b]$.

We have also the complementary result:

Theorem 3. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$f\left(\sqrt{ab}\right) \leq (\geq) \frac{\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt}{\int_a^b \frac{1}{t}f(t)dt}. \tag{2.4}$$

We observe that by Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt \leq \left(\int_a^b \frac{1}{t^2}f^2(t)\,dt\right)^{1/2}\left(\int_a^b f^2\left(\frac{ab}{t}\right)dt\right)^{1/2}. \tag{2.5}$$

If we change the variable $\frac{ab}{t}=s$, then $dt=-\frac{ab}{s^2}ds$ and we have

$$\int_a^b f^2\left(\frac{ab}{t}\right)dt = ab\int_a^b \frac{1}{s^2}f^2(s)\,ds.$$

From (2.5) we get

$$\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt \leq \left(\int_a^b \frac{1}{t^2}f^2(t)\,dt\right)^{1/2}\left(ab\int_a^b \frac{1}{s^2}f^2(s)\,ds\right)^{1/2}$$
$$= \sqrt{ab}\int_a^b \frac{1}{t^2}f^2(t)\,dt.$$

Now, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GH-convex, then from (2.4) we have

$$f\left(\sqrt{ab}\right) \leq \frac{\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt}{\int_a^b \frac{1}{t}f(t)\,dt} \leq \sqrt{ab}\,\frac{\int_a^b \frac{1}{t^2}f^2(t)\,dt}{\int_a^b \frac{1}{t}f(t)\,dt}. \tag{2.6}$$

If the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is monotonic either nonincreasing or nondecreasing, then the functions $f(\cdot)$ and $f\left(\frac{ab}{\cdot}\right)$ have opposite monotonicities. By the Čebyšev weighted integral inequality for asynchronous functions g and h and the positive weight $w\geq 0$, namely

$$\int_a^b w(t)\,dt\int_a^b w(t)g(t)h(t)\,dt \leq \int_a^b w(t)g(t)\,dt\int_a^b w(t)h(t)\,dt,$$

we have

$$\int_a^b \frac{1}{t}dt\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt \leq \int_a^b \frac{1}{t}f(t)\,dt\int_a^b \frac{1}{t}f\left(\frac{ab}{t}\right)dt,$$

i.e.,

$$\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}f(t)\,dt\int_a^b \frac{1}{t}f\left(\frac{ab}{t}\right)dt.$$

So, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GH-convex and monotonic on $[a,b]$, then from (2.4) we have

$$f\left(\sqrt{ab}\right)\leq\frac{\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b\frac{1}{t}f(t)\,dt}\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f\left(\frac{ab}{t}\right)dt \tag{2.7}$$

or, equivalently

$$f\left(\sqrt{ab}\right)\leq\frac{\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b\frac{1}{t}f(t)\,dt}\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f(t)\,dt. \tag{2.8}$$

Theorem 4. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on I. If $x,y\in\mathring{I}$, the interior of I, then there exists $\varphi(y)\in\left[f'_-(y),f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)}-1\leq(\geq)\,\frac{\varphi(y)\,y}{f(y)}(\ln y-\ln x). \tag{2.9}$$

In particular, we have:

Corollary 1. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on I and differentiable on $\mathring{I}$. If $x,y\in\mathring{I}$, then*

$$\frac{f(y)}{f(x)}-1\leq(\geq)\,\frac{f'(y)\,y}{f(y)}(\ln y-\ln x). \tag{2.10}$$

We also have:

Theorem 5. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$\int_a^b\frac{1}{s}f^2(s)\,ds\leq(\geq)\left[(\ln b-\ln u)\,f(b)+(\ln u-\ln a)\,f(a)\right]f(u), \tag{2.11}$$

for any $u\in[a,b]$.

If we take in (2.11) $u=G(a,b)=\sqrt{ab}$, then we get

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{s}f^2(s)\,ds\leq(\geq)\,A(f(a),f(b))\,f(G(a,b)). \tag{2.12}$$

If we take in (2.11) either $u=a$ or $u=b$, then we have

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{s}f^2(s)\,ds\leq(\geq)\,f(b)\,f(a). \tag{2.13}$$

Also, by taking in (2.11) $u=I(a,b)$, the *identric mean*, that is defined by

$$I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b\neq a\\ a \text{ if } b=a,\end{cases}$$

then we get

$$\begin{aligned}&\int_a^b\frac{1}{s}f^2(s)\,ds\\&\leq(\geq)\left[(\ln b-\ln I(a,b))\,f(b)+(\ln I(a,b)-\ln a)\,f(a)\right]f(I(a,b)).\end{aligned} \tag{2.14}$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)}$$

and

$$\ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

then the inequality (2.14) is equivalent to

$$\int_a^b \frac{1}{s} f^2(s)\, ds \le (\ge)\, f(I(a,b)) \left[\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a) \right]. \tag{2.15}$$

3. Proofs

Since $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is GH-convex (concave) on $[a,b]$, hence $f \circ \exp$ is AH-convex (concave) on $[\ln a, \ln b]$. By the inequality (1.2) for $f \circ \exp$ and $\ln a$, $\ln b$ we have

$$\begin{aligned} \frac{f \circ \exp(\ln a) + f \circ \exp(\ln b)}{2 f \circ \exp(\ln a)\, f \circ \exp(\ln b)} &\le (\ge) \int_0^1 \frac{d\lambda}{f \circ \exp((1-\lambda)\ln a + \lambda \ln b)} \\ &\le (\ge) \frac{1}{f \circ \exp\left(\frac{\ln a + \ln b}{2}\right)} \end{aligned} \tag{3.1}$$

that is equivalent to

$$\frac{f(a) + f(b)}{2 f(a) f(b)} \le (\ge) \int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^\lambda)} \le (\ge) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{3.2}$$

If we change the variable $t = a^{1-\lambda} b^\lambda$, then $(1-\lambda)\ln a + \lambda \ln b = \ln t$, which gives $\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}$ and $d\lambda = \frac{1}{(\ln b - \ln a)t} dt$. We have then

$$\int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^\lambda)} = \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t f(t)} dt$$

and by (3.2) we obtain the desired result (2.1).

From the definition of GH-convex (concave) functions on $[a,b]$ and by integration we get

$$\int_0^1 f\left(a^{1-\lambda} b^\lambda\right) d\lambda \le (\ge)\, f(a) f(b) \int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)}. \tag{3.3}$$

If $f(a) = f(b)$, then the integral

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} \tag{3.4}$$

reduces to $\frac{1}{f(a)}$.

If $f(a) \neq f(b)$, then by changing the variable $u = (1-\lambda) f(a) + \lambda f(b)$ in (3.4) we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} = \frac{1}{f(b) - f(a)} \int_{f(a)}^{f(b)} \frac{du}{u} = \frac{1}{L(f(a), f(b))}.$$

Also, as above, if we change the variable $t = a^{1-\lambda} b^\lambda$, then

$$\int_0^1 f\left(a^{1-\lambda} b^\lambda\right) d\lambda = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt.$$

Replacing these values in (3.3), we get the desired result (2.2).

If we take in the definition of GH-convex functions $\lambda = \frac{1}{2}$, then we get

$$f\left(\sqrt{xy}\right) \leq (\geq) \frac{2f(x)f(y)}{f(y)+f(x)}. \tag{3.5}$$

If we replace in (3.5), $x = a^{1-\lambda}b^{\lambda}$ and $y = a^{\lambda}b^{1-\lambda}$, then we get

$$f\left(\sqrt{ab}\right)\left[f\left(a^{1-\lambda}b^{\lambda}\right) + f\left(a^{\lambda}b^{1-\lambda}\right)\right] \leq (\geq) 2f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right). \tag{3.6}$$

By integrating this inequality over λ on $[0,1]$ we obtain

$$\begin{aligned} &f\left(\sqrt{ab}\right)\left[\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda + \int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda\right] \\ &\leq (\geq) 2\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda. \end{aligned} \tag{3.7}$$

Observe that

$$\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda = \int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt$$

and

$$\begin{aligned} \int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda &= \int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) f\left(\frac{ab}{a^{1-\lambda}b^{\lambda}}\right)d\lambda = \\ &= \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)f\left(\frac{ab}{t}\right)}{t}dt. \end{aligned}$$

Making use of (3.7) we deduce the desired result (2.4).

The following lemma is of interest in itself:

Lemma 1. *Let $f : I \subset \mathbb{R} \to (0,\infty)$ be AH-convex (concave) on I. If $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)}(y-x) \tag{3.8}$$

holds.

Proof. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_-(y), f'_+(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \lambda(y)(y-x) = -\lambda(y)(x-y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \frac{\varphi(y)}{f^2(y)}(x-y) \tag{3.9}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (3.9) can be also written as

$$1 - \frac{f(y)}{f(x)} \geq (\leq) \frac{\varphi(y)}{f(y)}(x-y)$$

or as

$$\frac{f(y)}{f(x)}-1\leq(\geq)\frac{\varphi(y)}{f(y)}(y-x)$$

and the inequality (3.8) is proved. □

Now, since $f:I\subset(0,\infty)\to(0,\infty)$ is GH-convex (concave) on I, then the function $f\circ\exp$ is AH-convex (concave) on $\ln I$.

Let $u,v\in\ln\mathring{I}$, then by (3.8) we have

$$\frac{f(e^v)}{f(e^u)}-1\leq(\geq)\frac{\varphi(e^v)e^v}{f(e^v)}(v-u) \tag{3.10}$$

with $\varphi(e^v)\in\left[f'_-(e^v),f'_+(e^v)\right].$

If $x,y\in\mathring{I}$ and we take $u=\ln x$, $v=\ln y$ in (3.10) then we get

$$\frac{f(y)}{f(x)}-1\leq(\geq)\frac{\varphi(y)y}{f(y)}(\ln y-\ln x)$$

with $\varphi(y)\in\left[f'_-(y),f'_+(y)\right].$

This proves Theorem 4.

The following lemma is of interest in itself.

Lemma 2. *Let $g:[c,d]\subset(0,\infty)\to(0,\infty)$ be AH-convex (concave) on $[c,d]$, then we have the inequality*

$$\frac{1}{d-c}\int_c^d g^2(t)\,dt\leq(\geq)\left[\frac{d-s}{d-c}g(d)+\frac{s-c}{d-c}g(c)\right]g(s) \tag{3.11}$$

for any $s\in[c,d]$.

Proof. If the function $g:[c,d]\subset(0,\infty)\to(0,\infty)$ is AH-convex (concave) on $[c,d]$, then the function g is differentiable almost everywhere on $[c,d]$ and we have the inequality

$$\frac{g(t)}{g(s)}-1\leq(\geq)\frac{g'(t)}{g(t)}(t-s) \tag{3.12}$$

for every $s\in[c,d]$ and almost every $t\in[c,d]$.

Multiplying (3.12) by $g(t)>0$ and integrating over $t\in[c,d]$ we have

$$\frac{1}{g(s)}\int_c^d g^2(t)\,dt-\int_c^d g(t)\,dt\leq(\geq)\int_c^d g'(t)(t-s)\,dt. \tag{3.13}$$

Integrating by parts we also have

$$\int_c^d g'(t)(t-s)\,dt=g(d)(d-s)+g(c)(s-c)-\int_c^d g(t)\,dt$$

and by (3.13) we get the desired result (3.11). □

Now, since $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GH-convex (concave) on I, then the function $g=f\circ\exp$ is AH-convex (concave) on $[c,d]=[\ln a,\ln b]$.

From (3.11) we then have for $s=\ln u$, $u\in[a,b]$ that

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f^2\circ\exp(t)\,dt\\&\leq(\geq)\left[\frac{\ln b-\ln u}{\ln b-\ln a}f\circ\exp(\ln b)+\frac{\ln u-\ln a}{\ln b-\ln a}f\circ\exp(\ln a)\right]f\circ\exp(\ln u).\end{aligned}$$

This is equivalent to

$$\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f^{2}\circ \exp(t)\,dt \leq (\geq)\left[\frac{\ln b-\ln u}{\ln b-\ln a}f(b)+\frac{\ln u-\ln a}{\ln b-\ln a}f(a)\right]f(u), \tag{3.14}$$

for any $u\in[a,b]$.

If we make the change of variable $s=\exp(t)$, then $t=\ln s$, $dt=\frac{ds}{s}$ and by (3.14) we get the desired inequality (2.11).

4. Applications

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=t^{p}$, $p\in\mathbb{R}\setminus\{0\}$. By the weighted geometric mean-harmonic mean inequality, we have

$$\begin{aligned} f\left(x^{1-\lambda}y^{\lambda}\right) &= \left(x^{1-\lambda}y^{\lambda}\right)^{p} = (x^{p})^{1-\lambda}(y^{p})^{\lambda} \\ &\geq \frac{1}{\frac{1-\lambda}{x^{p}}+\frac{\lambda}{y^{p}}} = \frac{x^{p}y^{p}}{(1-\lambda)y^{p}+\lambda x^{p}} \\ &= \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}, \end{aligned} \tag{4.1}$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$, which shows that f is GG-concave on $[a,b]$.

For $p\in\mathbb{R}\setminus\{0,-1\}$, we define the *p-logarithmic mean* as

$$L_{p}(a,b):=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & \text{if } b\neq a\\ b \text{ if } b=a.\end{cases}$$

We observe that

$$L_{p}^{p}(a,b)=\frac{1}{b-a}\int_{a}^{b}t^{p}dt,\ p\in\mathbb{R}\setminus\{0,-1\}.$$

If we write the inequality (2.1) for the GG-concave function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=t^{p}$, $p\in\mathbb{R}\setminus\{0\}$, then

$$H^{-1}(a^{p},b^{p})\geq\frac{1}{\ln b-\ln a}\int_{a}^{b}t^{-p-1}dt\geq G^{-1}(a^{p},b^{p}). \tag{4.2}$$

Observe that

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_{a}^{b}t^{-p-1}dt &= \frac{b-a}{\ln b-\ln a}\frac{1}{b-a}\int_{a}^{b}t^{-p-1}dt\\ &= L(a,b)\,L_{-p-1}^{-p-1}(a,b)\end{aligned}$$

for $p\in\mathbb{R}\setminus\{0,-1\}$ and by (4.2) we get

$$H^{-1}(a^{p},b^{p})\geq L(a,b)\,L_{-p-1}^{-p-1}(a,b)\geq G^{-1}(a^{p},b^{p}), \tag{4.3}$$

for $p\in\mathbb{R}\setminus\{0,-1\}$.

Now, if we use (2.2), then we get

$$L(a,b)\,L_{p-1}^{p-1}(a,b)\geq\frac{G^{2p}(a,b)}{L(a^{p},b^{p})}, \tag{4.4}$$

for $p\in\mathbb{R}\setminus\{0,1\}$.

From (2.4) we also have

$$L_{p-1}^{p-1}(a,b) \geq G^{p}(a,b)\, L(a,b), \tag{4.5}$$

for $p \in \mathbb{R} \setminus \{0,1\}$.

Moreover, if we use the inequality (2.12) we have

$$L(a,b)\, L_{2p-1}^{2p-1}(a,b) \geq A(a^{p},b^{p})\, G^{p}(a,b), \tag{4.6}$$

for $p \in \mathbb{R} \setminus \left\{0,\frac{1}{2}\right\}$.

From (2.15) we finally have

$$L(a,b)\, L_{2p-1}^{2p-1}(a,b) \geq I^{p}(a,b)\left[\frac{L(a,b)-a}{b-a}b^{p}+\frac{b-L(a,b)}{b-a}a^{p}\right], \tag{4.7}$$

for $p \in \mathbb{R} \setminus \left\{0,\frac{1}{2}\right\}$.

Now, for $q>0$ consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(-qt)$. Then by the weighted arithmetic mean - geometric mean - harmonic mean inequality, we have for any $x,y\in[a,b]$ and $\lambda\in[0,1]$ that

$$\begin{aligned} f\left(x^{1-\lambda}y^{\lambda}\right) &= \exp\left(-qx^{1-\lambda}y^{\lambda}\right) \geq \exp\left(-q\left[(1-\lambda)x+\lambda y\right]\right) \\ &= \left[\exp(-qx)\right]^{1-\lambda}\left[\exp(-qy)\right]^{\lambda} \\ &\geq \frac{1}{(1-\lambda)\frac{1}{\exp(-qx)}+\lambda\frac{1}{\exp(-qy)}} \\ &= \frac{\exp(-qx)\exp(-qy)}{(1-\lambda)\exp(-qy)+\lambda\exp(-qx)} \\ &= \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}, \end{aligned}$$

which shows that f is GH-concave on $[a,b]$.

We consider the following α-*exponential integral mean*

$$\mathrm{Ei}_{\alpha}(a,b) := \frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{\exp(\alpha t)}{t}dt,$$

where $b>a>0$ and $\alpha\in\mathbb{R}$.

By (2.1) for the GH-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(-qt)$ where $q>0$, we get that

$$\frac{\exp(-qa)+\exp(-qb)}{2\exp(-q(a+b))} \geq \mathrm{Ei}_{q}(a,b) \geq \exp\left(q\sqrt{ab}\right). \tag{4.8}$$

From (2.2) we have for $q>0$ that

$$\mathrm{Ei}_{-q}(a,b) \geq \frac{\exp(-q(a+b))}{L(\exp(-qa),\exp(-qb))}. \tag{4.9}$$

Observe, however, that

$$\begin{aligned} L(\exp(-qa),\exp(-qb)) &= \frac{\exp(-qb)-\exp(-qa)}{q(a-b)} \\ &= \frac{\exp(qa)-\exp(qb)}{q(a-b)\exp(q(a+b))} \\ &= \frac{E(qa,qb)}{\exp(q(a+b))}, \end{aligned}$$

where E is defined by

$$E(c,d) := \frac{\exp d - \exp c}{d-c},\ c \neq d.$$

Then by (4.9) we get

$$E(qa,qb)\,\mathrm{Ei}_{-q}(a,b) \geq 1. \tag{4.10}$$

From (2.12) we also have

$$\mathrm{Ei}_{-2q}(a,b) \geq A(\exp(-qa), \exp(-qb)) \exp\left(-q\sqrt{ab}\right), \tag{4.11}$$

where $q > 0$ and $b > a > 0$.

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[5] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[6] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[7] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[8] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[9] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematică, Cluj-Napoca, 1978. viii+92 pp.

[10] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[11] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[12] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[13] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[14] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[15] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[16] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[17] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[18] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_c^d f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[19] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[20] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[21] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[22] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No.3, Article 35.

[23] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[24] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[25] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[26] S. S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[27] S. S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[28] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[29] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[30] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[31] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[32] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[33] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[34] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[35] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[36] A. El Farissi, Simple proof and refeinment of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.

[37] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[38] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.

[39] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* (in press)

[40] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[41] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[42] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
[43] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.
[44] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.
[45] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.
[46] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.
[47] M. Rădulescu, S. Rădulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.
[48] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
[49] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
[50] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.
[51] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.
[52] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

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