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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *HG*-CONVEX FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for HG-convex functions defined on positive intervals are given. Applications for special means are also provided.

1. Introduction

Following [4] (see also [42]) we say that the function $f:I\subset\mathbb{R}\setminus\{0\}\rightarrow\mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq(1-t)f(x)+tf(y) \tag{1.1}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave.*

If $I\subset(0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

If $[a,b]\subset I\subset(0,\infty)$ and if we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\rightarrow\mathbb{R}$, defined by $g(t)=f\left(\frac{1}{t}\right)$, then we can state the following fact [4]:

Lemma 1. *The function f is HA-convex on $[a,b]$ if and only if g is convex in the usual sense on* $\left[\frac{1}{b},\frac{1}{a}\right]$.

Therefore, as examples of HA-convex functions we can take $f(t)=g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b},\frac{1}{a}\right]$.

In the recent paper [27] we obtained the following characterization result as well:

Lemma 2. *Let $f,h:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ be so that $h(t)=tf(t)$ for $t\in[a,b]$. Then f is HA-convex on the interval $[a,b]$ if and only if h is convex on* $[a,b]$.

Following [4] (see also [42]) we say that the function $f:I\subset\mathbb{R}\setminus\{0\}\rightarrow(0,\infty)$ is *HG-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq[f(x)]^{1-t}[f(y)]^{t} \tag{1.2}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.2) is reversed, then f is said to be *HG-concave.*

By the geometric-mean - arithmetic mean inequality we have that any HG-convex function is HA-convex. The converse is obviously not true.

We observe that $f:I\subset\mathbb{R}\setminus\{0\}\rightarrow(0,\infty)$ is HG-convex *if and only if* the function $\ln f:I\subset\mathbb{R}\setminus\{0\}\rightarrow\mathbb{R}$ is HA-convex on I.

Using Lemmas 1 and 2 we have:

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Proposition 1. *Let* $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ *and define the associated functions* $G_f:\left[\frac{1}{b},\frac{1}{a}\right]\rightarrow\mathbb{R}$ *defined by* $G_f(t)=\ln f\left(\frac{1}{t}\right)$ *and* $H_f:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ *defined by* $H_f(t)=t\ln f(t)$. *Then the following statements are equivalent:*

(i) The function f *is HG-convex on* $[a,b]$;

(ii) The function G_f *is convex on* $\left[\frac{1}{b},\frac{1}{a}\right]$;

(iii) The function H_f *is convex on* $[a,b]$.

For a convex function $h:[c,d]\rightarrow\mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d h(t)\,dt\leq\frac{h(c)+h(d)}{2}. \tag{1.3}$$

For related results, see [1]-[20], [23]-[28], [29]-[38] and [39]-[50].

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for *HG*-convex functions defined on positive intervals. Applications for special means are also provided.

2. Main Results

The following result holds.

Theorem 1. *Let* $f:[a,b]\subset(0,\infty)\rightarrow(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then for any* $\lambda\in[0,1]$ *we have the inequalities*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq\left[f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right]^{1-\lambda}\left[f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\right]^{\lambda} \\ &\leq\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right) \\ &\leq\sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}}\leq\sqrt{f(a)f(b)}. \end{aligned} \tag{2.1}$$

If we take $\lambda=\frac{1}{2}$ in (2.1), then we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq\sqrt{f\left(\frac{4ab}{a+3b}\right)f\left(\frac{4ab}{3a+b}\right)} \\ &\leq\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right) \\ &\leq\sqrt{f\left(\frac{2ab}{a+b}\right)\sqrt{f(a)f(b)}}\leq\sqrt{f(a)f(b)}. \end{aligned} \tag{2.2}$$

The *identric mean* $I(a,b)$ for two distinct positive numbers a,b is defined by

$$I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b):=\frac{b-a}{\ln b-\ln a}.$$

Theorem 2. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then*

$$f\left(L\left(a,b\right)\right)\leq\exp\left(\frac{1}{b-a}\int_a^b\ln f\left(t\right)dt\right)\leq\left[f\left(b\right)\right]^{\frac{\left(L(a,b)-a\right)b}{(b-a)L(a,b)}}\left[f\left(a\right)\right]^{\frac{\left(b-L(a,b)\right)a}{(b-a)L(a,b)}}. \tag{2.3}$$

If we write the classical Hermite-Hadamard inequality for the function H_f that is convex on $[a,b]$ when $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on $[a,b]$ and perform the required calculations, we get:

Theorem 3. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then we have*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}\leq\exp\left(\frac{1}{b-a}\int_a^b t\ln f\left(t\right)dx\right)\leq\sqrt{\left[f\left(b\right)\right]^b\left[f\left(a\right)\right]^a}. \tag{2.4}$$

We have the reverse inequalities as well:

Theorem 4. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then we have*

$$\begin{aligned}1&\leq\frac{\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)}{f\left(\frac{2ab}{a+b}\right)}\\&\leq\exp\left(\frac{1}{8}\left[\frac{f'_-\left(b\right)}{f\left(b\right)}b^2-\frac{f'_+\left(a\right)a^2}{f\left(a\right)}\right]\left(\frac{b-a}{ab}\right)\right)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}1&\leq\frac{\sqrt{f\left(a\right)f\left(b\right)}}{\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)}\\&\leq\exp\left(\frac{1}{8}\left[\frac{f'_-\left(b\right)}{f\left(b\right)}b^2-\frac{f'_+\left(a\right)a^2}{f\left(a\right)}\right]\left(\frac{b-a}{ab}\right)\right).\end{aligned} \tag{2.6}$$

The following related result also holds:

Theorem 5. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then we have*

$$\begin{aligned}1&\leq\frac{\sqrt{\left[f\left(a\right)\right]^a\left[f\left(b\right)\right]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f\left(t\right)dt\right)}\\&\leq\left(\frac{f\left(b\right)}{f\left(a\right)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}\left(b-a\right)\left(\frac{bf'_-\left(b\right)}{f\left(b\right)}-\frac{af'_+\left(a\right)}{f\left(a\right)}\right)\right)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}1&\leq\frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f\left(t\right)dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}}\\&\leq\left(\frac{f\left(b\right)}{f\left(a\right)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}\left(b-a\right)\left(\frac{bf'_-\left(b\right)}{f\left(b\right)}-\frac{af'_+\left(a\right)}{f\left(a\right)}\right)\right).\end{aligned} \tag{2.8}$$

From a different perspective we have:

Theorem 6. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then*

$$\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)\le\sqrt{f(x)\,[f(b)]^{\frac{a(b-x)}{x(b-a)}}\,[f(a)]^{\frac{b(x-a)}{x(b-a)}}}\tag{2.9}$$

for any $x\in[a,b]$.

If we take in (2.9), $x=\frac{a+b}{2}$, then we get from (2.9) that

$$\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)\le\sqrt{f\left(\frac{a+b}{2}\right)[f(b)]^{\frac{a}{a+b}}\,[f(a)]^{\frac{b}{a+b}}}.\tag{2.10}$$

3. Proofs

In [26], in order to improve Işcan's inequality [42] for HA-convex functions $g:[a,b]\subset(0,\infty)\to\mathbb{R}$,

$$g\left(\frac{2ab}{a+b}\right)\le\frac{ab}{b-a}\int_a^b\frac{g(t)}{t^2}dt\le\frac{g(a)+g(b)}{2},\tag{3.1}$$

we obtained the following result:

$$\begin{aligned}g\left(\frac{2ab}{a+b}\right)&\le(1-\lambda)\,g\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda g\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\\&\le\frac{ab}{b-a}\int_a^b\frac{g(t)}{t^2}dt\\&\le\frac{1}{2}\left[g\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)\,g(a)+\lambda g(b)\right]\\&\le\frac{g(a)+g(b)}{2},\end{aligned}\tag{3.2}$$

where $\lambda\in[0,1]$.

Now, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an HG-convex function on the interval $[a,b]$, then $g:=\ln f$ is HA-convex on $[a,b]$, and by (3.2) we get

$$\begin{aligned}&\ln f\left(\frac{2ab}{a+b}\right)\\&\le(1-\lambda)\ln f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda\ln f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\\&\le\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\\&\le\frac{1}{2}\left[\ln f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)\ln f(a)+\lambda\ln f(b)\right]\\&\le\frac{\ln f(a)+\ln f(b)}{2},\end{aligned}\tag{3.3}$$

that is equivalent to

$$\begin{aligned}\ln f\left(\frac{2ab}{a+b}\right) &\leq \ln\left(\left[f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right]^{1-\lambda}\left[f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\right]^{\lambda}\right)\\ &\leq \frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt \leq \ln\sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}}\\ &\leq \ln\sqrt{f(a)f(b)},\end{aligned}$$

and by taking the exponential we get the desired result (2.1).

We have the following result for HA-convex functions [26]:

Lemma 3. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$g(L(a,b)) \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{(L(a,b)-a)\,bg(b)+(b-L(a,b))\,ag(a)}{(b-a)\,L(a,b)}. \tag{3.4}$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an HG-convex function on the interval $[a,b]$, then $g:=\ln f$ is HA-convex on $[a,b]$, and by (3.4) we have

$$\begin{aligned}\ln f(L(a,b)) &\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx\\ &\leq \frac{(L(a,b)-a)\,b\ln f(b)+(b-L(a,b))\,a\ln f(a)}{(b-a)\,L(a,b)}\\ &= \ln\left([f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}[f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}\right).\end{aligned}\tag{3.5}$$

By taking the exponential in (3.5) we get the desired result (3.4).

We use the following results obtained by the author in [21] and [22]:

Lemma 4. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\begin{aligned}0 &\leq \frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt\\ &\leq \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha)\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}0 &\leq \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt-h\left(\frac{\alpha+\beta}{2}\right)\\ &\leq \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha).\end{aligned}\tag{3.7}$$

The constant $\frac{1}{8}$ is best possible in (3.6) and (3.7).

If $\ell:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an HA-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=\ell\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

Now, by (3.6) and (3.7) we have

$$0 \le \frac{g\left(\frac{1}{a}\right)+g\left(\frac{1}{b}\right)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} g(t)\,dt \le \frac{1}{8}\left[g'_{-}\left(\frac{1}{a}\right)-g'_{+}\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right) \tag{3.8}$$

and

$$0 \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} g(t)\,d - g\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right) \le \frac{1}{8}\left[g'_{-}\left(\frac{1}{a}\right)-g'_{+}\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right). \tag{3.9}$$

We also have

$$g'_{\pm}(s)=\ell'_{\mp}\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right)$$

and then

$$g'_{-}\left(\frac{1}{a}\right)=-\ell'_{+}(a)\,a^2 \text{ and } g'_{+}\left(\frac{1}{b}\right)=-\ell'_{-}(b)\,b^2.$$

From (3.8) and (3.9) we have

$$0 \le \frac{\ell(a)+\ell(b)}{2}-\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}\ell\left(\frac{1}{s}\right)ds \le \frac{1}{8}\left[\ell'_{-}(b)\,b^2-\ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right) \tag{3.10}$$

and

$$0 \le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}\ell\left(\frac{1}{s}\right)ds-\ell\left(\frac{2ab}{a+b}\right) \le \frac{1}{8}\left[\ell'_{-}(b)\,b^2-\ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right). \tag{3.11}$$

If we change the variable $\frac{1}{s}=u$, then $ds=-\frac{du}{u^2}$ and (3.10) and (3.11) can be written as

$$0 \le \frac{\ell(a)+\ell(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{\ell(t)}{t^2}dt \le \frac{1}{8}\left[\ell'_{-}(b)\,b^2-\ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right) \tag{3.12}$$

and

$$0 \le \frac{ab}{b-a}\int_{a}^{b}\frac{\ell(t)}{t^2}dt-\ell\left(\frac{2ab}{a+b}\right) \le \frac{1}{8}\left[\ell'_{-}(b)\,b^2-\ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right). \tag{3.13}$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an HG-convex function on the interval $[a,b]$, then $\ell:=\ln f$ is HA-convex on $[a,b]$, and by (3.12) and (3.13) we have

$$0\le \ln\sqrt{f(a)f(b)}-\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt \le \frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2-\frac{f'_+(a)a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right) \tag{3.14}$$

and

$$0\le \frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt-\ln f\left(\frac{2ab}{a+b}\right) \le \frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2-\frac{f'_+(a)a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right), \tag{3.15}$$

and the Theorem 4 is proved.

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an HG-convex function on the interval $[a,b]$, then H_f is convex on $[a,b]$ and by (3.6) and (3.7) we have after appropriate calculations

$$\begin{aligned}0&\le \ln\sqrt{[f(a)]^a[f(b)]^b}-\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\\&\le\frac{1}{8}\left[\ln f(b)+\frac{bf'_-(b)}{f(b)}-\ln f(a)-\frac{af'_+(a)}{f(a)}\right](b-a)\\&=\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}+\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\end{aligned}$$

and

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b t\ln f(t)\,dt-\ln\left(\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}\right)\\&\le\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}+\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right).\end{aligned}$$

These inequalities are equivalent to

$$\begin{aligned}0&\le\ln\left(\frac{\sqrt{[f(a)]^a[f(b)]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}\right)\\&\le\ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\right)\right]\end{aligned}$$

and

$$\begin{aligned}0&\le\ln\left(\frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}}\right)\\&\le\ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\right)\right]\end{aligned}$$

and by taking the exponential we get the desired results (2.7) and (2.8).

The following lemma is of interest in itself:

Lemma 5. *Let* $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$\frac{1}{2x}\left(\frac{g(b)a(b-x)+g(a)b(x-a)}{b-a}+xg(x)\right)\geq\frac{ab}{b-a}\int_a^b\frac{g(y)}{y^2}dy \tag{3.16}$$

for any $x\in[a,b]$.

Proof. Since $h(t)=tg(t)$ for $t\in[a,b]$ is convex, then by the gradient inequality for convex functions we have

$$xg(x)-yg(y)\geq\big(g(y)+yg'_-(y)\big)(x-y)$$

for any $x,y\in(a,b)$.

This is equivalent to

$$xg(x)-xg(y)\geq yg'_-(y)(x-y) \tag{3.17}$$

for any $x,y\in(a,b)$.

From (3.17) we have, by division with $xy^2>0$, that

$$\frac{1}{y^2}g(x)-\frac{1}{y^2}g(y)\geq\frac{g'_-(y)}{y}\left(1-\frac{y}{x}\right)$$

for any $x,y\in(a,b)$.

Taking the integral mean over y we have

$$\begin{aligned}&g(x)\frac{1}{b-a}\int_a^b\frac{1}{y^2}dy-\frac{1}{b-a}\int_a^b\frac{g(y)}{y^2}dy\\&\geq\frac{1}{b-a}\int_a^b\frac{g'_-(y)}{y}dy-\frac{1}{x}\frac{1}{b-a}\int_a^b g'_-(y)\,dy\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{g(x)}{ab}-\frac{1}{b-a}\int_a^b\frac{g(y)}{y^2}dy\\&\geq\frac{1}{b-a}\left[\frac{g(b)}{b}-\frac{g(a)}{a}+\int_a^b\frac{g(y)}{y^2}dy\right]-\frac{1}{x}\frac{g(b)-g(a)}{b-a}\\&=\frac{1}{b-a}\left(\frac{g(b)}{b}-\frac{g(a)}{a}\right)+\frac{1}{b-a}\int_a^b\frac{g(y)}{y^2}dy-\frac{1}{x}\frac{g(b)-g(a)}{b-a},\end{aligned}$$

for any $x\in(a,b)$. This can be written as

$$\frac{1}{x}\frac{g(b)-g(a)}{b-a}-\frac{1}{b-a}\left(\frac{g(b)}{b}-\frac{g(a)}{a}\right)\geq\frac{2}{b-a}\int_a^b\frac{g(y)}{y^2}dy-\frac{g(x)}{ab}$$

or as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[g(b)\frac{b-x}{xb}+g(a)\frac{x-a}{ax}\right]+\frac{g(x)}{ab}\right)\geq\frac{1}{b-a}\int_a^b\frac{g(y)}{y^2}dy.$$

This is equivalent to the desired result (3.16). □

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then $g:=\ln f$ is *HA*-convex on $[a,b]$, and by (3.16) we have

$$\frac{1}{2x}\left(\frac{a(b-x)\ln f(b)+b(x-a)\ln f(a)}{b-a}+x\ln f(x)\right)\geq\frac{ab}{b-a}\int_a^b\frac{\ln f(y)}{y^2}dy$$

for any $x \in [a,b]$.

This is clearly equivalent to

$$\ln\left(\sqrt{[f(b)]^{\frac{a(b-x)}{x(b-a)}}[f(a)]^{\frac{b(x-a)}{x(b-a)}}}\sqrt{f(x)}\right) \geq \frac{ab}{b-a}\int_a^b \frac{\ln f(y)}{y^2}dy \tag{3.18}$$

for any $x \in [a,b]$.

If we take the exponential in (3.18), then we get the desired result (2.9).

4. Applications

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(t) = t$. Using the geometric mean - harmonic mean inequality, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right) = \frac{xy}{tx+(1-t)y} \leq x^{1-t}y^t = [f(x)]^{1-t}[f(y)]^t,$$

which shows that f is HG-convex on $[a,b]$.

We need the following integrals

$$\frac{1}{b-a}\int_a^b \ln f(t)\,dt = \frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b)$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b t\ln f(t)\,dt &= \frac{1}{b-a}\int_a^b t\ln t dt \\ &= \frac{1}{2}A(a,b)\ln I\left(a^2,b^2\right) = \ln\left[I\left(a^2,b^2\right)^{\frac{1}{2}A(a,b)}\right]\end{aligned}$$

and

$$\int_a^b \frac{\ln f(t)}{t^2}dt = \int_a^b \frac{\ln t}{t^2}dt = \frac{b-a}{ab}\ln\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}$$

giving that

$$\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt = \ln\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}.$$

Now, if we write the inequality (2.1) for the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(t) = t$, we get

$$\begin{aligned}H(a,b) &\leq \left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)^{1-\lambda}\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)^{\lambda} \\ &\leq \left[I\left(a^{-1},b^{-1}\right)\right]^{-1} \\ &\leq \sqrt{\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)a^{1-\lambda}b^{\lambda}} \leq G(a,b),\end{aligned} \tag{4.1}$$

where $H(a,b) := \frac{2ab}{a+b}$ is the *harmonic mean.*

If we use the inequality (2.3) for $f(t) = t$, then we have

$$(L(a,b) \leq)\, I(a,b) \leq b^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}a^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{4.2}$$

If we use the inequality (2.4) for $f(t) = t$, $t \in [a,b]$, then we also get

$$[A(a,b)]^{A(a,b)} \leq I\left(a^2,b^2\right)^{\frac{1}{2}A(a,b)} \leq G\left(b^b,a^a\right). \tag{4.3}$$

From (2.5) and (2.6) for $f(t)=t$ we have

$$1\leq\frac{\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}}{H(a,b)}\leq\exp\left(\frac{(b-a)^{2}}{8ab}\right)\tag{4.4}$$

and

$$1\leq\frac{G(a,b)}{\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}}\leq\exp\left(\frac{(b-a)^{2}}{8ab}\right).\tag{4.5}$$

From (2.7) and (2.8) we also have

$$1\leq\frac{G\left(a^{a},b^{b}\right)}{\left[I\left(a^{2},b^{2}\right)\right]^{\frac{1}{2}A(a,b)}}\leq\left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}\tag{4.6}$$

and

$$1\leq\frac{\left[I\left(a^{2},b^{2}\right)\right]^{\frac{1}{2}A(a,b)}}{\left[A(a,b)\right]^{A(a,b)}}\leq\left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}.\tag{4.7}$$

Finally, from (2.10) we obtain

$$\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}\leq\sqrt{A(a,b)\,b^{\frac{a}{a+b}}a^{\frac{b}{a+b}}}.\tag{4.8}$$

Now consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(t)$. Using the harmonic mean-arithmetic mean inequality we have

$$\begin{aligned}f\left(\frac{xy}{tx+(1-t)y}\right)&=\exp\left(\frac{xy}{tx+(1-t)y}\right)\leq\exp((1-t)x+ty)\\&=[\exp(x)]^{1-t}[\exp(y)]^{t}=[f(x)]^{1-t}[f(y)]^{t}\end{aligned}$$

for any $x,y\in[a,b]$ and $t\in[0,1]$.

Now, if we use the inequality (2.1) for the HG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(t)$, then we get, after suitable calculations, that

$$\begin{aligned}H(a,b)&\leq\frac{2(1-\lambda)ab}{(1-\lambda)a+(\lambda+1)b}+\frac{2\lambda ab}{(2-\lambda)a+\lambda b}\\&\leq\frac{G^{2}(a,b)}{L(a,b)}\\&\leq\frac{1}{2}\left(\frac{ab}{(1-\lambda)a+\lambda b}+(1-\lambda)a+\lambda b\right)\leq A(a,b),\end{aligned}\tag{4.9}$$

for any $\lambda\in[0,1]$.

If we use the inequalities (2.5) and (2.6) for the HG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=\exp(t)$, then, by performing the required calculations, we get

$$0\leq\frac{G^{2}(a,b)}{L(a,b)}-H(a,b)\leq\frac{1}{4}\frac{A(a,b)}{G^{2}(a,b)}(b-a)^{2}\tag{4.10}$$

and

$$0\leq A(a,b)-\frac{G^{2}(a,b)}{L(a,b)}\leq\frac{1}{4}\frac{A(a,b)}{G^{2}(a,b)}(b-a)^{2}.\tag{4.11}$$

From the inequality (2.10) we also have

$$\frac{G^2(a,b)}{L(a,b)} \leq \frac{1}{2}\left(A(a,b) + b^{\frac{a}{a+b}} a^{\frac{b}{a+b}}\right). \tag{4.12}$$

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